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Periodical Coefficient Linear Systems with Random Stationary Input

VLADIMÍR KRACÍK

This paper examines the problems of the transfer functions of periodic parameter systems in relation to the computation of the correlation function of the output process at the stationary input. These systems are described by systems of difference, differential and differential- difference equations.

1. SYSTEMS OF DIFFERENCE EQUATIONS IN DISCRETE CASE

Let the system of difference equations be given

$$\mathbf{y}_k = \mathbf{A}_k \mathbf{y}_{k-1} + \mathbf{B}_k \mathbf{x}_k \,, \tag{1}$$

where $\mathbf{A}_k$, $\mathbf{B}_k$, $\mathbf{y}_k$, $\mathbf{x}_k$ are matrices of the dimensions $m \times m$, $m \times l$, $m \times r$, $l \times r$, respectively.

Let $\mathbf{A}_{k+p} = \mathbf{A}_k$, $\mathbf{B}_{k+p} = \mathbf{B}_k$ be valid; let us write $\mathbf{A}_{np+j} = \mathbf{A}_j$, $\mathbf{B}_{np+j} = \mathbf{B}_j$ for $n = 0, 1, 2, \ldots$

Let us further define the matrices in the following way:

$$\mathbf{Y}_n = \begin{bmatrix} \mathbf{y}_{np} \\ \mathbf{y}_{np+1} \\ \vdots \\ \mathbf{y}_{np+p-1} \end{bmatrix}, \quad \mathbf{X}_n = \begin{bmatrix} \mathbf{x}_{(n-1)p+1} \\ \mathbf{x}_{(n-1)p+2} \\ \vdots \\ \mathbf{x}_{np} \\ \mathbf{x}_{np+1} \\ \vdots \\ \mathbf{x}_{np+p-1} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} [\mathbf{A}_0 \mathbf{A}_{p-1} \ldots \mathbf{A}_1], & 0, & & \ldots, 0 \\ 0, & [\mathbf{A}_1 \mathbf{A}_0 \mathbf{A}_{p-1} \ldots \mathbf{A}_2], & 0, & \ldots, 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0, & \ldots, & & 0, \; [\mathbf{A}_{p-1} \mathbf{A}_{p-2} \ldots \mathbf{A}_0] \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} [A_0A_{p-1} \dots A_2B_1], & [A_0A_{p-1} \dots A_3B_2], & \dots, & [A_0B_{p-1}], & B_0, & 0, & \dots, & 0 \\ 0, & [A_1A_0 \dots A_3B_2], & [A_1A_0 \dots A_4B_3], & \dots, & [A_1B_0], & B_1, & 0, \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & \dots, & 0, & [A_{p-1} \dots A_1B_0], & [A_{p-1} \dots A_2B_1], & \dots, & [A_{p-1}B_{p-2}], & B_{p-1} \end{bmatrix}.$$

If we express by a successive application of the recurrent relation (1) $\mathbf{y}_n$ depending on $\mathbf{y}_{n-p}$, then the given system with periodical coefficients may be transferred to the system with constant coefficients, if we write

$$\mathbf{y}_n = \mathbf{A}\mathbf{y}_{n-1} + \mathbf{B}\mathbf{x}_n \tag{2}$$

This system is stable if all eigenvalues of the matrix $\mathbf{A}$ i.e. all eigenvalues of the matrix $[A_{p-1}A_{p-2} \dots A_0]$ lie inside the unit circle. The transfer function of the system (2) using z-transformation is

$$\mathbf{Y}_{\mathbf{X}}(z^{-1}) = (\mathbf{E} - z^{-1}\mathbf{A})^{-1}\,\mathbf{B}\,;$$

let us write for convenience $z^{-1} = s$.

Let there further $\{\mathbf{x}_n\}$ be a stationary sequence ($\mathbf{x}_n$ is a matrix of the dimension $(2p - 1)\,l \times r$) with the correlation matrix $\mathbf{K}_{\mathbf{x}}(k) = \mathrm{M}[\mathbf{x}_n\mathbf{x}^{\mathrm{T}}_{n+k}]$ and the matrix of the spectral density $\mathbf{S}_{\mathbf{x}}(s) = \sum_{-\infty}^{\infty} \mathbf{K}_{\mathbf{x}}(k)\, s^k$, then

$$\mathbf{S}_{\mathbf{y}}(s) = \mathbf{Y}_{\mathbf{X}}\left(\frac{1}{s}\right) \mathbf{S}_{\mathbf{x}}(s)\, \mathbf{Y}^{\mathrm{T}}_{\mathbf{X}}(s) \tag{3}$$

(indeed, let us write $\mathbf{Y}_{\mathbf{X}}(s) = \sum_0^{\infty} \mathbf{W}_k s^k$; $\mathbf{W}_k$ is the weighting sequence of the system; $\mathbf{W}_k = \mathbf{Y}^{(k)}_{\mathbf{X}}(0)/k!$, then $\mathbf{y}_n = \sum_0^{\infty} \mathbf{W}_j \mathbf{x}_{n-j}$ and therefore

$$\mathbf{K}_{\mathbf{y}}(k) = \mathrm{M}\{\sum_i \mathbf{W}_i \mathbf{x}_{n-i} (\sum_j \mathbf{W}_j \mathbf{x}_{n+k-j})^{\mathrm{T}}\} = \sum_i \sum_j \mathbf{W}_i \mathbf{K}_x(k - j + i)\, \mathbf{W}^{\mathrm{T}}_j\,;$$

M is the mean value operator, T is the symbol of transposition; further $\mathbf{S}_{\mathbf{y}}(s) = \sum_{-\infty}^{\infty} \mathbf{K}_{\mathbf{y}}(k)\, s^k$ hence there follows (3)).

For the correlation function of the output matrix we obtain*

$$\mathbf{K}_{\mathbf{y}}(l) = \frac{1}{2\pi \mathrm{i}} \int_k \mathbf{Y}_{\mathbf{X}}\left(\frac{1}{s}\right) \mathbf{S}_{\mathbf{x}}(s)\, \mathbf{Y}^{\mathrm{T}}_{\mathbf{X}}(s)\, s^{-l} \frac{\mathrm{d}s}{s}\,,$$

where k is the unit circle.

* Some remarks concerning numerical computation of this integral see in Section 3.

If there is given a higher order difference equation or a system of such equations, then it is not necessary to do the transformation to the form (1) but the periodical transfer function may be calculated directly as is shown in the following example:

$$(4) \qquad y_n + a_{1n}y_{n-1} + a_{2n}y_{n-2} = b_{0n}x_n + b_{1n}x_{n-1} ;$$

let there be $a_{in} = a_{in+3}$, $b_{in} = b_{in+3}$ and let the equation (4) be stable in the sense of the stability of the system (2).

Let $\{w_{nk}\}_{k=0}^{\infty}$ be for every n the weighting sequence of the system (4) (i.e. w_{nk} is the response of the system at the step n to the unite impulse in the step $n - k$) so that generally $y_n = \sum_{k=0}^{\infty} w_{nk}x_{n-k}$. Evidently $w_{nk} = w_{n+3,k}$ holds. If the input has the form $x_n = z^n = s^{-n}$, the output is $y_n = \sum_{k=0}^{\infty} w_{nk}z^{n-k} = z^n \sum_{k=0}^{\infty} w_{nk}z^{-k} = s^{-n} \sum_{k=0}^{\infty} w_{nk}s^k = s^{-n} Y_X(n, s)$, where $Y_X(n, s) = \sum_{k=0}^{\infty} w_{nk}s^k$ is the periodical transfer function. If we substitute into (4) s^{-n} for the input and $Y_X(n, s)\, s^{-n}$ for the output we get

$$Y_X(n, s)\, s^{-n} + a_{1n}Y_X(n-1, s)\, s^{-(n-1)} + a_{2n}Y_X(n-2, s)\, s^{-(n-2)} = b_{0n}s^{-n} + b_{1n}s^{-(n-1)}$$

and therefore

$$(5) \qquad Y_X(n, s) + a_{1n}Y_X(n-1, s)\, s + a_{2n}Y_X(n-2, s)\, s^2 = b_{0n} + b_{1n}s .$$

By substituting into (5) $n = 0, 1, 2$ we get a system of linear algebraic equations for $Y_X(0, s)$, $Y_X(1, s)$, $Y_X(2, s)$.

If further $n_1 = 3k_1 + h$, $n_2 = 3k_2 + j$, it may be easily shown:

$$K_y(n_1, n_2) = \frac{1}{2\pi i} \int_k S_x(s)\, Y_X\left(h, \frac{1}{s}\right) Y_X(j, s)\, s^{n_2 - n_1} \frac{ds}{s} .$$

It is further possible to solve easily the inverse problem: to determine the corresponding difference equation to the given $Y_X(0, s)$, $Y_X(1, s)$. $Y_X(2, s)$. According to (5) the transfer function are rational functions with common denominator, hence $Y_X(j, s) = P_j/Q$; let us denote the highest degree of polynoms P_j, Q by m.

The difference equation with unknown coefficients is

$$(6) \qquad a_{0n}y_n + a_{1n}y_{n-1} + \ldots + a_{mn}y_{n-m} = b_{0n} + \ldots + b_{nn}x_{n-m} .$$

If we substitute in (6) s^{-n} for x_n, $Y_X(n, s)\, s^{-n}$ for y_n we get

$$(7) \qquad \begin{aligned} a_{00}P_0 + sa_{10}P_2 + \ldots + s^m a_{m0}P_{3-m} &= (b_{00} + sb_{10} + \ldots + s^m b_{m0})\, Q , \\ a_{01}P_1 + sa_{11}P_0 + \ldots + s^m a_{m1}P_{3-m+1} &= (b_{01} + sb_{11} + \ldots + s^m b_{m1})\, Q , \\ a_{02}P_2 + sa_{12}P_1 + \ldots + s^m a_{m2}P_{3-m+2} &= (b_{02} + sb_{12} + \ldots + s^m b_{m2})\, Q . \end{aligned}$$

Each of identities in "s" (7) represents a homogeneous linear system of equations for the corresponding coefficients a, b.

2. DIFFERENCE EQUATIONS SYSTEM IN CONTINUOUS CASE

Let us consider a difference equations system in the form

$$\mathbf{y}(t) = \sum_{j=1}^{m} \mathbf{C}_j(t)\, \mathbf{y}(t - T_j(t)) + \mathbf{B}(t)\, \mathbf{x}(t)\,, \tag{8}$$

where $\mathbf{C}_j(t)$, $T_j(t)$, $\mathbf{B}(t)$ are periodical with the period length T, $\mathbf{C}_j(t)$, $\mathbf{B}(t)$, $\mathbf{y}(t)$, $x(t)$ being matrices of the dimensions $m \times m$, $m \times l$, $m \times r$, $l \times r$ respectively. The difference equation for the transfer function $\mathbf{Y}_x(\mathrm{i}\omega, \gamma)$ (the response to the input $\mathbf{x}(t) = \mathrm{e}^{\mathrm{i}\omega(nT+\gamma)}\mathbf{E}$ is $\mathbf{Y}_x(\mathrm{i}\omega, \gamma)\, \mathrm{e}^{\mathrm{i}\omega(nT+\gamma)}$; the dimensions of the matrices $\mathbf{E}$ or $\mathbf{Y}_x(\mathrm{i}\omega, \gamma)$ are $l \times l$ or $m \times l$ respectively) is obtained by substituting the input and output into (8):

$$\mathbf{Y}_{\mathbf{x}}(\mathrm{i}\omega, \gamma)\, \mathrm{e}^{\mathrm{i}\omega(nT+\gamma)} = \sum_{j=1}^{m} \mathbf{C}_j(\gamma)\, \mathbf{Y}_{\mathbf{x}}(\mathrm{i}\omega, \gamma - T_j(\gamma)\, \mathrm{e}^{\mathrm{i}\omega(nT+\gamma)-T_j(\gamma)} + \mathbf{B}(\gamma)\, \mathrm{e}^{\mathrm{i}\omega(nT+\gamma)}$$

and finally

$$\mathbf{Y}_{\mathbf{x}}(\mathrm{i}\omega, \gamma) = \sum_{j=1}^{m} \mathbf{C}_j(\gamma)\, \mathbf{Y}_{\mathbf{x}}(\mathrm{i}\omega, \gamma - T_j(\gamma))\, \mathrm{e}^{-\mathrm{i}\omega(T_j(\gamma))} + \mathbf{B}(\gamma)\,. \tag{9}$$

In some simpler case the equation (9) for the transfer function may be solved explicitely.

Example. There is given the equation

$$y(nT + \gamma) = k\gamma y(nT - 0) + x(nT + \gamma)\,, \quad 0 \leqq \gamma < T\,.$$

According to (8), (9)

$$Y_X(\mathrm{i}\omega, \gamma) = k\gamma Y_X(\mathrm{i}\omega, T - 0)\, \mathrm{e}^{-\mathrm{i}\omega\gamma} + 1$$

hence

$$Y_X(\mathrm{i}\omega, T - 0) = kTY_X(\mathrm{i}\omega, T - 0)\, \mathrm{e}^{-\mathrm{i}\omega T} + 1\,,$$

$$Y_x(\mathrm{i}\omega, T - 0) = \frac{1}{1 - kTs}\,, \quad \mathrm{e}^{-\mathrm{i}\omega T} = s\,,$$

and finally

$$Y_X(\mathrm{i}\omega, \gamma) = \frac{k\mathrm{e}^{-\mathrm{i}\omega T}}{1 - kTs} + 1 \tag{11}$$

(the equation (10) is stable for $|kT| < 1$).

From (11) we see:

$$(12)\qquad y(nT + \gamma) = x(nT + \gamma) + k\gamma \sum_{j=0}^{\infty} (kT)^j \, x(nT - jT) .$$

((12) may be also attained directly from the recurrent relation (10)). If e.g.

$$K_x(\tau) = \mathrm{e}^{-a|\tau|} ,$$

$$S_{T,x}(s) = \sum_{-\infty}^{\infty} \mathrm{e}^{-a|nT|} s^n = 1 + \frac{\mathrm{e}^{-aT} s}{1 - \mathrm{e}^{-aT} s} + \frac{\mathrm{e}^{-aT}}{s - \mathrm{e}^{-aT}} ,$$

the mean square output value in the phase γ is

$$\mathrm{M}[y_\gamma^2] = 1 + k^2\gamma^2 \int_k \frac{1}{1 - kTs} \frac{s}{s - kT} S_{T,x}(s) \frac{\mathrm{d}s}{s} + 2k\gamma \mathrm{e}^{-a\gamma} \frac{1}{1 - \mathrm{e}^{-aT} kT} .$$

3. DIFFERENTIAL EQUATIONS SYSTEM

Let there be given a differential equations system

$$(13)\qquad \frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = \mathbf{A}(t)\, \mathbf{y}(t) + \mathbf{B}(t)\, \mathbf{x}(t) ,$$

where $\mathbf{A}(t)$, $\mathbf{B}(t)$ are periodical with the period length T, the dimensions of $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{y}(t)$, $\mathbf{x}(t)$ being $m \times m$, $m \times l$, $m \times r$, $l \times r$ respectively; as regards the correlation function of the process $\mathbf{x}(t)$ without loss of generality let us suppose

$$\mathbf{K}_x(\tau) = \mathrm{M}[\mathbf{x}(t + \tau)\, \mathbf{x}^*(t)] = \delta(\tau)\, \mathbf{E} .$$

Here the symbol * designates the operations of transposition and complex conjugation. Let further $\mathbf{W}(t, \tau)$ be the weighting function (transition matrix) of the system

$$(14)\qquad \frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = \mathbf{A}(t)\, \mathbf{y}(t) + \mathbf{x}(t)$$

(the dimension of $\mathbf{x}(t)$ is $m \times r$).

Then for the output from the system (13) the following relation holds

$$\mathrm{M}[\mathbf{y}(t_1)\, \mathbf{y}^*(t_2)] = \mathrm{M} \int_0^\infty \mathbf{W}(t_1, \tau)\, \mathbf{B}(t_1 - \tau)\, \mathbf{x}(t_1 - \tau)\, \mathrm{d}\tau \int_0^\infty \mathbf{x}^*(t_2 - u) \,.$$

$$. \, \mathbf{B}^*(t_2 - u)\, \mathbf{W}^*(t_2, u)\, \mathrm{d}u =$$

$$= \int_0^\infty \mathbf{W}(t_1, \tau + t_1 - t_2)\, \mathbf{B}(t_2 - \tau)\, \mathbf{B}^*(t_2 - \tau)\, \mathbf{W}^*(t_2, \tau)\, \mathrm{d}\tau ,$$

 specially

$$\mathrm{M}[\boldsymbol{y}(t)\,\boldsymbol{y}^*(t)] = \int_0^\infty \boldsymbol{W}(t,\tau)\,\boldsymbol{B}(t-\tau)\,\boldsymbol{B}^*(t-\tau)\,\boldsymbol{W}^*(t,\tau)\,\mathrm{d}\tau\,.$$

If we denote $\boldsymbol{W}(t_1,\tau)\,\boldsymbol{B}(t_1-\tau) = \mathbf{W}(t_1,\tau)$, is

$$\mathrm{M}[\mathbf{y}(t_1)\,\mathbf{y}^*(t_2)] = \int_0^\infty \mathbf{W}(t_1,\tau+t_1-t_2)\,\mathbf{W}^*(t_2,\tau)\,\mathrm{d}\tau\,, \tag{15}$$

$$\mathrm{M}[\boldsymbol{y}(t)\,\boldsymbol{y}^*(t)] = \int_0^\infty \mathbf{W}(t,\tau)\,\mathbf{W}^*(t,\tau)\,\mathrm{d}\tau\,. \tag{16}$$

As it is known, $\boldsymbol{W}(t,\tau)$ satisfies the equation (the dimension of $\boldsymbol{E}$ being $m \times m$)

$$\frac{\partial}{\partial t}\boldsymbol{W}(t,\tau) = \boldsymbol{A}(t)\,\boldsymbol{W}(t,\tau);\ \boldsymbol{W}(t-\tau,\tau) = \boldsymbol{E}$$

and the adjoined equation

$$\frac{\partial}{\partial\tau}\boldsymbol{W}^*(t,\tau) = \boldsymbol{A}^*(t-\tau)\,\boldsymbol{W}^*(t,\tau)\,,\quad \boldsymbol{W}^*(t,0) = \boldsymbol{E}\,. \tag{17}$$

From the periodicity of $\boldsymbol{A}(t)$ there follows also the periodicity of $\boldsymbol{W}(t,\tau)$ with respect to t. Then $\boldsymbol{W}(t,\tau) = \boldsymbol{W}(\gamma,\tau)$ where $0 \leqq \gamma < T$, $t = nT + \gamma$. Further for $\tau_1 < \tau_2$ with respect to what has been said above the following relations are valid

$$\boldsymbol{W}^*(t,\tau_2) = \boldsymbol{W}^*(t-\tau_1,\tau_2-\tau_1)\,\boldsymbol{W}^*(t,\tau_1)$$

and specially

$$\boldsymbol{W}^*(t,\tau+T) = \boldsymbol{W}^*(t-T,\tau)\,\boldsymbol{W}^*(t,T) = \boldsymbol{W}^*(\gamma,\tau)\,\boldsymbol{W}^*(\gamma,T)$$

and hence by the induction

$$\boldsymbol{W}^*(t,\tau+nT) = \boldsymbol{W}^*(\gamma,\tau)\,[\boldsymbol{W}^*(\gamma,T)]^n\,.$$

Hence also

$$\mathbf{W}^*(t,\tau+nT) = \mathbf{W}^*(\gamma,\tau)\,[\boldsymbol{W}^*(\gamma,T)]^n\,.$$

If further $t_1 - t_2 = mT + \alpha$, it is possible to write

$$\mathbf{W}^*(\gamma_1,\tau+t_1-t_2) = \mathbf{W}^*(\gamma_1,\tau+\alpha)\,[\boldsymbol{W}^*(\gamma_1,T)]^m\,.$$

Thus from (15) we get the output correlation function in the following form

$$\mathrm{M}[\boldsymbol{y}(t_1)\,\boldsymbol{y}^*(t_2)] = \tag{18}$$
$$= \sum_{n=0}^{\infty}[\boldsymbol{W}(\gamma_1,T)]^n\,[\boldsymbol{W}(\gamma_1,T)]^m\left[\int_0^T \mathbf{W}(\gamma_1,\tau+\alpha)\,\mathbf{W}^*(\gamma_2,\tau)\,\mathrm{d}\tau\right][\boldsymbol{W}^*(\gamma_2,T)]^n$$

and specially the mean square output value in phase γ is

(18a) $$\mathrm{M}[\mathbf{y}(\gamma)\,\mathbf{y}^*(\gamma)] = \sum_{n=0}^{\infty} [\mathbf{W}(\gamma, T)]^n \left[\int_0^T \mathbf{W}(\gamma, \tau)\,\mathbf{W}^*(\gamma, \tau)\,\mathrm{d}\tau\right] [\mathbf{W}^*(\gamma, T)]^n .$$

The expressions (18), (18a) may be also transcribed in the form

(19) $$\mathrm{M}[\mathbf{y}(t_1), \mathbf{y}^*(t_2)] =$$
$$= \frac{1}{2\pi \mathrm{i}} \int_k [\mathbf{E} - z\mathbf{W}(\gamma_1, T)]^{-1} [\mathbf{W}(\gamma_1, T)]^m \left[\int_0^T \mathbf{W}(\gamma_1, \tau + \alpha)\,\mathbf{W}^*(\gamma_2, \tau)\,\mathrm{d}\tau\right] \times$$
$$\times \left[\mathbf{E} - \frac{1}{z}\,\mathbf{W}^*(\gamma_2, T)\right]^{-1} \frac{\mathrm{d}z}{z},$$

(19a) $$\mathrm{M}[\mathbf{y}(\gamma)\,\mathbf{y}^*(\gamma)] =$$
$$= \frac{1}{2\pi \mathrm{i}} \int_k [\mathbf{E} - z\mathbf{W}(\gamma, T)]^{-1} \left[\int_0^T \mathbf{W}(\gamma, \tau)\,\mathbf{W}^*(\gamma, \tau)\,\mathrm{d}\tau\right] \left[\mathbf{E} - \frac{1}{z}\,\mathbf{W}^*(\gamma, T)\right]^{-1} \frac{\mathrm{d}z}{z}$$

where z is a complex variable.

Some remarks concerning numerical computation. We can use the relations

(20) $$\mathbf{W}^*(\gamma, \tau) = \mathbf{W}^*(0, T + \tau - \gamma)\,\mathbf{W}^{*-1}(0, T - \gamma) \quad \text{for} \quad 0 \leqq \tau \leqq \gamma$$

(21) $$\mathbf{W}^*(\gamma, \tau) = \mathbf{W}^*(0, \tau - \gamma)\,\mathbf{W}^*(0, T)\,\mathbf{W}^{*-1}(0, T - \gamma) \quad \text{for} \quad \gamma < \tau \leqq T$$

so that it is not necessary to compute $\mathbf{W}^*(\gamma, \tau)$, $0 \leqq \tau < T$ for each γ separately. Then e.g. the inner integral of (18a) or (19a) becomes

$$\int_0^T \mathbf{W}(\gamma, \tau)\,\mathbf{W}^*(\gamma, \tau)\,\mathrm{d}\tau = \mathbf{W}^{-1}(0; T - \gamma)\left[\mathbf{W}(0, T)\left(\int_0^{T-\gamma} \mathbf{W}(0, u)\,\mathbf{W}^*(0, u)\,\mathrm{d}u\right).\right.$$
$$\left. .\,\mathbf{W}^*(0, T) + \int_{T-\gamma}^T \mathbf{W}(0, u)\,\mathbf{W}^*(0, u)\,\mathrm{d}u\right] \mathbf{W}^{-1*}(0, T - \gamma).$$

Similarly it is possible to express the inner integral of (18) or (19) in a more complicated manner.

If the spectra of the matrix $\mathbf{W}(0, T)$ are known ($\mathbf{W}(\gamma, T)$ being mutually similar for different γ-see (21)), the external integral of (19), (19a) can be computed by means or residues. Generally it is convenient to use (18), (18a).

If we denote

$$\mathbf{R} = [\mathbf{W}(\gamma_1, T)]^m \int_0^T \mathbf{W}(\gamma, \tau + \alpha)\,\mathbf{W}^*(\gamma_2, \tau)\,\mathrm{d}\tau ,$$

it is easily seen that for the sequence $\mathbf{Q}_n$ of partial sums of (18) the following relation is valid:

$$\mathbf{Q}_{n+1} = \mathbf{R} + \mathbf{W}(\gamma_1, T)\,\mathbf{Q}_n\mathbf{W}^*(\gamma_2, T)\,, \quad \mathbf{Q}_0 = 0\,.$$

Further we suppose that for some norm $\|.\|$

$$\|\mathbf{W}(\gamma, T)\| \leqq \alpha < 1\,, \quad 0 \leqq \gamma < T$$

is valid. Then $\lim_{n\to\infty} \mathbf{Q}_n$ exists and

$$\lim_{n\to\infty} \mathbf{Q}_n = \mathbf{Q} = \mathrm{M}(\mathbf{y}(t_1)\,\mathbf{y}^*(t_2))$$

and

$$\|\mathbf{Q}_n - \mathbf{Q}\| \leqq \|\mathbf{Q}_1 - \mathbf{Q}_0\| \frac{\alpha^{2n}}{1-\alpha^2} = \|\mathbf{R}\| \frac{\alpha^{2n}}{1-\alpha^2}\,.$$

Further it follows that $\mathbf{Q}$ satisfies the following matrix equation

$$\mathbf{Q} = \mathbf{R} + \mathbf{W}(\gamma_1, T)\,\mathbf{Q}\mathbf{W}^*(\gamma_2, T)\,. \tag{22}$$

In case α is near to 1, it is convenient to solve directly the equation (22).

If we know the periodical transfer function (p is a complex variable):

$$\mathbf{Y}_{\mathbf{X}}(\gamma, p) = \int_0^\infty \mathbf{W}(\gamma, \tau)\,\mathrm{e}^{-p\tau}\,\mathrm{d}\tau = (\mathbf{E} - \mathrm{e}^{-pT}\,\mathbf{W}(\gamma, T))^{-1} \int_0^T \mathbf{W}(\gamma, \tau)\,\mathrm{e}^{-p\tau}\,\mathrm{d}\tau \tag{23}$$

we may express analogously to the discrete case

$$\mathrm{M}[\mathbf{y}(t_1)\,\mathbf{y}^*(t_2)] = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Im}} \mathbf{Y}_{\mathbf{X}}(\gamma_1, p)\,\mathbf{S}_x(p)\,\mathbf{Y}^*_{\mathbf{X}}(\gamma_2, p)\,\mathrm{e}^{p(t_1 - t_2)}\,\mathrm{d}p$$

and specially

$$\mathrm{M}[\mathbf{y}(\gamma)\,\mathbf{y}^*(\gamma)] = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Im}} \mathbf{Y}_{\mathbf{X}}(\gamma, p)\,\mathbf{S}_x(p)\,\mathbf{Y}^*_{\mathbf{X}}(\gamma, p)\,\mathrm{d}p\,.$$

Here the path of integration is the imaginary axis, $\mathbf{S}_x(i\omega) = \mathbf{S}_x(p)$ is the matrix of spectral densities of the process $\mathbf{x}(t)$, i.e.

$$\mathbf{K}_x(\tau) = \mathrm{M}[\mathbf{x}(t+\tau)\,\mathbf{x}^*(t)]\,, \quad \mathbf{S}_x(\mathrm{i}\omega) = \int_{-\infty}^{\infty} \mathbf{K}_x(\tau)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\tau\,.$$

The periodical transfer function may not be generally obtained in an explicit form. Generally the problem of finding the periodical transfer function leads to a difference infinite order equation with non-constant coefficients [1], [2].

Here we give a simple method for approximate determination of the periodical transfer function for the event of $\mathbf{A}(t)$ having a finite number of "small" harmonics.

Let

$$\mathscr{A}_c = \left\{\mathbf{A}(p) : \operatorname{Re} p_i \leqq c < 0;\ p_i \text{ are singularities of } \mathbf{A}(p); \right.$$
$$\left. \left\| \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Im}} \mathbf{A}(p)\, \mathbf{A}^*(p)\, \mathrm{d}p \right\| < \infty \right\}.$$

Let us introduce the norm $\|\mathbf{A}(p)\|_p = \sqrt{(\lambda_M)}$ where λ_M is the maximum eigenvalue of the matrix $1/2\pi\mathrm{i} \int_{\mathrm{Im}} \mathbf{A}(p)\, \mathbf{A}^*(p)\, \mathrm{d}p$. (The given norm is that of the operator $\mathbf{A}(p)$ induced by the norm in the n-dimensional Euclidean space.) $\|\mathbf{C}\|$ is the radical from the maximum eigenvalue of the matrix $\mathbf{CC}^*$. It can be shown that for the "scalar product"

$$(\mathbf{A}(p), \mathbf{B}(p)) = \frac{1}{2\pi\mathrm{i}} \int_{\mathrm{Im}} \mathbf{A}(p)\, \mathbf{B}^*(p)\, \mathrm{d}p$$

an analogue of Schwarz's inequality holds:

$$\|(\mathbf{A}(p), \mathbf{B}(p))\| \leqq \|\mathbf{A}(p)\|_p\, \|\mathbf{B}(p)\|_p .$$

Let us now consider the equation (the period length is 2π)

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = (\mathbf{A}_0 + \sum_{k=-n}^{n}{}^{0} \mathbf{A}_k \mathrm{e}^{\mathrm{i}kt})\, \mathbf{y}(t) + (\mathbf{B}_0 + \sum_{k=-n}^{n}{}^{0} \mathbf{B}_k \mathrm{e}^{\mathrm{i}kt})\, \mathbf{x}(t)\,; \quad \mathbf{S}_x(p) = \mathbf{E}\,; \tag{24}$$

$\sum^0$ denotes the sum with omitted index 0.

According to (23)

$$\mathbf{Y}_{\mathbf{X}}(\gamma, p) = \mathscr{L}[\mathbf{W}(\gamma, \tau)] = \mathscr{L}[\mathbf{W}(\gamma, \tau)\, \mathbf{B}(\gamma - \tau)]\,;$$

$\mathbf{W}^*(\gamma, \tau)$ is the solution of the equation (17) and hence for

$$\mathbf{Y}^*_{\mathbf{X}}(\gamma, p) = (\mathscr{L}[\mathbf{W}(\gamma, \tau)])^* = \int_0^\infty \mathbf{W}^*(\gamma, \tau)\, \mathrm{e}^{-p^*\tau}\, \mathrm{d}\tau$$

we get

$$p^* \mathbf{Y}^*_{\mathbf{X}}(\gamma, p) = (\mathbf{A}^*_0 + \sum_{k=-n}^{n}{}^{0} \mathbf{A}^*_k \mathrm{e}^{-\mathrm{i}k\gamma} \mathrm{D}^k)\, \mathbf{Y}^*_{\mathbf{X}}(\gamma, p) + \mathbf{E}\,,$$

where $\mathrm{D}\mathbf{F}(p) = \mathbf{F}(p + i)$; hence it follows

$$\mathbf{Y}^*_{\mathbf{X}}(\gamma, p) = (p^*\mathbf{E} - \mathbf{A}^*_0)^{-1}\, [(\sum_{k=-n}^{n}{}^{0} \mathbf{A}^*_k \mathrm{e}^{-\mathrm{i}k\gamma} \mathrm{D}^k)\, \mathbf{Y}^*_{\mathbf{X}}(\gamma, p) + \mathbf{E}]$$

or

$$\mathbf{Y}_{\mathbf{X}}(\gamma, p) = [\sum_{k=-n}^{n}{}^{0} \mathrm{D}^k(\mathbf{Y}_{\mathbf{X}}(\gamma, p))\, \mathbf{A}_k \mathrm{e}^{\mathrm{i}k\gamma} + \mathbf{E}]\, (p\mathbf{E} - \mathbf{A}_0)^{-1}\,; \tag{25}$$

For the operator

$$\mathbf{H} = \left[\sum_{k=-n}^{n}{}^{0} \mathrm{D}^k(\cdot)\, \boldsymbol{A}_k \mathrm{e}^{ik\gamma}\right] (p\boldsymbol{E} - \boldsymbol{A}_0)^{-1}$$

the following relation holds

$$\|\mathbf{H}\|_p \leqq \|(p\boldsymbol{E} - \boldsymbol{A}_0)^{-1}\|_p \sum_{k=-n}^{n}{}^{0} \|\boldsymbol{A}_k\| = \alpha$$

(obviously $\|\mathrm{D}\|_p = 1$).

As D represents a shift in the direction of the imaginary axis, the operator $\mathbf{H}$ maps the set $\mathscr{A}_C$ into. If then $\alpha < 1$ the sequence $\{\boldsymbol{Y}_{\boldsymbol{X}}^{(m)}(\gamma, p)\}_{m=0}^{\infty}$:

$$\boldsymbol{Y}_{\boldsymbol{X}}^{(m+1)}(\gamma, p) = \mathbf{H}\boldsymbol{Y}_{\boldsymbol{X}}^{(m)}(\gamma, p) + (p\boldsymbol{E} - \boldsymbol{A}_0)^{-1}\,; \quad \boldsymbol{Y}_{\boldsymbol{X}}^{(0)}(\gamma, p) = 0 \tag{26}$$

is fundamental. Let further $\boldsymbol{Y}_{\boldsymbol{X}}(\gamma, p)$ exists and $\boldsymbol{Y}_{\boldsymbol{X}}(\gamma, p) \in \mathscr{A}_c$ for every γ. Then obviously

$$\lim_{m\to\infty} \boldsymbol{Y}_{\boldsymbol{X}}^{(m)}(\gamma, p) = \boldsymbol{Y}_{\boldsymbol{X}}(\gamma, p)\,,$$

$\boldsymbol{Y}_{\boldsymbol{X}}(\gamma, p)$ is the solution of the equation (25) and

$$\varepsilon_m = \|\boldsymbol{N}_m(\gamma, p)\|_p = \|\boldsymbol{Y}_{\boldsymbol{X}}^{(m)}(\gamma, p) - \boldsymbol{Y}_{\boldsymbol{X}}(\gamma, p)\|_p \leqq$$

$$\leqq \frac{\alpha^m}{1-\alpha} \|\boldsymbol{Y}_{\boldsymbol{X}}^{(1)} - \boldsymbol{Y}_{\boldsymbol{X}}^{(0)}\|_p = \frac{\alpha^m}{1-\alpha} \|\boldsymbol{Y}_{\boldsymbol{X}}^{(1)}\|_p\,.$$

From (26) it follows that

$$\boldsymbol{Y}_{\boldsymbol{X}}^{(m)} = \boldsymbol{Y}_{\boldsymbol{X}}^{(1)} + \mathbf{H}\boldsymbol{Y}_{\boldsymbol{X}}^{(1)} + \ldots + \mathbf{H}^{m-1}\boldsymbol{Y}_{\boldsymbol{X}}^{(1)}\,.$$

Let us remark that $\boldsymbol{Y}_{\boldsymbol{X}}^{(1)} = (p\boldsymbol{E} - \boldsymbol{A}_0)^{-1}$ is the "basic" transfer function for the system with constant matrix $\boldsymbol{A}_0$, the further members being successive corrections depending, however, on γ.

For $\mathbf{Y}_{\boldsymbol{X}}(\gamma, p)$ we get further

$$\mathbf{Y}_{\boldsymbol{X}}(\gamma, p) = \mathbf{C}\boldsymbol{Y}_{\boldsymbol{X}}(\gamma, p)\,,$$

where

$$\boldsymbol{N} = \sum_{k=-n}^{n} \mathrm{e}^{ik\gamma} \mathrm{D}^n(\cdot)\, \boldsymbol{B}_k\,;$$

$$\mathbf{N}_m(\gamma, p) = \mathbf{Y}_{\boldsymbol{X}}^{(m)}(\gamma, p) - \mathbf{Y}_{\boldsymbol{X}}(\gamma, p) = \mathbf{C}\boldsymbol{N}_m(\gamma, p)\,,$$

$$\tilde{\varepsilon}_m = \|\mathbf{N}_m(\gamma, p)\|_p \leqq \varepsilon_m \|\mathbf{C}\|_p \leqq \varepsilon_m \sum_{k=-n}^{n} \|\boldsymbol{B}_k\|\,.$$

Let us remark that $\|\mathbf{Y}_{\boldsymbol{X}}(\gamma, p)\|_p$ in the unidimensional case is rms $(y(\gamma))$ where $y(t)$ is the output of the system (24), similarly $\tilde{\varepsilon}_m(\gamma) = \text{rms}\,(y_m(\gamma) - y(\gamma))$ where $y_m(\gamma)$ is the output from the system with the estimated transfer function $Y_{\boldsymbol{X}}^{(m)}$. Specially

$\|Y_X^{(1)}\|_p$ is rms $(y(t))$, where $y(t)$ is the output at the basic transfer function $(A(t) = A_0,$
$B(t) = 1)$.

For the error $\tilde{\varepsilon}_m(t_1, t_2)$ in norm of the correlation function $\mathrm{M}[\mathbf{y}(t_1), \mathbf{y}^*(t_2)]$ when $\mathbf{Y}_{\mathbf{X}}^{(m)}$ is used the relation

$$\tilde{\varepsilon}_m(t_1, t_2) \leqq \tilde{\varepsilon}_m(\gamma_1) \|\mathbf{Y}_{\mathbf{X}}^{(m)}(\gamma_2, p)\|_p + \tilde{\varepsilon}_m(\gamma_2) \|\mathbf{Y}_{\mathbf{X}}^{(m)}(\gamma_1, p)\|_p + \tilde{\varepsilon}_m(\gamma_1)\, \tilde{\varepsilon}_m(\gamma_2)$$

holds.

Example.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2, & \lambda \cos t \\ \lambda \sin t, & -3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} x_1 \\ 0 \end{bmatrix}; \tag{27}$$

$x_1(t)$ is a white noise. The equation (27) is transcribed

$$\frac{\mathrm{d}\mathbf{y}(t)}{\mathrm{d}t} = (\mathbf{A}_0 + \mathbf{A}_1 \mathrm{e}^{\mathrm{i}t} + \mathbf{A}_{-1} \mathrm{e}^{-\mathrm{i}t})\, \mathbf{y}(t) + \mathbf{B}_0\, \mathbf{x}(t)$$

where $\mathbf{x}(t)$ is a two-dimensional white noise,

$$\mathbf{A}_0 = \begin{bmatrix} -2, & 0 \\ 0, & -3 \end{bmatrix}, \quad \mathbf{B}_0 = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix}, \quad \mathbf{A}_1 = \frac{\lambda}{2} \begin{bmatrix} 0, & 1 \\ -\mathrm{i}, & 0 \end{bmatrix}, \quad \mathbf{A}_{-1} = \frac{\lambda}{2} \begin{bmatrix} 0, & 1 \\ \mathrm{i}, & 0 \end{bmatrix},$$

$$\mathbf{Y}_{\mathbf{X}}^{(1)}(p) = \begin{bmatrix} \dfrac{1}{p+2}, & 0 \\ 0\ , & \dfrac{1}{p+3} \end{bmatrix}; \quad \|\mathbf{Y}_{\mathbf{X}}^{(1)}\|_p = \max\left(\sqrt{\tfrac{1}{4}}, \sqrt{\tfrac{1}{6}}\right) = \tfrac{1}{2};$$

$$\mathbf{H}\mathbf{Y}_{\mathbf{X}}^{(1)} = (\mathrm{D}^1(\mathbf{Y}_{\mathbf{X}}^{(1)})\, \mathbf{A}_1 \mathrm{e}^{\mathrm{i}\gamma} + \mathrm{D}^{-1}(\mathbf{Y}_{\mathbf{X}}^{(1)})\, \mathbf{A}_{-1} \mathrm{e}^{-\mathrm{i}\gamma})\, \mathbf{Y}_{\mathbf{X}}^{(1)} =$$

$$= \frac{\lambda}{2} \left(\begin{bmatrix} \dfrac{1}{p+\mathrm{i}+2}, & 0 \\ 0\ , & \dfrac{1}{p+\mathrm{i}+3} \end{bmatrix} \begin{bmatrix} 0, & 1 \\ -\mathrm{i}, & 0 \end{bmatrix} \mathrm{e}^{\mathrm{i}\gamma} + \right.$$

$$\left. + \begin{bmatrix} \dfrac{1}{p-\mathrm{i}+2}, & 0 \\ 0\ , & \dfrac{1}{p-\mathrm{i}+3} \end{bmatrix} \begin{bmatrix} 0\ , & 1 \\ \mathrm{i}\ , & 0 \end{bmatrix} \mathrm{e}^{-\mathrm{i}\gamma} \right) \mathbf{Y}_{\mathbf{X}}^{(1)} =$$

$$= \lambda \begin{bmatrix} 0 & , & \dfrac{(p+2)\cos\gamma + \sin\gamma}{(p+3)\,[(p+2)^2+1]} \\ \dfrac{(p+3)\sin\gamma - \cos\gamma}{(p+2)\,[(p+3)^2+1]}, & & 0 \end{bmatrix},$$

$$\mathbf{Y}_X^{(2)} = \mathbf{H}\mathbf{Y}_X^{(1)} + \mathbf{Y}_X^{(1)},$$

$$\mathbf{Y}_X^{(2)} = (\mathbf{H}\mathbf{Y}_X^{(1)} + \mathbf{Y}_X^{(1)})\,\mathbf{B}_0 = \begin{bmatrix} \dfrac{1}{p+2}, & 0 \\ \lambda \dfrac{(p+3)\sin\gamma - \cos\gamma}{(p+2)\left[(p+3)^2+1\right]}, & 0 \end{bmatrix},$$

$$\alpha = \|\mathbf{Y}_X^{(1)}\|_p \left(\sum_{k=-n}^{n}{}^0 \|\mathbf{A}_k\|\right) = \frac{\lambda}{2},$$

$$\varepsilon_2 = \|\mathbf{Y}_X^{(2)} - \mathbf{Y}_X\|_p \leqq \frac{\left(\frac{\lambda}{2}\right)^2}{1 - \frac{\lambda}{2}} \frac{1}{2} = \left(\frac{\lambda}{2}\right)^2 \frac{1}{2-\lambda};$$

as $\|\mathbf{B}_0\| = 1$, then also

$$\tilde{\varepsilon}_2 \leqq \left(\frac{\lambda}{2}\right)^2 \frac{1}{2-\lambda},$$

so that e. g.

$$\|\mathbf{Y}_X^{(2)}\|_p - \varepsilon_2 \leqq \|\mathbf{Y}_X\|_p \leqq \|\mathbf{Y}_X^{(2)}\|_p + \tilde{\varepsilon}_2 .$$

In a unidimensional case when there is given an equation of a general order it is not necessary to do the conversion to the system of first order equations. For the sake of simplicity we shall illustrate this fact on an second order equation.

Let there be given the equation ($\sum_k^0$ denotes a finite sum with omitted index 0):

$$\ddot{y} + \left(a_0 + \sum_k{}^0 a_k e^{ikt}\right)\dot{y} + \left(b_0 + \sum_k{}^0 b_k e^{ikt}\right) y = \left(c_0 + \sum_k{}^0 c_k e^{ikt}\right) x(t),$$

where $x(t)$ is white noise. The weighting function $w(\gamma, \tau)$ is the solution of the adjoined equation:

$$w_{\tau\tau} + \left[\left(a_0 + \sum_k{}^0 a_k e^{ik(\gamma-\tau)}\right) w\right]_\tau + \left(b_0 + \sum_k{}^0 b_k e^{ik(\gamma-\tau)}\right) w = \delta(\tau)$$

and for $Y_X(\gamma, p) = \mathscr{L}(w(\gamma, \tau))$ we get

$$p^2 Y_X + a_0 p Y_X + b_0 Y_X + p \sum_k{}^0 a_k e^{ik\gamma} D^k Y_X + \sum_k{}^0 b_k e^{ik\gamma} D^k Y_X = 1 . \tag{28}$$

We transcribe the relation (28) into the form

$$Y_X = -\frac{1}{p^2 + a_0 p + b_0}\left[p \sum_k{}^0 a_k e^{ik\gamma} D^k Y_X + \sum_k{}^0 b_k e^{ik\gamma} D^k Y_X - 1\right].$$

For the norm of the operator

$$H = -\frac{1}{p^2 + a_0 p + b_0}\left[p{\sum_k}^0 a_k e^{ik\gamma} D^k + {\sum_k}^0 b_k e^{ik\gamma} D^k\right]$$

the following relation holds

$$\|H\|_p \leqq \left\|\frac{p}{p^2 + a_0 p + b_0}\right\|_p {\sum_k}^0 |a_k| + \left\|\frac{1}{p^2 + a_0 p + b_0}\right\|_p {\sum_k}^0 |b_k| = \alpha \,.$$

The iteration may be done in an analogous way (if $\alpha < 1$); finally

$$Y_X(\gamma, p) = \sum_k c_k D^k Y_X(\gamma, p)\,.$$

Example.

(29) $$\ddot{y} + 7\dot{y} + (12 - \cos t)\, y = x(t)\,,$$

$x(t)$ is white noise.

Let us transcribe the equation (29):

$$\ddot{y} + 7\dot{y} + (12 - \tfrac{1}{2}e^{it} - \tfrac{1}{2}e^{-it})\, y = x(t)\,,$$

$$Y_X^{(1)} = \frac{1}{p^2 + 7p + 12}\,, \quad \|Y_X^{(1)}\|_p = \sqrt{\left(\frac{1}{2\pi i}\int_{\mathrm{Im}} Y_X^{(1)}(p)\, Y_X^{(1)}(-p)\, dp\right)} = \frac{1}{\sqrt{168}}\,,$$

$$Y_X^{(2)} = \frac{1}{p^2 + 7p + 12}\left[\frac{e^{i\gamma}}{2}\frac{1}{(p + i + 3)(p + i + 4)} + \right.$$

$$\left. + \frac{e^{-i\gamma}}{2}\frac{1}{(p + 3 - i)(p + 4 - i)} + 1\right] =$$

$$= \frac{[(p + 3)(p + 4) - 1]\cos\gamma + (2p + 7)\sin\gamma}{(p^2 + 7p + 12)[(p + 3)^2 + 1][(p + 4)^2 + 1]} + \frac{1}{p^2 + 7p + 12}\,,$$

$$\alpha = \left\|\frac{1}{p^2 + a_0 p + b_0}\right\|_p (|b_1| + |b_{-1}|) = \frac{1}{\sqrt{168}}$$

and therefore

$$\varepsilon_2 \leqq \frac{\dfrac{1}{168}\dfrac{1}{\sqrt{168}}}{1 - \dfrac{1}{\sqrt{168}}} = \frac{1}{168(\sqrt{(168)} - 1)}$$

 so that e.g.

$$\sqrt{(\sum_{p_i \in L} \text{Res}\, Y_X^{(2)}(\gamma, p)\, Y_X^{(2)}(\gamma, -p)} - \varepsilon_2 \leqq \sqrt{M(y^2(\gamma))} \leqq$$
$$\leqq \sqrt{(\sum_{p_i \in L} \text{Res}\, Y_X^{(2)}(\gamma, p)\, Y_X^{(2)}(\gamma, -p))} + \varepsilon_2$$

(here p_i are poles, L is the left halfplane).

4. SYSTEM OF DIFFERENTIAL-DIFFERENCE EQUATIONS

The results of the paragraph 3. may be formally extended also to differential equations with delay.

Let us consider the equation (the dimensions are analogous as in Section 3.)

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}(t)\,\mathbf{y}(t) + \mathbf{C}(t)\,\mathbf{y}(t - T) + \mathbf{B}(t)\,\mathbf{x}(t)\,, \tag{30}$$

where again the matrices $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ are periodical, the delay T equals the period length. The case when the ratio of the period length and of the delay is rational may be converted into the forme (30) ([5]). The relation (30) can be written in the following way

$$\frac{d\mathbf{y}}{dt} = (\mathbf{A}(t) + \mathbf{C}(t)\,\mathrm{Z}^{-1})\,\mathbf{y}(t) + \mathbf{B}(t)\,\mathbf{x}(t)\,, \tag{31}$$

where Z is operator defined by

$$\mathrm{Z}(\mathbf{f}(t)) = \mathbf{f}(t + T)\,.$$

Let us further denote the product integral

$$\lim_{\Delta \to 0} [\mathbf{E} + \Delta\mathbf{G}(t_{n-1})]\,[\mathbf{E} + \Delta\mathbf{G}(t_{n-2})] \ldots [\mathbf{E} + \Delta\mathbf{G}(t_0)] = \tag{32}$$
$$= \mathrm{EXP} \int_a^b \mathbf{G}(t)\,dt\,,$$

where $a = t_0 < t_1 < \ldots < t_n = b$, $\Delta = t_k - t_{k-1}$ for $k = 1, \ldots, n$. The operator EXP$\int$ has similar properties as exp $\int$ e.g. (see [6]):

$$\mathrm{EXP} \int_a^c \mathbf{G}(t)\,dt = \left(\mathrm{EXP} \int_b^c \mathbf{G}(t)\,dt\right)\left(\mathrm{EXP} \int_a^b \mathbf{G}(t)\,dt\right) \quad \text{for} \quad a < b < c\,,$$

$$\mathrm{EXP} \int_a^b (\mathbf{G}(t) + \mathbf{H}(t))\,dt = \left(\mathrm{EXP} \int_a^b \mathbf{G}(t)\,dt\right)\left(\mathrm{EXP} \int_a^b \mathbf{H}_s(t)\,dt\right),$$

where

$$\mathbf{H}_s(t) = \mathbf{F}^{-1}(t)\,\mathbf{H}(t)\,\mathbf{F}(t)\,, \quad \mathbf{F}(t) = \mathrm{EXP}\!\!\int_a^t \mathbf{G}(u)\,\mathrm{d}u\,.$$

If for every t_1, t_2 the relation $\mathbf{G}(t_1)\,\mathbf{G}(t_2) = \mathbf{G}(t_2)\,\mathbf{G}(t_1)$ holds (which occurs specially in one-dimensional case), then

$$\mathrm{EXP}\!\!\int_a^b \mathbf{G}(t)\,\mathrm{d}t = \exp\left(\int_a^b \mathbf{G}(t)\,\mathrm{d}t\right).$$

The weighting function for the equation (31) for $\mathbf{B}(t) = \mathbf{E}$ is the solution of the adjoined equation

(33)
$$\mathbf{W}^*_\tau(\gamma, \tau) = (\mathbf{A}^*(\gamma - \tau) + \mathbf{C}^*(\gamma - \tau)\,Z^{-1})\,\mathbf{W}^*(\gamma, \tau)\,;$$
$$\mathbf{W}^*(\gamma, \tau) = 0 \text{ for } \tau < 0\,, \quad \mathbf{W}^*(\gamma, 0) = \mathbf{E}\,.$$

Then the following relation holds

$$\mathbf{W}^*(\gamma, nT + \tau) = \left(\mathrm{EXP}\!\!\int_0^\tau [\mathbf{A}^*(\gamma - u) + \mathbf{C}^*(\gamma - u)\,Z^{-1}]\,\mathrm{d}u\right)\mathbf{W}^*(\gamma, nT)\,.$$

For the sequence $\{\mathbf{W}^*(\gamma, nT)\}_{n=0}^\infty$ the following relation holds

$$\mathbf{W}^*(\gamma, nT) = \left(\mathrm{EXP}\!\!\int_0^T [\mathbf{A}^*(\gamma - u) + \mathbf{C}^\circ(\gamma - u)\,Z^{-1}]\,\mathrm{d}u\right)Z^{-1}\mathbf{W}^*(\gamma, nT)\,;$$
$$\mathbf{W}^*(\gamma, 0) = \mathbf{E}\,,$$

and then the z-transformation of the sequence $\{\mathbf{W}^*(\gamma, nT)\}_{n=0}^\infty$ or $\{\mathbf{W}^*(\gamma, nT + \tau)\}_{n=0}^\infty$ is $(\mathbf{E} - z^{-1}\mathbf{G}^*(\gamma, T, z))^{-1}$ or $\mathbf{G}^*(\gamma, \tau, z)\,(\mathbf{E} - z^{-1}\mathbf{G}^*(\gamma, T, z))^{-1}$ respectively, where

$$\mathbf{G}^*(\gamma, \tau, z) = \mathrm{EXP}\!\!\int_0^\tau [\mathbf{A}^*(\gamma - u) + \mathbf{C}^*(\gamma - u)\,z^{-1}]\,\mathrm{d}u\,,$$

z is a complex variable.

If we further consider $\mathbf{B}(t) \neq \mathbf{E}$, we have similarly as in Section 3:

$$\mathbf{B}^*(\gamma - \tau)\,\mathbf{W}^*(\gamma, nT + \tau) = \mathbf{W}^*(\gamma, nT + \tau)\,; \quad \mathbf{G}^*(\gamma, \tau, z) = \mathbf{B}^*(\gamma - \tau)\,\mathbf{G}^*(\gamma, \tau, z)\,.$$

The z-transformation of the sequence $\{\mathbf{W}^*(\gamma, nT + \tau)\}_{n=0}^\infty$ is

$$\mathbf{G}^*(\gamma, \tau, z)\,(\mathbf{E} - z^{-1}\mathbf{G}^*(\gamma, T, z))^{-1}\,.$$

Similarly according to (15) and (19) we get for $t_1 - t_2 = mT + \alpha$, γ_1 and γ_2 being phases corresponding to t_1 and t_2 respectively:

$$\text{(34)} \qquad \mathrm{M}[\mathbf{y}(t_1)\,\mathbf{y}^*(t_2)] = \frac{1}{2\pi \mathrm{i}} \int_k [\mathbf{E} - z\mathbf{G}(\gamma_1, T, z)]^{-1} [\mathbf{G}(\gamma_1, T, z)]^m \times$$
$$\times \left(\int_0^T \mathbf{G}(\gamma_1, \tau + \alpha, z)\, \mathbf{G}^*(\gamma_2, \tau, z)\, \mathrm{d}\tau \right) [\mathbf{E} - z^{-1}\mathbf{G}^*(\gamma_2, T, z)]^{-1} \frac{\mathrm{d}z}{z}$$

and specially

$$\text{(35)} \quad \mathrm{M}[\mathbf{y}(\gamma)\,\mathbf{y}^*(\gamma)] = \frac{1}{2\pi \mathrm{i}} \int_k [\mathbf{E} - z\mathbf{G}(\gamma, T, z)]^{-1} \left(\int_k^T \mathbf{G}(\gamma, \tau, z)\, \mathbf{G}^*(\gamma, \tau, z)\, \mathrm{d}\tau \right) \times$$
$$\times [\mathbf{E} - z^{-1}\mathbf{G}^*(\gamma, T, z)]^{-1} \frac{\mathrm{d}z}{z}.$$

Remark 1. For $\mathbf{C} = 0$ (see (30)) the expressions (34), (35) are reduced to the expressions (19), (19a).

Remark 2. In a special case when $\mathbf{A}(t_1)$, $\mathbf{A}(t_2)$, $\mathbf{C}(t_3)$, $\mathbf{C}(t_4)$ are mutually commutative (Z^{-1} being commutative with regard to the periodical matrix) EXP = exp and the following relation holds:

$$\mathbf{G}^*(\gamma, \tau, z) = \left(\exp \int_0^\tau \mathbf{A}^*(\gamma - u)\, \mathrm{d}u \right) \left[\exp \left(\int_0^\tau \mathbf{C}^*(\gamma - u)\, \mathrm{d}u z^{-1} \right) \right].$$

In a general case

$$\mathbf{G}^*(\gamma, \tau, z) = \left(\mathrm{EXP} \int_0^\tau \mathbf{A}^*(\gamma - u)\, \mathrm{d}u \right) \left(\mathrm{EXP} \int_0^\tau (\mathbf{C}_s^*(\gamma, u)\, \mathrm{d}u z^{-1}) \right)$$

where

$$\mathbf{C}_s^*(\gamma, u) = [\mathbf{W}_0^*(\gamma, u)]^{-1}\, \mathbf{C}^*(\gamma - u)\, \mathbf{W}_0^*(\gamma, u);$$
$$\mathbf{W}_0^*(\gamma, \tau) = \mathrm{EXP} \int_0^\tau \mathbf{A}^*(\gamma - u)\, \mathrm{d}u,$$

$\mathbf{W}_0^*(\gamma, \tau)$ of course being the weighting function for the system with zero delay member.

Remark 3. In a general case the product integral may be replaced approximately by a product of the type (47). If e.g. $\mathbf{C}_s^*(\gamma - u) = \mathbf{K}_0 + \mathbf{K}_1(\gamma - u)$ where $\mathbf{K}_0$ is constant matrix and $\mathbf{K}_1$ is a "small" periodical matrix, it is advantageous to express

$$\mathrm{EXP} \int_0^\tau (\mathbf{C}_s^*(\gamma - u)\, \mathrm{d}u z^{-1}) = \exp(\mathbf{K}_0 \tau z^{-1})\, \mathrm{EXP} \int_0^\tau (\mathbf{K}_{1s}(\gamma, u)\, \mathrm{d}u z^{-1}),$$

where

$$\mathbf{K}_{1s}(\gamma, u) = \exp(-\mathbf{K}_0 u z^{-1})\, \mathbf{K}_1(\gamma - u) \exp(\mathbf{K}_0 u z^{-1})$$

and further

$$(36)\qquad \mathrm{EXP}\!\!\!\!\!\!\int_0^\tau (\mathbf{K}_{1s}(\gamma, u)\,\mathrm{d}u z^{-1}) = \mathbf{E} + \left(\int_0^\tau \mathbf{K}_{1s}(\gamma, u)\,\mathrm{d}u\right) z^{-1} + \\ + \left(\int_0^\tau \mathbf{K}_{1s}(\gamma, u)\,\mathrm{d}u \int_0^u \mathbf{K}_{1s}(\gamma, v)\,\mathrm{d}v\right) z^{-2} + \ldots$$

where several first members of the series may be taken as approximation. The error of this approximation may be easily determined ([6]). A detailed analysis of problems of an approximate numerical computation which is very laborious will not be carried out here.

It is also possible to use the transfer function technique as in par. 3. In case $\mathbf{A}(t)$, $\mathbf{B}(t)$, $\mathbf{C}(t)$ have again a finite number of harmonics, the transfer functional equation according (33) will be as follows:

$$p^*\mathbf{Y}^*_{\mathbf{X}}(\gamma, p) = [\mathbf{A}^*_0 + \sum_{k=-n}^{n}{}^0 \mathbf{A}^*_k \mathrm{e}^{-\mathrm{i}k\gamma}\mathrm{D}^k + (\mathbf{C}^*_0 + \sum_{k=-n}^{n}{}^0 \mathbf{C}^*_k \mathrm{e}^{-\mathrm{i}k\gamma}\mathrm{D}^k)\,\mathrm{e}^{-p^*T}]\,\mathbf{Y}^*_{\mathbf{X}}(\gamma, p) + \mathbf{E}\,.$$

If the harmonics are "small", iteration again can be made:

$$\mathbf{Y}^{(m+1)}_{\mathbf{X}}(\gamma, p) = \mathbf{H}\mathbf{Y}^{(m)}_{\mathbf{X}}(\gamma, p) + (p\mathbf{E} - \mathbf{A}_0 - \mathrm{e}^{-pT}\mathbf{C}_0)^{-1}\,, \quad \mathbf{Y}^{(0)}_{\mathbf{X}}(\gamma, p) = 0\,,$$

where

$$\mathbf{H} = [\sum_{k=-n}^{n}{}^0 \mathrm{D}^k(\cdot)\,\mathbf{A}_k \mathrm{e}^{\mathrm{i}k\gamma} + \sum_{k=-n}^{n}{}^0 \mathrm{D}^k(\mathrm{e}^{-pT}\cdot)\,\mathbf{C}_k \mathrm{e}^{\mathrm{i}k\gamma}]\,(p\mathbf{E} - \mathbf{A}_0 - \mathrm{e}^{-pT}\mathbf{C}_0)^{-1}\,,$$

$$\|\mathbf{H}\|_p \leqq \|(p\mathbf{E} - \mathbf{A}_0 - \mathrm{e}^{-pT}\mathbf{C}_0)^{-1}\|_p\,(\sum_{k=-n}^{n}{}^0 (\|\mathbf{A}_k\| + \|\mathbf{C}_k\|))\,.$$

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REFERENCES

[1] Тафт В. А.: Основы спектральной теории и расчет цепей с периодическими параметрами. „Наука" 1966.

[2] Endrass H., Schweizer G.: Zusammenhange zwischen der Übertragungsfunktion und der Theorie von Floquet bei linearen Regelsystemen mit periodischen Parametern. Regelungstechnik *11* (1969), 501—504.

[3] Schiehlen W., Kolbe O.: Ein Verfahren zur Untersuchung von linearen Regelsystemen mit periodischen Parametern. Regelungstechnik *15* (1967), 451—455.

[4] Еругин Н. П.: Метод Лаппо — Данилевского в теории линейных дифференциальных уравнений. Изд. Лен. унив. 1956.

[5] Беллман Р., Кук К.: Дифференциально-разностные уравнения. (Добавление: Зверкин А. М.: Дифференциально-разностные уравнения с периодическим коэфициентами.) Москва 1967.

[6] Гантмахер Ф. Р.: Теория матриц. Москва 1966.

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