Petr Tichavský

Estimating the angles of arrival of multiple plane waves. The statistical performance of the music and the minimum norm algorithms

ESTIMATING THE ANGLES OF ARRIVAL OF MULTIPLE PLANE WAVES

The Statistical Performance of the Music and the Minimum Norm Algorithms

PETR TICHAVSKÝ

The problem of estimating the angles of arrival of M plane waves incident simultaneously on a line array with $L > M$ sensors utilizing the special eigenstructure of the covariance matrix R of the signal plus noise at the output of the array is considered. The asymptotical analysis of the two most popular — MUSIC and Minimum-Norm — methods following the paper [1] by Kaveh and Barabell is completed, i.e. an approximate expression is derived for the resolution threshold of two independent closely spaced plane waves with equal power in noise in both methods. The results are verified by Monte Carlo simulations.

1. INTRODUCTION

In radar and sonar applications, seismology and other fields one is frequently interested in estimating the directions of arrival (DOA) of radiating sources from measurements provided by a passive array of sensors (antennas). The signals received by the sources very often consist of scaled and delayed replicas of the waveforms, radiated by the sources.

There has been a great deal of recent interest in the use of signal-subspace processing methods for the estimation of the DOA of multiple plane waves [1]–[4]. These methods in the simpler case of the narrow-band sources first form an estimate of the covariance matrix of the observation. The number of signal components is determined from the eigenvalues of this covariance matrix [5] and the angles are estimated from its eigenvectors.

This paper presents a statistical analysis of the two most popular methods as reported in [2] and [3], namely, the MUSIC and the Minimum-Norm methods, with the aim of determining their resolving properties. It makes the pioneer work [1] of Kaveh and Barabell complete.

A common feature of the two methods discussed here is the decomposition of the column space of the received signal covariance matrix into orthogonal "signal"

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and "noise" subspaces and formulating the DOA estimator in one or the other subspace. There is a certain function, called the null-spectrum, formed, so that it contains minima (nulls) at or in the immediate neighbourhoods of the true DOA.

When the spatial covariance matrix is estimated from a finite number of independent snapshots, the eigen-assisted methods also exhibit deviations from zero in their null-spectra at the true angles, resulting in a loss of resolution. This deviation is due to the statistical sampling perturbation of the signal and noise subspaces. This perturbation depends on the signal-to-noise ratios, signal parameters, and any array specifications, which together determine the resolving capability of the estimation method used. In this paper we shall examine the finite-sample bias in the null-spectra of the two eigen-assisted methods mentioned above.

2. FORMULATION OF THE METHODS

Consider an L-element line array of equally spaced sensors upon which are simultaneously incident $M(<L)$ plane wave signals radiated from M narrowband sources located at angles $\theta_1, \ldots, \theta_M$ relative to the array normal. If $a_k(i)$ denotes the complex amplitude of the ith signal at the origin at time instant t_k the observations at the mth element can be written as

$$
(1) \qquad x_k(m) = \sum_{i=1}^{M} a_k(i) \, e^{j(m-1)\omega_i} + n_k(m)
$$

where

$$
\omega_i = \frac{2\pi}{\lambda} \, d \sin \theta_i, \quad m = 1, \ldots, L, \quad k = 1, \ldots, N, \quad i = 1, \ldots, M,
$$

λ is the wavelength of the signal radiated, d is the distance of the sensors and $n_k(m)$ represents the additive, zero-mean complex Gaussian noise with $E|n_k(m)|^2 = \sigma^2$, which is assumed to be mutually independent for different m and k. We also assume that amplitudes $a_k(i)$ with varying k form an M-dimensional zero mean stationary Gaussian process with the joint (source) covariance matrix R_S of elements

$$
(R_S)_{i,j} = E[a_k(i) \, a_k^*(j)] .
$$

Here "*" denotes complex conjugate. In what follows, we will suppose that the matrix R_S is nonsingular. By defining an M-vector $\theta = (\theta_1, \ldots, \theta_M)$, and L-vector $X_k = (x_k(1), \ldots, x_k(L))^T$ an L-vector $n_k = (n_k(1), \ldots, n_k(L))^T$ an M-vector $a_k = (a_k(1), \ldots, a_k(M))^T$ and an $N \times M$ matrix of direction vectors

$$
D(\theta) = (d(\theta_1), \ldots, d(\theta_M)),
$$

whose ith column is given by

$$
d(\theta_i) = (1, e^{j\omega_i}, \ldots, e^{j(L-1)\omega_i})^T ,
$$

(1) can be written compactly as

$$
(2) \qquad\qquad X_k = D(\theta) \, a_k + n_k .
$$

The observation covariance matrix $\mathbf{R}$, and its estimate $\hat{\mathbf{R}}$ are given by

$$\mathbf{R} = \mathsf{E}[X_k X_k^{\mathrm{H}}] = \mathbf{D}(\theta) \, \mathbf{R}_S \, \mathbf{D}(\theta)^{\mathrm{H}} + \sigma^2 \mathbf{I} , \tag{3}$$

where H denotes Hermitian transpose, and $\mathbf{I}$ is the identity matrix

$$\hat{\mathbf{R}} = \frac{1}{N} \sum_{k=1}^{N} X_k X_k^{\mathrm{H}} . \tag{4}$$

$\hat{\mathbf{R}}$ is the statistic on which the angular spectral estimates discussed in this paper are based. Throughout this presentation "$\hat{}$" will denote the estimate of the quantity over which it appears. This estimate, in turn, is a result of using $\hat{\mathbf{R}}$ in place of $\mathbf{R}$.

Both methods are based on an eigendecomposition of the matrix $\mathbf{R}$ as follows:

$$\mathbf{R} = \sum_{i=1}^{L} \lambda_i \mathbf{S}_i \mathbf{S}_i^{\mathrm{H}} , \tag{5}$$

where $\lambda_1, \ldots, \lambda_L$ are eigenvalues of $\mathbf{R}$ in nonincreasing order and $\mathbf{S}_i$ are the corresponding orthonormal eigenvectors. It is easy to show that the equation (3) implies $\lambda_1 \geqq \ldots \geqq \lambda_M > \lambda_{M+1} = \ldots = \lambda_L = \sigma^2$. The eigenvectors $\mathbf{S}_i$, $i \leq M$, form the base of the signal subspace. All the directions vectors $\mathbf{d}(\theta_1), \ldots, \mathbf{d}(\theta_M)$ are shown to be elements of the signal subspace and they are orthogonal to the noise subspace. These properties allow to find different null-spectra $D(\omega)$, nonnegative real functions satisfying relations $D(\omega_i) = 0$ and $D(\omega) > 0$, $\omega \neq \omega_i$, $i = 1, \ldots, M$.

A. MUSIC

Under the plane wave model, null-spectrum $D_{MUS}(\omega)$ for the MUSIC method is given in terms of signal-space quantities by

$$D_{MUS}(\omega) = 1 - \sum_{i=1}^{M} |\mathbf{V}^{\mathrm{H}}(\omega) \, \mathbf{S}_i|^2 , \tag{5}$$

where the steering vector is given by

$$\mathbf{V}(\omega) = \frac{1}{\sqrt{L}} (1, e^{j\omega}, \ldots, e^{j(L-1)\omega})^{\mathrm{T}} . \tag{7}$$

The sense of this definition is that $D_{MUS}(\omega)$ is the squared Euclidean distance from the steering vector $\mathbf{V}(\omega)$ to the signal subspace. When $\mathbf{V}(\omega)$ coincides with a signal direction vector at angular frequency ω_i, then $D_{MUS}(\omega_i) = 0$ as desired.

B. Minimum-Norm

This technique finds the vector A with a unit first element which is entirely in the noise-subspace and has the minimum Euclidean norm. In [1] and [3] the null-spectrum is defined by

$$D_{MN}(\omega) = |\mathbf{V}^{\mathrm{H}}(\omega) \, A|^2 . \tag{8}$$

We reproduce a useful expression for A, which is given in [3]. Let the matrix $\mathbf{G}$

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be constructed as

$$G = (S_1, \ldots, S_M) = \begin{pmatrix} g^{\mathrm{T}} \\ G' \end{pmatrix},$$

where g^{T} is the first row of G and G' denotes the rest of G. Then

$$A = \left(1, \, -g^{\mathrm{H}}G'^{\mathrm{T}}/(1 - g^{\mathrm{H}}g)\right)^{\mathrm{T}}.$$

We change the definition of the null-spectra by using the $\left(g^{\mathrm{H}}g - 1\right)$ — factor of this vector. Such multiplicative factor has no influence for resolution properties of the method, but the null-spectrum has a simple form, as follows:

$$(9) \qquad D_{MN}(\omega) = \left| V^{\mathrm{H}}(\omega) \begin{pmatrix} g^{\mathrm{T}}g^* - 1 \\ G'g^* \end{pmatrix} \right|^2 = \left| V^{\mathrm{H}}(\omega) \, Gg^* - L^{-1/2} \right|^2.$$

Also in this case $D_{MN}(\omega_i) = 0$, $i = 1, \ldots, M$, as desired.

3. THE STATISTICAL BEHAVIOUR OF METHODS

In this section, approximate statistical behaviour of the MUSIC and the Minimum-Norm methods are examined. Statistical bias, especially in the neighbourhood of ω_i can be interpreted as indicators of the resolving capabilities of these techniques. We evaluate $\mathsf{E}\big[\hat{D}_{MUS}(\omega)\big]$ and $\mathsf{E}\big[\hat{D}_{MN}(\omega)\big]$ and relate them to the angular separation, the source covariance matrix, array signal-to-noise ratios and number of snapshots. We close the section by deriving an expression for the resolution threshold of MUSIC and Minimum-Norm method in the case of two independent equipowered sources.

In the following analysis we make use of the asymptotic statistics for the eigenvalues and eigenvectors of the sample covariance matrix $\hat{R}$ of a complex Gaussian process. They are derived in [1] using perturbation methods results in the eigenvalue problem [6] and the known second moments of Wishart distribution of the matrix $\hat{R}$. The statistics are valid under the additional assumption that the "signal" eigenvalues are distinct, i.e. $\lambda_1 > \lambda_2 > \ldots > \lambda_M$. Let us denote $\eta_i = \hat{S}_i - S_i$ and $\beta_i = \hat{\lambda}_i - \lambda_i$. Then η_i and β_i have the following asymptotic properties:

$$(9) \qquad \mathsf{E}\big[\beta_i\beta_j\big] = \frac{\lambda_i^2}{N} \, \delta_{ij} + o\left(\frac{1}{N}\right),$$

$$(10) \qquad \mathsf{E}\big[\eta_i\eta_j^{\mathrm{H}}\big] = \frac{\lambda_i}{N} \, \delta_{ij} \sum_{\substack{k=1 \\ k \neq 1}}^{L} \frac{\lambda_k}{\left(\lambda_i - \lambda_k\right)^2} \, S_k S_k^{\mathrm{H}} + o\left(\frac{1}{N}\right),$$

$$(11) \qquad \mathsf{E}\big[\eta_i\big] = -\frac{\lambda_i}{2N} \sum_{\substack{k=1 \\ k \neq i}}^{L} \frac{\lambda_k}{\left(\lambda_i - \lambda_k\right)^2} \, S_i + o\left(\frac{1}{N}\right) = a_i S_i + o\left(\frac{1}{N}\right),$$

where δ_{ij} is the Kronecker delta, $i, j = 1, \ldots, M$ and

$$(12) \qquad a_i = -\frac{\lambda_i}{2N} \sum_{\substack{k=1 \\ k \neq i}}^{L} \frac{\lambda_k}{\left(\lambda_i - \lambda_k\right)^2}.$$

We are interested especially in the case when $M = 2$. Introducing the notation

$$(13) \qquad b = \frac{\lambda_1 \lambda_2}{N(\lambda_1 - \lambda_2)^2} \,,$$

$$(14) \qquad c_i = \frac{\lambda_i \sigma^2}{N(\lambda_i - \sigma^2)^2} \,, \quad i = 1, 2 \,,$$

$$(15) \qquad c = c_1 + c_2 \,,$$

we can write

$$(16) \qquad a_i = -\tfrac{1}{2}b - \tfrac{1}{2}c_i(L - 2) \,, \quad i = 1, 2,$$

$$(17) \qquad \mathsf{E}\big[\boldsymbol{\eta}_i \boldsymbol{\eta}_i^{\mathrm{H}}\big] = b\boldsymbol{S}_{3-i}\boldsymbol{S}_{3-i}^{\mathrm{H}} + c_i\big[\boldsymbol{I} - \boldsymbol{S}_1\boldsymbol{S}_1^{\mathrm{H}} - \boldsymbol{S}_2\boldsymbol{S}_2^{\mathrm{H}}\big] \,.$$

Now, we are ready to evaluate the mean of the MUSIC and the Minimum-Norm null-spectrum.

A. MUSIC

The estimated null-spectrum for this method is given by

$$(18) \qquad \hat{D}_{MUS}(\omega) = 1 - \boldsymbol{V}^{\mathrm{H}}(\omega)\,\Big(\sum_{i=1}^{M} \hat{\boldsymbol{S}}_i \hat{\boldsymbol{S}}_i^{\mathrm{H}}\Big)\,\boldsymbol{V}(\omega) \,.$$

The expected value of $\hat{D}_{MUS}(\omega)$, using the definition of $\boldsymbol{\eta}$ is

$$(19) \qquad \mathsf{E}\big[\hat{D}_{MUS}(\omega)\big] = 1 - \boldsymbol{V}^{\mathrm{H}}(\omega)\,\Big(\sum_{i=1}^{M} \boldsymbol{S}_i \boldsymbol{S}_i^{\mathrm{H}}\Big)\,\boldsymbol{V}(\omega) - \boldsymbol{V}^{\mathrm{H}}(\omega)\,\mathsf{E}\Big(\sum_{i=1}^{M} \boldsymbol{\eta}_i \boldsymbol{\eta}_i^{\mathrm{H}}\Big)\,\boldsymbol{V}(\omega) -$$
$$- 2\,\mathrm{Re}\,\Big[\boldsymbol{V}^{\mathrm{H}}(\omega)\,\Big(\sum_{i=1}^{M} \boldsymbol{S}_i\, \mathsf{E}\boldsymbol{\eta}_i^{\mathrm{H}}\Big)\,\boldsymbol{V}(\omega)\Big] \,.$$

Substituting for the expectations in (19) gives

$$(20) \qquad \mathsf{E}\big[\hat{D}_{MUS}(\omega)\big] = D_{MUS}(\omega) - \boldsymbol{V}^{\mathrm{H}}(\omega)\Big[\sum_{i=1}^{M} \sum_{\substack{j=1 \\ j \neq i}}^{L} \frac{\lambda_i \lambda_j}{N(\lambda_i - \lambda_j)^2}\,\big(\boldsymbol{S}_j \boldsymbol{S}_j^{\mathrm{H}} - \boldsymbol{S}_i \boldsymbol{S}_i^{\mathrm{H}}\big)\Big]\boldsymbol{V}(\omega) \,.$$

In the case of $M = 2$ after some simplifications we get

$$(21) \qquad \mathsf{E}\big[\hat{D}_{MUS}(\omega)\big] = (1 - c)\,D_{MUS}(\omega) + (L - 2)\big[c_1|\boldsymbol{V}^{\mathrm{H}}\boldsymbol{S}_1|^2 + c_2|\boldsymbol{V}^{\mathrm{H}}\boldsymbol{S}_2|^2\big] \,.$$

B. Minimum-Norm

The evaluation of the mean of the null-spectrum for this method is in general a little bit more complex than the previous one. Let s_i be the first element of the vector $\boldsymbol{S}_i$, $i = 1, \ldots, M$. We can write

$$(22) \qquad s_i = \mathrm{e}^{\mathrm{T}}\boldsymbol{S}_i \,,$$

where $\mathbf{e} = (1, 0, \ldots, 0)^{\mathrm{T}}$. Following (9) we have

$$(23) \qquad \hat{D}_{MN}(\omega) = \big|\boldsymbol{V}^{\mathrm{H}}\hat{\boldsymbol{G}}\hat{\boldsymbol{g}}^* - L^{-1/2}\big|^2 = \Big|\sum_{i=1}^{M} \hat{s}_i^* \boldsymbol{V}^{\mathrm{H}}\hat{\boldsymbol{S}}_i - L^{-1/2}\Big|^2 =$$

$$= \frac{1}{L} + \sum_{i=1}^{M} |\hat{s}_i|^2 \, |V^H \hat{S}_i|^2 - \frac{2}{\sqrt{L}} \sum_{i=1}^{M} \text{Re} \left(\hat{s}_i^* V^H \hat{S}_i \right) .$$

Substituting $\hat{S}_i = S_i + \eta_i$, $\hat{s}_i = s_i + \varrho_i$ and neglecting higher-than-second-order powers of η_i and ϱ_i we obtain

$$(24) \qquad \hat{D}_{MN}(\omega) = D_{MN}(\omega) + \sum_{i=1}^{M} \{ |s_i|^2 \left[2 \, \text{Re} \left(V^H S_i \eta_i^H V \right) + |V^H \eta_i|^2 \right] +$$

$$+ \, 2 \, \text{Re} \left(s_i^* \varrho_i \right) \left[|V^H S_i|^2 + 2 \, \text{Re} \left(V^H S_i \eta_i^H V \right) \right] + |\varrho_i|^2 \, |V\|S_i|^2 \} -$$

$$- \, \frac{2}{\sqrt{L}} \sum_{i=1}^{M} \text{Re} \left(s_i^* V^H \eta_i + \varrho_i^* V^H S_i + \varrho_i^* V^H \eta_i \right) .$$

The expression for $\mathsf{E}[\hat{D}_{MN}(\omega)]$ we can obtain using the relation $\varrho_i = \mathbf{e}^T \eta_i$ and substituting the expectation values (10) and (11) in the places of η_i and $\eta_i \eta_i^H$. The resultant expression for $M = 2$ is

$$(25) \qquad \mathsf{E}[\hat{D}_{MN}(\omega)] = D_{MN}(\omega) + c_1 |s_1|^2 + c_2 |s_2|^2 - \frac{2}{L} c +$$

$$+ \, |V^H S_1|^2 \left[2b \left(|s_2|^2 - |s_1|^2 \right) - 2L c_1 |s_1|^2 + c_1 - c |s_2|^2 \right] +$$

$$+ \, |V^H S_2|^2 \left[2b \left(|s_1|^2 - |s_2|^2 \right) - 2L c_2 |s_2|^2 + c_2 - c |s_1|^2 \right] +$$

$$+ \, 2 \, \text{Re} \left(V^H S_1 S_2^H V s_1^* s_2 \right) \left(2b - c \right) + 2L^{-1/2} \, \text{Re} \left(s_1^* V^H S_1 \right) \left[(L - 2) c_1 - c \right] +$$

$$+ \, 2L^{-1/2} \, \text{Re} \left(s_2^* V^H S_2 \right) \left[(L - 2) c_2 - c \right] .$$

Now, we would like to express the eigenvalues λ_i, the eigenvectors S_i, $i = 1, \ldots, M$, and other constants in terms of the signal, noise and array parameters of interest.

We begin by writing the covariance matrix R via (3) in the form

$$(26) \qquad R = D(\theta) \, R_S \, D(\theta)^H + \sigma^2 I =$$

$$= \sum_{i=1}^{M} \sum_{j=1}^{M} (R_S)_{i,j} d(\theta_i) \, d(\theta_j)^H + \sigma^2 I .$$

We can look for the signal eigenvectors of this matrix as for linear combinations of $d(\theta_1), \ldots, d(\theta_M)$ or equivalently $V(\omega_1), \ldots, V(\omega_M)$. We will do it in Appendix A for the case of two independent equipowered sources, i.e. for

$$(27) \qquad (R_S)_{1,1} = (R_S)_{2,2} = P ,$$

$$(R_S)_{1,2} = (R_S)_{2,1} = 0 .$$

We define signal-to-noise ratio in this case as $\xi = P/\sigma^2$. Following Kaveh and Barabell in order to obtain a quantitative measure of the resolution threshold for two closely spaced sources we use a nonprobabilistic approach based on the mean of the null-spectrum. They have proposed that the signal-to-noise ratio at which $\mathsf{E}[\hat{D}(\omega_1)] = \mathsf{E}[\hat{D}(\omega_2)] = \mathsf{E}[\hat{D}(\omega_m)]$, where $\omega_m = \frac{1}{2}(\omega_1 + \omega_2)$, is approximately this threshold.

This proposition allows us to compute resolution thresholds (Appendix B) as follows:

$$(28) \qquad \xi_{MUS} = \frac{180\omega_d^{-4}}{NL(L^2-1)(L+2)} \left(1 + \sqrt{\left(1 + \tfrac{1}{15}N(L+2)\,\omega_d^2\right)}\right),$$

$$(29) \qquad \xi_{MN} = \frac{36(L-3)\,\omega_d^{-4}}{NL(L-1)(L^2-1)(L-2)} \left(1 + \sqrt{\left(1 + \frac{(L-1)(L-2)}{3(L-3)}\,N\omega_d^2\right)}\right),$$

where $\omega_d = \tfrac{1}{2}(\omega_1 - \omega_2)$, one half of the angular separation. The thresholds are shown to have very similar form, especially in the case of large L. For $N \ll L^{-1}\omega_d^{-2}$ they vary as $N^{-1}L^{-4}\omega_d^{-4}$, whereas for $N \gg L^{-1}\omega_d^{-2}$ they vary as $N^{-1/2}L^{-7/2}\omega_d^{-3}$. The resolution threshold for the Minimum-Norm method is approximately 5 times smaller for low N and $\sqrt{5}$ times smaller for large N, than the MUSIC one. In usual logarithmic scale it means the difference in resolution about $3\cdot5-7$ dB. We did not compute ξ_{MN} for $L = 3$. The Minimum-Norm resolution threshold seems to exhibit another kind of dependence on parameters N, ω_d in this case.

Finally, to verify the accuracy of this approximate analysis, ξ_{MUS} and ξ_{MN} were calculated for two combinations of array and signal parameters according to (28) and (29) and compared to probability of resolution as a function of the signal-to-noise ratio from Monte Carlo simulations. In each simulation trial, two sources were considered resolved if the null-spectrum of the algorithm under test satisfies the inequality

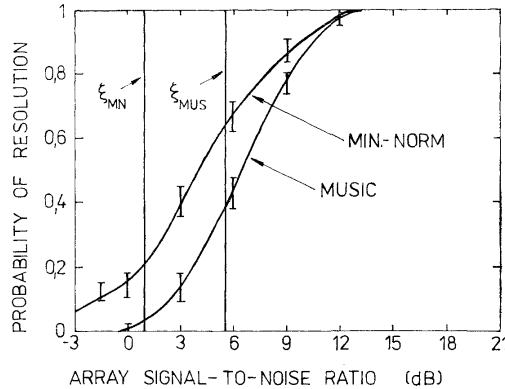$$\hat{D}(\omega_m) > \max\left\{\hat{D}(\omega_1), \hat{D}(\omega_2)\right\}.$$

Fig. 1. Probability of resolution as a function of the ASNR for two equipowered emitters spaced $0\cdot4$ beamwidths apart. 100 looks, 100 trials, five-element array. The perpendicular abscissae mark the confidence intervals with respect to the finite number of trials.

Figures 1 and 2 show close agreement between the theoretical predictions and simulation results; the resolution thresholds (28) and (29) correspond with $0.2 - 0.35$ probability of resolution in the sense described above.
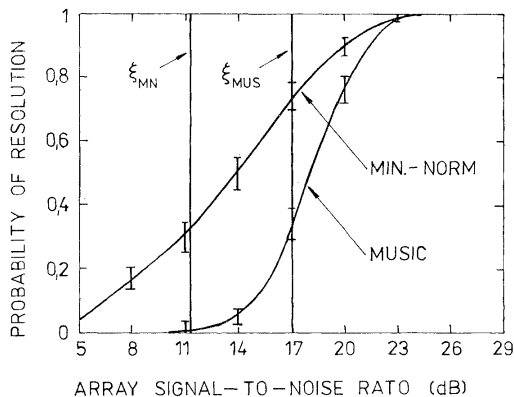

Fig. 2. Probability of resolution as a function of the ASNR for two equipowered emitters spaced 0.1 beamwidths apart. 100 looks, 100 trials, ten element array. The perpendicular abscissae mark the confidence intervals with respect to the finite number of trials.

4. CONCLUSION

This paper presents an asymptotic evaluation of the resolving capability of two eigen-assisted spectral domain estimators of the directions of arrival of closely spaced, narrow-band plane waves. The mean of the null-spectra of the MUSIC and Minimum-Norm algorithms, including $O(N^{-1})$ errors were derived, following the paper [1] by Kaveh and Barabell. Their work was completed in the sense that for *both* algorithms expressions for a plausible detection threshold were derived and compared. The theoretical results show close agreement with the results from Monte Carlo simulations.

APPENDIX A

This appendix develops expressions for the eigenvalues, eigenvectors, several associated inner products and constants that are needed for the approximate evalua-tion of the mean of the null-spectra for two closely spaced equipowered sources.

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Let us simplify the notation: $V_1 = V(\omega_1)$, $V_2 = V(\omega_2)$, $V_m = V(\omega_m)$, where $\omega_m = \frac{1}{2}(\omega_1 + \omega_2)$. The covariance matrix R has the form

$$R = PL(V_1 V_1^H + V_2 V_2^H) + \sigma^2 I \,.$$

We will look for the eigenvectors of R of the form

(A1) $$S = V_1 + \alpha V_2$$

Substituting it into eigen-equation $RS = \lambda S$ and comparing the coefficients standing before V_1 and V_2 we obtain a set of equations

(A2) $$PL(1 + \alpha d) + \sigma^2 = \lambda$$
$$PL(d^* + \alpha) + \alpha \sigma^2 = \alpha \lambda \,,$$

where $d = V_1^H V_2$. There are two solutions of this set:

(A3) $$\alpha_{1,2} = \pm \sqrt{\left(\frac{d^*}{d}\right)} = \pm \frac{|d|}{d}$$

(A4) $$\lambda_{1,2} = PL(1 \pm |d|) + \sigma^2$$

The corresponding normed eigenvectors are as follows:

(A5) $$S_{1,2} = \frac{V_1 \pm \dfrac{|d|}{d} V_2}{\sqrt{(2(1 \pm |d|))}} \,.$$

Let $\omega_d = \frac{1}{2}(\omega_1 - \omega_2)$. Then

(A6) $$d = V_1^H V_2 = \sum_{k=1}^{L} \frac{1}{L} e^{-j(k-1)2\omega_d} = \frac{1}{L} e^{-j(L-1)\omega_d} \frac{\sin(L\omega_d)}{\sin \omega_d}$$

Let us assume that the angle separation $2\omega_d$ is small now. (A6) can be expanded as

(A7) $$d = e^{-j(L-1)\omega_d}\{1 - \tfrac{1}{6}(L^2 - 1) \omega_d^2 + [\tfrac{1}{120}(L^4 - 1) - \tfrac{1}{36}(L^2 - 1)] \,.$$
$$\cdot \, \omega_d^4 + O(\omega_d^6)\}$$

Similarly we compute

(A8) $$h = V_m^H V_1 = \frac{1}{L} e^{j(L-1)\omega_d/2} \frac{\sin(L\omega_d/2)}{\sin(\omega_d/2)}$$

(A9) $$V_m^H V_2 = h^* \,.$$

Using (A4), (A5), (A7) and (A8) we obtain

(A10) $$|V_1^H S_1|^2 = |V_2^H S_1|^2 = \tfrac{1}{2}(1 + |d|) = 1 - \tfrac{1}{12}(L^2 - 1) \omega_d^2 + O(\omega_d^4)$$

(A11) $$|V_1^H S_2|^2 = |V_2^H S_2|^2 = \tfrac{1}{2}(1 - |d|) = \tfrac{1}{12}(L^2 - 1) \omega_d^2 + O(\omega_d^4)$$

(A12) $$|V_m^H S_1|^2 = \frac{2|h|^2}{1 + |d|} = 1 - \frac{1}{720}(L^2 - 1)(L^2 - 4) \omega_d^4 + O(\omega_d^6)$$

(A13) $$|V_m^H S_2|^2 = 0$$

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(A14) $$c_1 = \frac{PL(1 + |d|) + \sigma^2}{NL^2 P^2(1 + |d|)^2} \sigma^2 = \frac{1 + 2\xi L}{4NL^2\xi^2} + O(\omega_d^2)$$

(A15) $$c_2 = \frac{PL(1 - |d|) + \sigma^2}{NL^2 P^2(1 - |d|)^2} \sigma^2 \doteq 6 \frac{L(L^2 - 1)\,\xi\omega_d^2 + 6}{NL^2(L^2 - 1)^2\,\xi^2\omega_d^4}$$

(A16) $$s_{1,2} = \frac{1 \pm \dfrac{|d|}{d}}{\sqrt{(2(1 \pm |d|))}}$$

APPENDIX B

In this appendix, we evaluate the mean of the null-spectra and the resolution thresholds for two close spaced plane waves. We simply substitute the results of Appendix A into expressions (21) and (25). We obtain

(B1) $$\mathsf{E}[\hat{D}_{MUS}(\omega_{1,2})] = (L - 2) c_1 + c_2 \tfrac{1}{12}(L^2 - 1)\,\omega_d^2 = \frac{L - 2}{NL^2\xi^2}\left[L\xi + \frac{3}{(L^2 - 1)\,\omega_d^2} \right],$$

(B2) $$\mathsf{E}[\hat{D}_{MUS}(\omega_m)] = (1 - c)\,D_{MUS}(\omega_m) + (L - 2)\,c_1 =$$
$$= \frac{1}{720}\,(L^2 - 1)\,(L^2 - 4)\,\omega_d^4 + (L - 2)\,\frac{1 + 2\xi L}{4NL^2\xi^2}\,.$$

The resolution threshold condition

$$\mathsf{E}[\hat{D}_{MUS}(\omega_{1,2})] = \mathsf{E}[\hat{D}_{MUS}(\omega_m)]$$

has now the form of quadratic equation:

(B3) $$\frac{1}{720}\,(L^2 - 1)\,(L^2 - 4)\,\omega_d^4\xi^2 - \frac{L - 2}{2NL}\,\xi - \frac{3(L - 2)}{NL^2(L^2 - 1)\,\omega_d^2} = 0\,.$$

The result (28) is obtained as a solution of this equation.

The computation in the case of Minimum-Norm method requires more work than the previous one. At first we have to compute $D_{MN}(\omega_m)$ using (23), (A12), (A13) and (A16) with the Taylor expansions in the places of d and h. The result is

(B4) $$D_{MN}(\omega_m) = \frac{1}{144L}\,(L - 1)^2\,(L - 2)^2\,\omega_d^4 + O(\omega_d^6)$$

Computing the resolution threshold we evaluate the difference $\mathsf{E}[\hat{D}_{MN}(\omega_m)] - \mathsf{E}[\hat{D}_{MN}(\omega_{1,2})]$ rather than both of the values separately. In order to use the equation (25), at first we prepare the differences

(B5) $$|V_m^H S_1|^2 - |V_{1,2}^H S_1|^2 = \tfrac{1}{12}(L^2 - 1)\,\omega_d^2 + O(\omega_d^4)$$

(B6) $$|V_m^H S_2|^2 - |V_{1,2}^H S_2|^2 = -\tfrac{1}{12}(L^2 - 1)\,\omega_d^2 + O(\omega_d^4)$$

(B7) $$\mathrm{Re}\,(V_m^H S_1 S_2^H V_m s_1^* s_2) - \mathrm{Re}\,(V_{1,2}^H S_1 S_2^H V_{1,2} s_1^* s_2) = 0$$

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$$(B8) \quad \mathrm{Re}\left(s_1^* V_m^H S_1\right) - \mathrm{Re}\left(s_1^* V_{1,2}^H S_1\right) = \frac{1}{12\sqrt{L}}\,(L-1)(2L-1)\,\omega_d^2 + O(\omega_d^4)$$

$$(B9) \quad \mathrm{Re}\left(s_2^* V_m^H S_2\right) - \mathrm{Re}\left(s_2^* V_{1,2}^H S_2\right) = -\frac{1}{4\sqrt{L}}\,(L-1)^2\,\omega_d^2 + O(\omega_d^4)$$

The assumption $\omega_d \ll 1$ allows us to neglect b and c_1 against c_2. After some simplifications we obtain

$$(B10)$$
$$\mathrm{E}\big[\hat{D}_{MN}(\omega_m)\big] - \mathrm{E}\big[\hat{D}_{MN}(\omega_{1,2})\big] = D_{MN}(\omega_m) - \frac{1}{12L}\,(L-1)(L-2)(L-3)\,c_2\omega_d^2 = 0$$

Substituting for $D_{MN}(\omega_m)$ and c_2 we have a quadratic equation with the solution (29), similarly to the MUSIC method case.

ACKNOWLEDGEMENT

REFERENCES

[1] M. Kaveh and A. J. Barabell: The statistical performance of the MUSIC and the Minimum-Norm algorithms in resolving plane waves in noise. IEEE Trans. Acoust. Speech Signal Process. *ASSP-34* (1986), 2, 331—340.
[2] R. Schmidt: Multiple emitter location and signal parameter estimation. Proc. RADC Spectrum Estimation Workshop, Roma, 1979, 243—258.
[3] R. Kumaresan and D. W. Tufts: Estimating the angles of arrival of multiple plane waves. IEEE Trans. Aerospace, Electron. Systems *AES-19* (1983), 1, 123—133.
[4] G. Bienvenu and L. Kopp: Optimality of high resolution array processing using the eigen-system approach. IEEE Trans. Acoust. Speech Signal Process. *ASSP-31* (1983), 5, 1235 to 1248.
[5] M. Wax and T. Kailath: Detection of signals by information theoretic criteria. IEEE Trans. Acoust. Speech Signal Process. *ASSP-33* (1985), 2, 387—392.
[6] J. H. Wilkinson: The Algebraic Eigenvalue Problem. Claredon Press, Oxford 1965.
[7] P. Tichavský: Odhad směrů šíření rovinných vln (Estimating the Directions of Propagating Plane Waves). Diploma work FJFI ČVUT, Praha 1987.

Ing. Petr Tichavský, Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.