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ROBUST EIGENVALUE ASSIGNMENT BY PERIODIC FEEDBACK[1]

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In this paper a robust periodic eigenvalue assignment algorithm is proposed for linear, time-invariant, discrete-time systems. The condition numbers characterizing the eigenstructure of the closed-loop system are assumed as a robustness measure. Alternative robustness measures are also introduced. The proposed periodic eigenvalue assignment algorithm has been tested on many different examples, giving satisfactory results.

1. INTRODUCTION

The problem of robust eigenvalue assignment for linear systems has been deeply investigated (see, e.g., [2], [7]) and a set of numerical algorithms is available in several computer aided control design systems. Moreover the use of a periodic controller for improving the robustness properties of closed-loop systems has been also investigated (see, e.g., [6], [9], [13]).

Consider a linear discrete-time system S described by:

$$x(k+1) = A\,x(k) + B\,u(k), \tag{1}$$

where $x(\cdot) \in \mathbb{R}^n$ is the state, $u(\cdot) \in \mathbb{R}^p$ is the control input, A and B are constant matrices of proper dimensions and $k \in \mathbb{Z}$. Matrix B is assumed to be full column-rank.

By means of a time-invariant state-feedback of the following form:

$$u(k) = F\,x(k) + \overline{u}(k), \tag{2}$$

where $\overline{u}(\cdot) \in \mathbb{R}^p$ and $F \in \mathbb{R}^{p\times n}$, it is possible to choose the eigenvalues of the closed-loop system described by

$$x(k+1) = A^F\,x(k) + B\,\overline{u}(k), \tag{3}$$

where $A^F := A + B\,F$. More precisely, for any prescribed symmetric set of n complex numbers $\mathcal{L}$ (a set of complex numbers subject to the requirement that

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nonreal elements appear in conjugate pairs) there exists a matrix F such that the set of the eigenvalues of A^F coincides with $\mathcal{L}$, if and only if system S is reachable [14].

For the case $p \geq 2$ (the multi-input case), under the hypothesis of reachability of S, there exist different choices of matrix F such that the set of the eigenvalues of A^F coincides with the prescribed symmetric set $\mathcal{L}$ and additional requirements can be itroduced in order to choose a specific solution. In general the robustness requirement is considered, requiring the eigenvalues of A^F to be as insensitive as possible to perturbations of the entries of matrices A and B.

In the case that the prescribed symmetric set $\mathcal{L}$ contains all distinct elements, the sensitivity of the eigenvalue λ_i, $i = 1, \ldots, n$, to perturbations of the entries of matrices A, B and F depends on the magnitude of the condition number c_i defined by [12]

$$c_i := \frac{1}{s_i} = \frac{\|w_i\|_2 \, \|v_i\|_2}{|w_i^T \, v_i|} \geq 1, \quad i = 1, 2, \ldots, n, \tag{4}$$

where v_i and w_i are the right and left eigenvectors of matrix A^F corresponding to the eigenvalue λ_i. Denoting with V the matrix composed by the n right eigenvectors of A^F, it is possible to consider as a global measure of the sensitivity of all the eigenvalues of A^F the condition $\kappa_2(V)$ of V defined by [12]

$$\kappa_2(V) := \|V\|_2 \, \|V^{-1}\|_2, \tag{5}$$

which satisfies the relation $\kappa_2(V) \geq c_i$ for $i = 1, 2, \ldots, n$. On the basis of this sensitivity measure, some algorithms for robust eigenvalue assignment have been developed (see, e. g., [2], [7]). Such algorithms produce a matrix F which allows to assign the prescribed symmetric set $\mathcal{L}$ of eigenvalues of A^F and to minimize the condition $\kappa_2(V)$ of V, imposing an appropriate set of eigenvectors of A^F. In general, only in the case $p = n$ these algorithms are able to produce a matrix F such that the condition $\kappa_2(V)$ of V is equal to the minimal value. In the case $p < n$, the existing structural constraints in the choice of eigenvectors prevent the achievement of the minimal values of the condition numbers. In this case, a feasible way for avoiding such structural constraints is to use a time-varying periodic feedback strategy [13].

Recently, a number of numerical efficient algorithms , such as the periodic Schur decomposition [1], the periodic QR and Singular Value decompositions [10], [5] and the discrete-time Floquet transformation [4], have been developed for periodic systems. The application of these algorithms could lead to an alternative solution to the periodic eigenvalue assignment problem here considered.

The aim of this paper is to propose an algorithm for the synthesis of a periodic state feedback law able to assign a prescribed set of eigenvalues and to better the robustness properties of the closed-loop system. In fact, the additional degrees of freedom made available in the periodic case can be used in order to improve the robustness properties of the closed-loop system. The condition numbers of the eigenvector matrices of the closed-loop system are assumed as a robustness measure and the periodic state-feedback law is deduced by the minimization of the condition numbers associated to the eigenvectors of the monodromy matrix of the closed-loop system. Alternative robustness measures are also introduced and tested.

2. ROBUST PERIODIC EIGENVALUE ASSIGNMENT

Consider the following periodic feedback law applied to the time-invariant system S described by (1):

$$u(k) = F(k)\,x(k) + \overline{u}(k), \quad \forall\, k \in \mathbb{Z}, \tag{6}$$

where $F(\cdot) \in \mathbb{R}^{p\times n}$ is a periodic matrix of period ω, i.e. $F(\ell+\omega) = F(\ell)$ for all $\ell \in \mathbb{Z}$. Denote by S' the periodic closed-loop system described by (1) and (6). The state transition matrix of S' is expressed by $\Phi^F(k,k_0) := A^F(k-1)\cdots A^F(k_0)$, with $k > k_0$, $k \in \mathbb{Z}$, $k_0 \in \mathbb{Z}$ and $\Phi^F(k,k) := I_n$ for all $k \in \mathbb{Z}$, where $A^F(k) := A+B\,F(k)$, for all $k \in \mathbb{Z}$, and I_n is the identity matrix of dimension n. By the periodicity it follows that $\Phi^F(k+h\omega, k_0+h\omega) = \Phi^F(k,k_0)$ for all $h \in \mathbb{Z}$. For an arbitrary initial time k_0, the state of S' satisfies the following equation:

$$\begin{aligned} x(h\omega+k_0+j+1) &= \Phi^F(k_0+j+1,k_0)\,x(h\omega+k_0) \\ &+\textstyle\sum_{\ell=0}^{j} \Phi^F(k_0+j+1,k_0+\ell+1)\,B\overline{u}(h\omega+k_0+\ell), \quad j=0,1,\dots,\omega-1,\ \forall\, h\in\mathbf{Z}^+, \end{aligned} \tag{7}$$

and the free-state response of S' is completely characterized by the eigenvalues of matrix $\Phi^F(k_0+\omega,k_0)$. The matrix $\Phi^F(k+\omega,k)$ is called the monodromy matrix of $A^F(\cdot)$ at time k and its characteristic polynomial is independent of k and it characterizes the stability of S' [3].

The characteristic polynomial of the monodromy matrix $\Phi^F(k+\omega,k)$ is independent of k, nevertheless the eigenstructure of $\Phi^F(k+\omega,k)$ is generally dependent on time k. Therefore the periodic eigenvalue assignment problem can be formally stated in the following way.

Problem 2.1. Given a symmetric set of n distinct eigenvalues $\mathcal{L}$, an integer ω and an ω-periodic non singular matrix $V(\cdot)$, find, if it exists, an ω-periodic matrix $F(\cdot)$ for the control law (6) such that the following conditions are satisfied

$$A^F(h)\,V(h) = V(h+1), \quad h = 0,1,\dots,\omega-2, \tag{8}$$

$$A^F(\omega-1)\,V(\omega-1) = V(0)\,\Lambda, \tag{9}$$

where $\Lambda := \mathrm{diag}\{\lambda_i,\ i=1,\dots,n,\}$, with $\lambda_i \in \mathcal{L}$.

The existence of a solution $F(\cdot)$ to Problem 2.1 implies that

$$\Phi^F(k+\omega,k)\,V(k) = V(k)\,\Lambda, \quad \forall\, k \in \mathbb{Z}, \tag{10}$$

and from this relation it is evident that the columns of $V(k)$ represent the linearly independent eigenvectors of $\Phi^F(k+\omega,k)$. Under the assumption that the matrix B is full rank, the following decomposition is considered

$$B = [\,U_a \quad U_b\,]\begin{bmatrix} Z \\ 0 \end{bmatrix}, \tag{11}$$

where $U = [\,U_a \quad U_b\,]$ is orthogonal and Z non-singular. This decomposition can be performed by the QR decomposition or the singular-value decomposition (SVD).

Now, for $h = 0, 1, \ldots, \omega - 1$, consider the ith column $v_i(h)$ of matrix $V(h)$ and define

$$\tilde{v}_i := \begin{bmatrix} v_i^T(0) & v_i^T(1) & \cdots & v_i^T(\omega - 1) \end{bmatrix}^T \in \mathbb{C}^{n\omega}, \quad i = 1, \ldots, n. \tag{12}$$

The vector $\tilde{v}_i$ will be called the lifted representation of the ith column of matrices $V(h), h = 0, 1, \ldots, \omega - 1$. Note that $v_i(h)$ represents a right eigenvector of $\Phi^F(h + \omega, h)$ associated with the assigned eigenvalue $\lambda_i \in \mathcal{L}$, for $i = 1, \ldots, n$. Moreover, define the following matrices

$$R(\lambda) := \begin{bmatrix} 0 & I_{(\omega-1)n} \\ \lambda I_n & 0 \end{bmatrix}, \quad \lambda \in \mathbb{C}, \tag{13}$$

$$\mathcal{A} := \operatorname{diag} \underbrace{\{A, A, \ldots, A\}}_{\omega \text{ blocks}}, \tag{14}$$

$$\mathcal{U}_b := \operatorname{diag} \underbrace{\{U_b, U_b, \ldots, U_b\}}_{\omega \text{ blocks}}. \tag{15}$$

In order to introduce a solvability condition for the Problem 2.1, define

$$\mathcal{P}_i := \ker \mathcal{U}_b^T (R(\lambda_i) - \mathcal{A}) \subset \mathbb{C}^{n\omega}, \quad i = 1, 2, \ldots, n. \tag{16}$$

The dimension of the subspace $\mathcal{P}_i$ is stated by the following Lemma.

Lemma 2.1. If system S is reachable:

$$\dim(\mathcal{P}_i) = p\omega, \qquad i = 1, 2, \ldots, n. \tag{17}$$

Proof. Define the following matrices:

$$\mathcal{U} := \operatorname{diag} \underbrace{\{U, U, \ldots, U\}}_{\omega \text{ blocks}}, \qquad \mathcal{U}_a := \operatorname{diag} \underbrace{\{U_a, U_a, \ldots, U_a\}}_{\omega \text{ blocks}}, \tag{18}$$

$$\mathcal{Z} := \operatorname{diag} \underbrace{\{Z, Z, \ldots, Z\}}_{\omega \text{ blocks}}, \qquad \mathcal{B} := \operatorname{diag} \underbrace{\{B, B, \ldots, B\}}_{\omega \text{ blocks}}. \tag{19}$$

Consider the matrix Q_i defined by

$$Q_i := \mathcal{U}^T \left[\mathcal{B} \quad (\mathcal{A} - R(\lambda_i)) \right], \qquad i = 1, \ldots, n, \tag{20}$$

whose rank, by the hypothesis of reachability of S, is equal to $n\omega$. Using the decomposition (11) and the definitions (15), (18) and (19), performing elementary row operations on the matrix Q_i, the following matrix is obtained

$$\tilde{Q}_i = \begin{bmatrix} \mathcal{Z} & \mathcal{U}_a^T (\mathcal{A} - R(\lambda_i)) \\ 0 & \mathcal{U}_b^T (\mathcal{A} - R(\lambda_i)) \end{bmatrix}, \qquad i = 1, \ldots, n, \tag{21}$$

whose rank is still equal to $n\omega$. Hence the rank of matrix $\mathcal{U}_b^T \ (\mathcal{A} - R(\lambda_i))$, $i = 1, \ldots, n$, is equal to $n\omega - p\omega$, and this proves the Lemma. □

Theorem 2.1. There exists a solution $F(\cdot)$ to the Problem 2.1 if and only if for $i=1,\ldots,n$ the lifted representation $\tilde{v}_i$ belongs to the subspace $\mathcal{P}_i$.

Proof. Under the decomposition (11) of matrix B, the conditions (8) and (9) assume the following form:

$$\begin{bmatrix} Z \\ 0 \end{bmatrix} F(h)\,V(h) = \begin{bmatrix} U_a^T \\ U_b^T \end{bmatrix} (V(h+1) - A\,V(h)), \quad h=0,1,\ldots,\omega-2, \tag{22}$$

$$\begin{bmatrix} Z \\ 0 \end{bmatrix} F(\omega-1)\,V(\omega-1) = \begin{bmatrix} U_a^T \\ U_b^T \end{bmatrix} (V(0)\,\Lambda - A\,V(\omega-1)) \tag{23}$$

and these relations are satisfied if and only if the following conditions are verified

$$\begin{aligned} U_b^T(V(h+1) - A\,V(h)) &= 0, \quad h=0,1,\ldots,\omega-2, && (24)\\ U_b^T(V(0)\Lambda - A\,V(\omega-1)) &= 0. && (25) \end{aligned}$$

Therefore,

$$\begin{aligned} U_b^T(v_i(h+1) - A\,v_i(h)) &= 0, \quad h=0,1,\ldots,\omega-2, \quad i=1,\ldots,n, && (26)\\ U_b^T(v_i(0)\lambda_i - A\,v_i(\omega-1)) &= 0, \quad i=1,\ldots,n. && (27) \end{aligned}$$

By the definitions (12), (13), (14) and (15), the relations (26) and (27) are equivalent to the following ones

$$\mathcal{U}_b^T(\mathcal{R}(\lambda_i) - \mathcal{A})\,\tilde{v}_i = 0, \quad i=1,\ldots,n, \tag{28}$$

and this proves the theorem. □

Note that, if the conditions of Theorem 2.1 are satisfied a solution $F(\cdot)$ to Problem 2.1 is given by

$$\begin{aligned} F(h) &= Z^{-1}\,U_a^T(V(h+1)\,V^{-1}(h) - A), \quad h=0,1,\ldots,\omega-2, && (29)\\ F(\omega-1) &= Z^{-1}\,U_a^T(V(0)\,\Lambda\,V^{-1}(\omega-1) - A). && (30) \end{aligned}$$

Then, if a solution $F(\cdot)$ to Problem 2.1 exists, the matrix $V(h)$ is composed by the right eigenvectors associated to the eigenvalues of $\Phi^F(h+\omega,h)$. The eigenvalues of the monodromy matrix $\Phi^F(h+\omega,h)$ are independent of time h, while the eigenstructure is generally $\omega-$periodically time-varying. Therefore the sensitivity of the eigenvalues to perturbations of the elements of $\Phi^F(h+\omega,h)$, measured by the condition number $\kappa_2(V(h)) := \|V(h)\|_2\|V(h)^{-1}\|_2$, is generally $\omega-$periodically time-varying. Thus, it seems appropriate to consider as a time-independent robustness measure the following quantity:

$$\kappa_\omega := \sum_{h=0}^{\omega-1} \kappa_2(V(h)). \tag{31}$$

The minimization of κ_ω implies the simultaneous minimization of $\kappa_2(V(h))$, for $h=0,1,\ldots,\omega-1$. Therefore, the robustness of the solution $F(\cdot)$ to Problem 2.1 depends on the assigned ω-periodic matrix $V(\cdot)$. This remark leads to the introduction of the following robust periodic eigenvalue assignment problem.

Problem 2.2. Given a symmetric set of n distinct eigenvalues $\mathcal{L}$, find an integer ω and an ω-periodic non-singular matrix $V(\cdot)$ which minimize κ_ω, under the constraint that $\tilde{v}_i$ belongs to the subspace $\mathcal{P}_i$, $i = 1, \ldots, n$.

Once a solution $V(\cdot)$ of Problem 2.2 has been found, a robust ω-periodic solution $F(\cdot)$ of Problem 2.1 is calculated using equations (29) and (30).

In the following an algorithm for the computation of a ω-periodic solution $V(\cdot)$ to Problem 2.2 and of a robust real ω-periodic solution $F(\cdot)$ to Problem 2.1 is proposed.

Algorithm 2.1

Step 0. Assign matrices A, B defining a system S described by equation (1) and a symmetric set $\mathcal{L}$ of desired closed-loop eigenvalues. Verify that system S is reachable. Set the period ω of the feedback law (6) equal to one. Compute the matrices U_a, U_b and Z in order to obtain the decomposition of B expressed by (11). Sort the symmetric set $\mathcal{L}$ in such a way that the complex elements appear in consecutive conjugate pairs.

Step 1. Compute the matrices $\mathcal{A}$ and $\mathcal{U}_b$ defined respectively by (14) and (15).

Step 2. Set $i = 1$. While $i \leq n$, compute a matrix $T(i) \in \mathbb{C}^{n\omega \times p\omega}$ such that $\mathrm{Im}\, T(i) = \mathcal{P}_i$ where $\mathcal{P}_i$ is defined by (16). If $\lambda_i \in \mathbb{R}$ set $i = i + 1$, else compute $T(i+1) = T^*(i)$ and set $i = i + 2$.

Step 3. Denote with $\left[T_{1,j}^T(i) \;\; T_{2,j}^T(i) \;\; \ldots \;\; T_{\omega,j}^T(i)\right]^T$, where $T_{\ell,j}(i) \in \mathbb{C}^{n\times 1}$, a block partition of the jth column of matrix $T(i)$, and solve the following minimization problem in the unknown scalar quantities $x_j(i)$, $\quad j = 1, \ldots, p\omega, \quad i = 1, \ldots, n$:

$$\min \kappa_\omega = \min \sum_{h=1}^{\omega} \kappa_2 \left[\sum_{j=1}^{p\omega} [x_j(1)\, T_{h,j}(1) \;\; x_j(2)\, T_{h,j}(2) \;\; \ldots \;\; x_j(n) T_{h,j}(n)] \right] \tag{32}$$

where $x_j(i) \in \mathbb{R}$ if λ_i is a real element, $x_j(i) \in \mathbb{C}$ and $x_j(i+1) = x_j^*(i)$ if λ_i is a complex element.

Step 4. Compute the matrices $V(h)$, $h = 0, 1, \ldots, \omega - 1$, according to

$$V(h) = \sum_{j=1}^{p\omega} [\overline{x}_j(1) T_{h+1,j}(1) \;\; \overline{x}_j(2) T_{h+1,j}(2) \;\; \ldots \;\; \overline{x}_j(n) T_{h+1,j}(n)]\,, \quad h = 0, 1, \ldots, \omega - 1, \tag{33}$$

where $\overline{x}_j(i)$, $j = 1, \ldots, p\omega$, $i = 1, \ldots, n$ denote the solutions of the minimization problem of Step 3.

Step 5. If the value of κ_ω is satisfactory go to Step 6, else set $\omega = \omega + 1$ and go to Step 1.

Step 6. Compute the robust ω-periodic solution to Problem 2.1 on the basis of equations (29) and (30).

Note that the constraint that $\tilde{v}_i$ belongs to the subspace $\mathcal{P}_i$, $i = 1, \ldots, n$, has been implicitly considered in the objective function defined at Step 3. In fact, on the basis of the notations introduced in the above algorithm, the vector $\tilde{v}_i \in \mathcal{P}_i \subset \mathbb{C}^{n\omega}$ can be written as

$$\tilde{v}_i = \sum_{j=1}^{p\omega} x_j(i) \begin{bmatrix} T_{1,j}^T(i) & T_{2,j}^T(i) & \ldots & T_{\omega,j}^T(i) \end{bmatrix}^T, \quad i = 1, \ldots, n. \tag{34}$$

Remark 2.1. The minimization problem considered at Step 3 of the Algorithm 2.1 has been solved using the Nelder–Mead algorithm [11] with a proper choice of the scalar quantities $x_j(i)$, $j = 1, \ldots, p\omega$, $i = 1, \ldots, n$. The major weakness of the algorithm is that it can be slow for an high number of variables. The use of more sophisticated minimization algorithms [2] can improve the overall performances of Algorithm 2.1.

Now denote with ξ the $np\omega$–dimensional vector composed by the scalar quantities $x_j(i)$, $j = 1, \ldots, p\omega$, $i = 1, \ldots, n$. The proposed algorithm at the Step 3 deduces a vector ξ which minimizes the quantity $\kappa_\omega(\xi)$. In order to avoid local-minima problems and to obtain better convergence performances it is possible to use different objective functions, such as those described in [2] for the case of time-invariant feedback. A possible extension of these objective functions to the periodic feedback case is given by the following functions:

$$f_a(\xi) := \sum_{h=1}^{\omega} \left(\|V(h)\|_2 + \|V(h)^{-1}\|_2 \right) \tag{35}$$

$$f_b(\xi) := \frac{-1}{f_a(\xi)} \tag{36}$$

$$f_c(\xi) := \log(f_a(\xi)). \tag{37}$$

In the next Section the results obtained using the previously defined objective functions $f_a(\cdot)$, $f_b(\cdot)$, $f_c(\cdot)$ are introduced and compared with the results obtained using $k_\omega(\cdot)$.

3. NUMERICAL RESULTS

The proposed Algorithm 2.1 has been tested on many different numerical examples. In all these examples the robustness properties provided by the proposed periodic state-feedback are better than those provided by a time-invariant feedback. In particular, in this Section, the application of Algorithm 2.1 to two numerical examples will be presented.

In order to compare the proposed approach with classical time-invariant eigenvalue assignment algorithms, the symmetric set $\tilde{\mathcal{L}}$, composed by the ωth roots of the elements of $\mathcal{L}$, is considered as desired set of closed-loop eigenvalues with time-invariant feedback.

Example 3.1. Consider the linear discrete-time system S described by (1), with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{38}$$

and the prescribed set of closed-loop eigenvalues $\mathcal{L} = \{0.1, 0.1i, -0.1i\}$. In this example the function PLACE of the MATLAB Control Toolbox, implementing a robust time-invariant algorithm for eigenvalue assignment [7], is not able to assign the set of eigenvalues $\tilde{\mathcal{L}}$. The proposed Algorithm 2.1 gives the following 2-periodic state feedback

$$F(0) = \begin{bmatrix} -0.5888 & 0.7217 & 1.1387 \\ -0.0090 & -1.9551 & 0.1854 \end{bmatrix}, \quad F(1) = \begin{bmatrix} -0.2856 & 0.0023 & 1.3231 \\ 0.1486 & -1.9287 & 0.2989 \end{bmatrix}$$

which leads to a closed-loop system with the prescribed set $\mathcal{L}$ of eigenvalues. The same Algorithm 2.1, with $f_a(\cdot), f_b(\cdot)$ and $f_c(\cdot)$ as objective functions, gives respectively the following 2-periodic state feedbacks:

$$F_a(0) = \begin{bmatrix} -0.4647 & 0.7190 & 1.2381 \\ -0.6951 & -2.1351 & -0.7876 \end{bmatrix}, \quad F_a(1) = \begin{bmatrix} -0.4022 & -0.0423 & 1.1582 \\ 0.0655 & -1.9839 & 0.0934 \end{bmatrix},$$

$$F_b(0) = \begin{bmatrix} -0.0259 & -0.2833 & 0.8447 \\ 2.1241 & -2.1691 & 2.7736 \end{bmatrix}, \quad F_b(1) = \begin{bmatrix} -0.4368 & 0.0305 & 1.1779 \\ 1.2423 & -1.9184 & 2.7638 \end{bmatrix},$$

$$F_c(0) = \begin{bmatrix} 0.1430 & -0.6736 & 0.2979 \\ 0.0995 & -1.8699 & 0.2760 \end{bmatrix}, \quad F_c(1) = \begin{bmatrix} -0.2952 & 0.0122 & 1.3117 \\ 0.9535 & -1.9615 & 1.9951 \end{bmatrix}.$$

The robustness properties of the proposed solutions have been tested by the introduction of random perturbations to all the entries of the matrices A and B. The sensitivity of the closed-loop eigenvalues to the considered perturbations is shown in Fig. 1. In this example the four different objective functions give different solutions with similar robustness properties.

Example 3.2. Consider the linearized model of an aircraft reported in [8]. The matrices A and B of the discrete-time model obtained by sampling with a period of $0.5s$ are given by:

$$A = \begin{bmatrix} 0.8539 & 0.1748 & -3.0041 & -0.0047 \\ 0.0033 & 0.9479 & 0.6501 & 0.0010 \\ 0.0107 & -0.0966 & 0.9386 & 0.0030 \\ 0.0918 & 0.0208 & -0.1489 & 0.9998 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0782 & 0.4018 \\ 0.0217 & -0.1722 \\ 0.0052 & 0.0100 \\ 0.0548 & 0.0193 \end{bmatrix}.$$

The prescribed set of closed-loop eigenvalues $\mathcal{L}$ is equal to $\{0.5, 0.3, 0.6i, -0.6i\}$. In this example the function PLACE of the MATLAB Control Toolbox, implementing a robust time-invariant algorithm for eigenvalue assignment [7], is able to assign

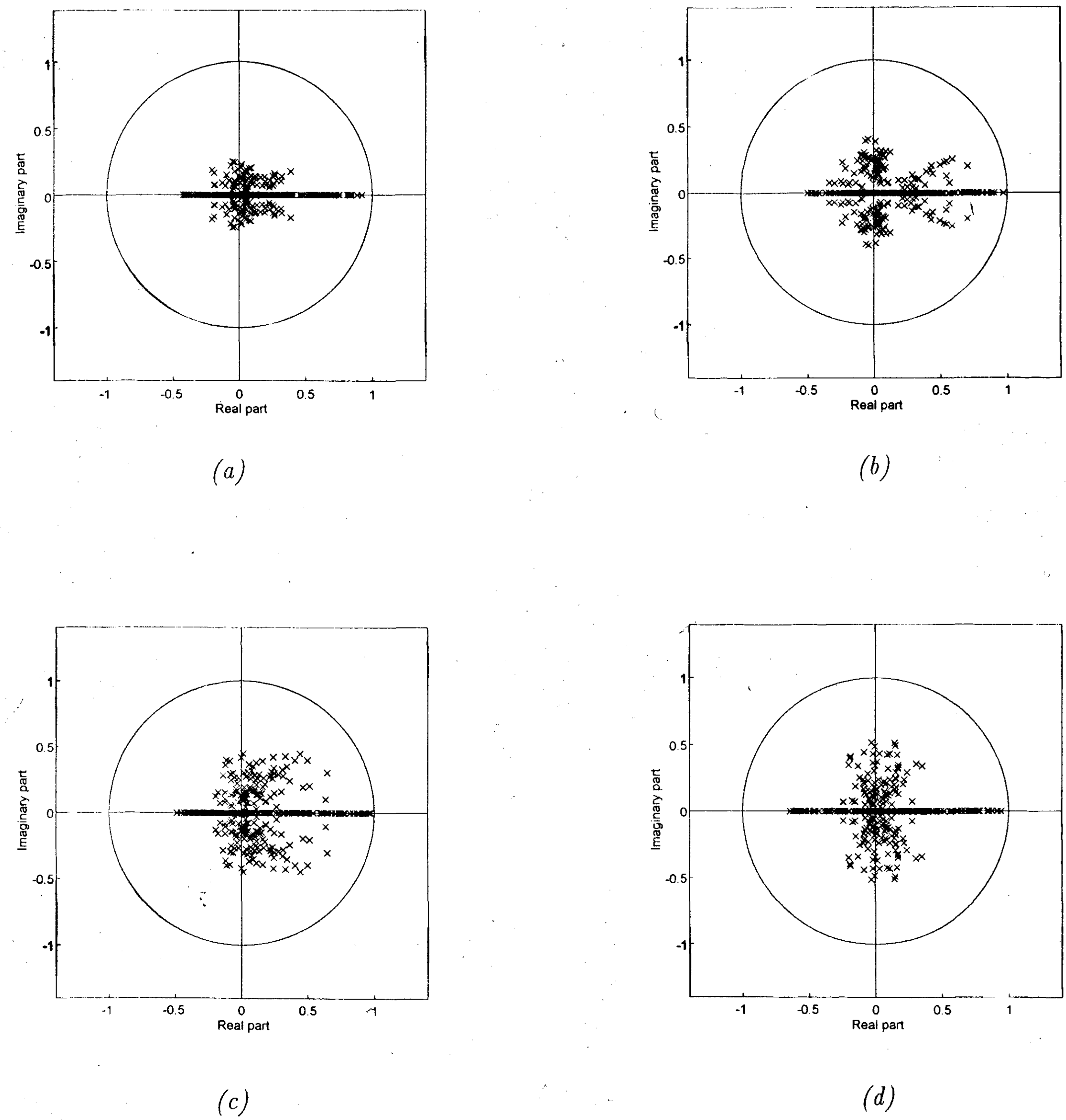

Fig. 1. Closed-loop eigenvalues of the Example 3.1: *(a)* 2-periodic feedback $F(\cdot)$ and 45 % random perturbations; *(b)* 2-periodic feedback $F_a(\cdot)$ and 45 % random perturbations; *(c)* 2-periodic feedback $F_b(\cdot)$ and 40 % random perturbations; *(d)* 2-periodic feedback $F_c(\cdot)$ and 35 % random perturbations.

the set of eigenvalues $\tilde{\mathcal{L}}$ and the resulting state feedback is given by:

$$F = \begin{bmatrix} 0.5703 & 1.3702 & -3.5671 & 1.3723 \\ 0.0091 & -2.7049 & 4.3471 & 9.0681 \end{bmatrix}.$$

The proposed Algorithm 2.1 gives the following 2-periodic state feedback

$$\begin{aligned} F(0) &= \begin{bmatrix} -2.5177 & -3.4347 & 10.0728 & -8.5975 \\ 4.3706 & 6.4019 & -14.5713 & 10.3575 \end{bmatrix}, \\ F(1) &= \begin{bmatrix} -1.4665 & -2.7157 & 9.8474 & -4.7430 \\ 1.4490 & 6.2176 & -24.4256 & -0.9734 \end{bmatrix}, \end{aligned}$$

which leads to a closed-loop system with the prescribed set $\mathcal{L}$ of eigenvalues. The same Algorithm 2.1 with the objective functions $f_a(\cdot)$ and $f_c(\cdot)$ gives respectively the following 2-periodic state feedbacks:

$$\begin{aligned} F_a(0) &= \begin{bmatrix} -2.2399 & -3.2332 & 10.0989 & -7.4906 \\ 4.7122 & 7.2882 & -20.7182 & 15.6913 \end{bmatrix}, \\ F_a(1) &= \begin{bmatrix} -1.6365 & -3.0502 & 11.2164 & -3.5850 \\ 1.9997 & 7.0154 & -27.0943 & -4.2167 \end{bmatrix}, \\ F_c(0) &= \begin{bmatrix} -1.6327 & -3.9379 & 9.8457 & -8.1547 \\ 3.9093 & 8.0391 & -17.9109 & 13.4841 \end{bmatrix}, \\ F_c(1) &= \begin{bmatrix} -1.1402 & -2.9885 & 9.1419 & -3.1157 \\ 0.7583 & 6.8432 & -19.1623 & -4.1670 \end{bmatrix}. \end{aligned}$$

Algorithm 2.1 with $f_b(\cdot)$ as objective function gives exactly the 2-periodic state feedback law $F_a(\cdot)$ obtained using $f_a(\cdot)$ as objective function. The robustness properties of the proposed solutions have been tested by the introduction of random perturbations to all the entries of matrices A and B. The sensitivity of the closed-loop eigenvalues to the considered perturbations is shown in Fig. 2. The results reported in this figure point out the better robustness properties obtained by periodic feedbacks. Furthermore, it is possible to observe that the use of different objective functions leads to periodic state feedbacks characterized by similar robustness properties.

4. CONCLUDING REMARKS

An algorithm for the synthesis of periodic state-feedback laws has been proposed. This algorithm produce state-feedback laws able to assign a prescribed set of eigenvalues and to minimize the condition numbers of the eigenvector matrices of the closed-loop system, representing the assumed robustness measure. Alternative robustness measures are also introduced. The proposed algorithm has been tested on many different examples and, in all these examples, the robustness properties provided by periodic state-feedbacks are better than those provided by time-invariant feedbacks. Moreover the use of different robustness measures generally leads to periodic state feedbacks characterized by similar robustness properties.

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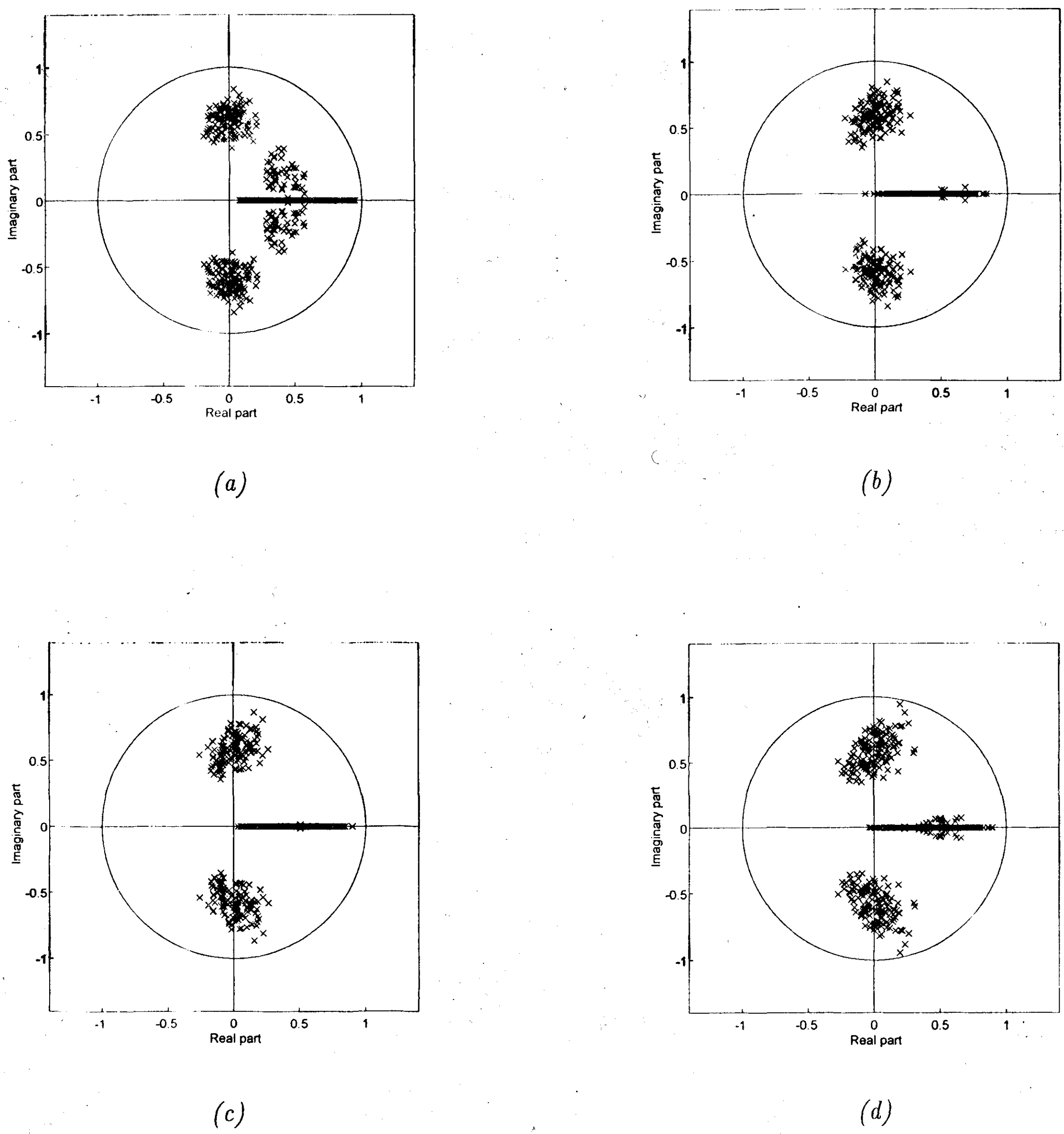

Fig. 2. Closed-loop eigenvalues of the Example 3.2: *(a)* time-invariant feedback F and 35 % random perturbations; *(b)* 2-periodic feedback $F(\cdot)$ and 50 % random perturbations; *(c)* 2-periodic feedback $F_a(\cdot)$ and 50 % random perturbations; *(d)* 2-periodic feedback $F_c(\cdot)$ and 50 % random perturbations.

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