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Hilbert-Space Methods in Experimental Design*)

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This is a partly review paper on the design of a regression experiment especially for the case of an infinite-dimensional set of responce levels. The estimability of a linear functional and the optimability of a design are expressed in terms of various forms of continuity of the functional. A new proof of the Elfving's theorem then follows. Results on designs for a nonlinear estimation are stated and a connection between a recent paper of Kiefer and Studden [10] and a previous result of the author on interpolation on $\langle -1, 1 \rangle$ are established.

1. INTRODUCTION

For about 25 years the regression experiment with uncorrelated observations has been in the centre of attention of the theory and practice of experimental design. During this period various approaches have made it possible to gain a deeper insight into the structure of an optimal regression experiment. We mention the game-theoretic approach (cf. e.g. [9, 11]) the information-theoretic approach (cf. e.g. [6, 19, 20]), the methods of approximation of functions (cf. e.g. [9, 10]), the methods of convex geometry (cf. e.g. [18]), the gradient and other methods of iterative computation of optimum designs (cf. e.g. [2, 5, 7, 19]), etc. Much has been done in the model choice of an experiment (cf. e.g. [1, 3]) and in the consideration of the proper place of experimental design in the technical and scientific sphere of activity (cf. e.g. [12]).

The list, of course, cannot be considered as complete and it reflects the personal choice of the author. However, what is intentionally omitted in it is the approach based on the Hilbert-space structure of a regression experiment. This structure is in the background of some papers (cf. e.g. [7, 9]) and certainly many authors were aware of it, however they did not put special emphasis on it. But, as we shall show

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in the following sections, the use of the Hilbert-space methods allows to gain a better insight into the design of a standard regression experiment (Proposition 1) and it allows to consider new problems, such as the design of experiments for the nonlinear estimation (Section 3) or the design of infinite-dimensional experiments.

Mathematically the following analysis is based on the geometry of Hilbert-spaces [8], on the methods of gaussian processes [13] and the chaos of Wiener [13] and on the integral and inner product representations of functionals [8, 14].

2. THE MODIFICATION OF THE REGRESSION EXPERIMENT

We recall briefly the structure of a standard regression experiment [5, 9]: On a compact metric space $\mathscr{X}$, m linearly independent continuous functions $f_1, \ldots, f_m$ are given. They span the set of all "response functions" $\Theta = \{\vartheta : \vartheta = \sum_{i=1}^{m} \alpha_i f_i ;\ \alpha_i \in R\}$, the parameters $\alpha_1, \ldots, \alpha_m$ of the actual response function $\sum_{i=1}^{m} \alpha_i f_i$ are supposed unknown. The points of $\mathscr{X}$ are referred to as "controlled variables". For each $x \in \mathscr{X}$ some elementary experiment can be performed whose outcome is a random variable $y(x)$ with the mean $\mathrm{E}_\vartheta(y(x)) = \vartheta(x)$ and the variance $\mathrm{D}_\vartheta(y(x)) = 1$. The parameters $\alpha_1, \ldots, \alpha_m$, or some linear functions of these parameters have to be estimated on the basis of N uncorrelated observations $y(x_1), \ldots, y(x_N)$. A *design* is a probability measure ξ assigning the probability $\xi(x^{(i)})$ at the points $x^{(1)}, \ldots, x^{(r)}$ where $N . \xi(x^{(i)}) = n_i$ is the number of those independent observations among $y(x_1), \ldots, y(x_N)$ which are taken at the same point $x^{(i)}$.

The following modifications of the standard regression experiment are unavoidable in order to obtain a description of an infinite-dimensional regression experiment.

Instead of the outcomes $y(x_1), \ldots, y(x_N)$ we shall (equivalently) consider the random variables $\tilde{Y}_\xi(B)$; $(B \in \mathscr{B})$ ($\mathscr{B}$ being the Borel σ-algebra on $\mathscr{X}$) which are defined by

$$\tilde{Y}_\xi(B) = \sum_{x_i \in B} y(x_i) ; \quad (B \in \mathscr{B}) .$$

Evidently $\mathrm{E}_\vartheta[\tilde{Y}_\xi(B)] = \int \chi_B \vartheta \, \mathrm{d}\xi$, $\mathrm{E}_\vartheta[(\tilde{Y}_\xi(B_1) - \mathrm{E}_\vartheta(\tilde{Y}_\xi(B_1)))(\tilde{Y}_\xi(B_2) - \mathrm{E}_\vartheta(\tilde{Y}_\xi . (B_2)))] = N\, \xi(B_1 \cap B_2)$ for every $B, B_1, B_2 \in \mathscr{B}, \vartheta \in \Theta$, and $\tilde{Y}_\xi(\bigcup_1^k B_i) = \sum_{i=1}^{k} \tilde{Y}_\xi(B_i)$ for mutually disjoint sets $B_1, \ldots, B_k \in \mathscr{B}$. Since the factor N in the covariance is of no importance for comparing experiments, we may say that a regression experiment in which observations are made according to a design ξ is equivalent to an experiment whose outcome are random variables $Y_\xi(B)$; $(B \in \mathscr{B})$ such that

$$E_\vartheta(Y_\xi(B)) = \int_B \vartheta \, \mathrm{d}\xi ; \quad (\vartheta \in \Theta,\ B \in \mathscr{B}) , \tag{1}$$

$$\text{(2)}\qquad \mathrm{E}_\vartheta[(Y_\xi(B_1) - \mathrm{E}_\vartheta(Y_\xi(B_1)))(Y_\xi(B_2) - \mathrm{E}_\vartheta(Y_\xi(B_2)))] = \xi(B_1 \cap B_2)\,; \quad (\vartheta \in \Theta, B_1, B_2 \in \mathscr{B})\,,$$

$$\text{(3)}\qquad Y_\xi\Big(\bigcup_{i=1}^{k} B_i\Big) = \sum_{i=1}^{k} Y_\xi(B_i)\,; \quad B_1, \ldots, B_k \in \mathscr{B}\,, \quad \text{disjoint.}$$

In this way the redefined outcomes of the experiment allow us to investigate also infinite-dimensional regression experiments, i.e. those with an infinite-dimensional set of response functions Θ, which remain supposed continuous ($\Theta \subset C(\mathscr{X})$). Then, instead of considering the estimates of the parameters $\alpha_1, \ldots, \alpha_m$ or of linear functions of these parameters, we shall consider estimates of linear (and in Section 4 also of nonlinear) functionals defined on Θ.

3. ESTIMATES OF FUNCTIONALS AND ELFVING'S THEOREM

The estimates of linear functionals which are defined on Θ are either finite linear combinations of the random variables $Y_\xi(B)$; $B \in \mathscr{B}$ or the L^2-limits of such linear combinations. For further purposes we denote by H the linear space of such potential estimates (under the hypothesis $\vartheta = 0$). H is a Hilbert space with the inner product

$$\langle Y_1, Y_2 \rangle_H = \mathrm{E}_0(Y_1 Y_2)\,.$$

The conditions needed for the estimability of a linear functional on Θ as well as an expression for the variance of the best linear estimate may be well expressed in terms of the Hilbert space $L^2(\mathscr{X}, \mathscr{B}, \xi)$ as follows (cf. [15]).

Proposition 1. A linear functional $g : \Theta \to R$ is (linearly) estimable without bias under the design ξ iff

$$\text{(4)}\qquad g(\vartheta) = \int l\vartheta \, \mathrm{d}\xi\,, \quad (\vartheta \in \Theta)$$

for some $l \in L^2(\mathscr{X}, \mathscr{B}, \xi)$. The orthogonal projection l_ξ of l onto $\bar{\Theta}^\xi$(= the closure of Θ in $L^2(\xi)$) is independent from the choice of l, and

$$\text{(5)}\qquad \mathrm{var}_\xi g = \int l_\xi^2 \, \mathrm{d}\xi$$

is the variance of the best (linear) estimate for g.

The functional g is estimable at least under one design iff g is continuous with respect to the $C(\mathscr{X})$-norm on Θ. A design ξ^* minimizing $\mathrm{var}_\xi\, g$ does always exist and $\mathrm{var}_{\xi^*}\, g$ is equal to the square of this norm.

Then there is at least one bounded signed Borel measure v such that $\int . \mathrm{d}v$ is an extention of $g(.)$ onto $C(\mathscr{X})$ preserving the $C(\mathscr{X})$-norm. Denote $v = v^+ - v^-$ the Jordan decomposition of v.

The design

$$\xi_\nu = \frac{\nu^+ + \nu^-}{\nu^+(\mathscr{X}) + \nu^-(\mathscr{X})}$$

minimizes $\mathrm{var}_\xi\, g$.

Proof. Under the hypothesis $\vartheta = 0$ there in an isomorphism of the Hilbert space H onto the Hilbert space $L^2(\mathscr{X}, \mathscr{B}, \xi)$, $Y \in H \to f_Y \in L^2(\mathscr{X}, \mathscr{B}, \xi)$, defined by the relations

$$\mathrm{E}_0(YY_\xi(B)) = \int_B f_Y \,\mathrm{d}\xi\,; \quad (B \in \mathscr{B})\,.$$

However, for an arbitrary $\vartheta \in \Theta$, the mapping $Y \to f_Y$ is only a linear bijection of H onto $L^2(\mathscr{X}, \mathscr{B}, \xi)$ such that

$$\mathrm{E}_\vartheta[(Y_1 - \mathrm{E}_\vartheta(Y_1))(Y_2 - \mathrm{E}_\vartheta(Y_2))] = \int f_{Y_1} f_{Y_2} \,\mathrm{d}\xi\,, \quad Y_1, Y_2 \in H\,.$$

Moreover, for every $\vartheta \in \Theta$, $Y \in H$ we have

$$\mathrm{E}_\vartheta(Y) = \int f_Y \vartheta \,\mathrm{d}\xi$$

since, according to (1), this is true for $Y = Y_\xi(B)$. Thus an estimate $Y \in H$ is an unbiassed estimate of g iff

$$\int f_Y \vartheta \,\mathrm{d}\xi = g(\vartheta)\,; \quad (\vartheta \in \Theta)$$

and its variance is $\int f_Y^2 \,\mathrm{d}\xi$.

Since $\int (f_Y - l)\, \vartheta \,\mathrm{d}\xi = 0$; $(\vartheta \in \Theta)$ implies $(f_Y - l) \perp \bar{\Theta}^\xi$, the projection of f_Y onto $\bar{\Theta}^\xi$ coincides with that of any l satisfying (4). Evidently the estimate $Z \in H$ such that $f_Z = l_\xi$, is the best unbiassed estimate.

If g is $C(\mathscr{X})$-continuous on Θ then, according to the Hahn-Banach theorem, it may be extended on $C(\mathscr{X})$ without a change of the $C(\mathscr{X})$-norm, and using the Riesz representation theorem [14], we may express this extention as the integral with respect to a conveniently chosen signed measure ν. From (4) it follows that g is estimable under ξ_ν. If g is estimable under some other design ξ then

$$\mathrm{var}_\xi\, g = \sup_{\substack{\vartheta \neq 0 \\ \vartheta \in \Theta}} \frac{|g(\vartheta)|^2}{\int \vartheta^2 \,\mathrm{d}\xi} \geqq \sup_{\substack{\vartheta \neq 0 \\ \vartheta \in \Theta}} \frac{|g(\vartheta)|^2}{\sup_{x \in \mathscr{X}} |\vartheta(x)|^2} =$$

$$= \sup_{\substack{f \neq 0 \\ f \in C(\mathscr{X})}} \frac{\left| \int f \frac{\mathrm{d}\nu}{\mathrm{d}\xi_\nu} \,\mathrm{d}\xi_\nu \right|^2}{\sup_{x \in \mathscr{X}} |f(x)|^2} \geqq \left(\frac{\mathrm{d}\nu}{\mathrm{d}\xi_\nu} \right)^2 = \int \left(\frac{\mathrm{d}\nu}{\mathrm{d}\xi_\nu} \right)^2 \mathrm{d}\xi_\nu$$

and

$$\int \left(\frac{d\nu}{d\xi_\nu}\right)^2 d\xi_\nu \geqq \mathrm{var}_{\xi_\nu} g$$

since

$$\int \frac{d\nu}{d\xi_\nu} \vartheta \, d\xi_\nu = g(\vartheta) ; \quad (\vartheta \in \Theta) . \qquad \square$$

Corollary (Elfving's theorem [4, 9]).
Let Θ be finite-dimensional:

$$\Theta = \{\vartheta : \vartheta = \sum_1^m \alpha_i f_i ; \ (\alpha_i \in R)\}$$

and let c be the vector defined by

$$g(\vartheta) = c'\alpha ; \quad (\vartheta = \sum_1^m \alpha_i f_i \in \Theta) .$$

Denote aco $\{f(x); x \in \mathscr{X}\}$ the convex hull of the set $\{f(x); x \in \mathscr{X}\} \cup \{-f(x); x \in \mathscr{X}\}$ where $f'(x) = (f_1(x), \ldots, f_m(x))$ is the vector of regression functions. Then a design μ minimizes $\mathrm{var}_\xi g$ iff there is a $\mathscr{B}$-measurable function φ on $\mathscr{X}$, with $|\varphi| \equiv 1$ such that

a) $\beta c = \int \varphi(x) f(x) \, \mu(dx)$ for some $\beta \in R$

b) $\int \varphi(x) f(x) \, \mu(dx)$ is a boundary point of aco $\{f(x); x \in \mathscr{X}\}$.

Then $\beta^{-2} = \mathrm{var}_\mu g$.

Proof. Let us denote by $\mathscr{N}$ the set of bounded signed measures $\varkappa$ on $(\mathscr{X}, \mathscr{B})$ such that

$$g(\vartheta) = \int \vartheta \, d\varkappa ; \quad (\vartheta \in \Theta) .$$

If ξ is a design which allows the estimation of g, then $\varkappa \equiv l_\xi \xi \in \mathscr{N}$, hence, according to Proposition 1,

$$(\mathrm{var}_{\xi_\varkappa} g)^2 \leqq \int \left(\frac{d\varkappa}{d\xi_\varkappa}\right)^2 d\xi_\varkappa = \left[\int \frac{d\varkappa}{d\xi_\varkappa} \frac{d\varkappa}{d\xi} d\xi\right]^2 \leqq$$

$$\leqq \left(\frac{d\varkappa}{d\xi_\varkappa}\right)^2 \int \left(\frac{d\varkappa}{d\xi}\right)^2 d\xi = \left(\frac{d\varkappa}{d\xi_\varkappa}\right)^2 \mathrm{var}_\xi g .$$

Thus $\mathrm{var}_{\xi_\varkappa} g \leqq \mathrm{var}_\xi g$ with the equality iff $\xi = \xi_\varkappa$; in the latter case $\mathrm{var}_\xi g = (d\varkappa/d\xi)^2$. It follows that a design μ minimizes $\mathrm{var}_\xi g$ iff

a) $\mu = \xi_\nu$ for some $\nu \in \mathcal{N}$,

b) $\left|\dfrac{d\nu}{d\xi_\nu}\right| = \min\limits_{\varkappa \in \mathcal{N}} \left|\dfrac{d\varkappa}{d\xi_\varkappa}\right|$;

then $|d\nu/d\xi_\nu|^2 = \min\limits_{\xi} \operatorname{var}_\xi g$. This statement is in fact a generalization of the statement in the Corollary (valid also for Θ being infinite-dimensional) as we see if we take

$$\Theta = \{\vartheta : \vartheta = \sum_1^m \alpha_i f_i\,;\ \ (\alpha_i \in R)\}\,, \quad g(\vartheta) = c'\alpha\,,$$

$$\varphi(x) = \operatorname{sign}\left(\frac{d\nu}{d\xi_\nu}(x)\right) \quad \text{and} \quad \mu = \xi_\nu\,. \qquad \square$$

We note here that the set aco $\{f(x);\ x \in \mathscr{X}\}$ is a convex, balanced and absorbing set in R^m, and so its Minkowski function

$$\varrho(c) = \inf\{\lambda : (1/\lambda)\, c \in \operatorname{aco}\{f(x);\ x \in \mathscr{X}\}\}$$

is a seminorm in R^m, which is a norm in the span of $\{f(x);\ x \in \mathscr{X}\}$. According to Proposition 1 and to the Corollary, $\varrho(c)$ is the $C(\mathscr{X})$ norm of the functional $g(\vartheta) = c'\alpha$ and both are equal to $(\min\limits_{\xi} \operatorname{var}_\xi g)^{1/2}$.

4. DESIGNS FOR THE ESTIMATION OF HOMOGENEOUS POLYNOMIALS

The Hilbert – space methods can be used advantageously also for the design of experiments in which homogeneous polynomials on Θ are to be estimated. An n-th degree homogeneous polynomial h is a function on Θ which may be expressed as

$$h(\vartheta) = q_n(\vartheta, \ldots, \vartheta)\,; \quad (\vartheta \in \Theta)\,,$$

where q_n is an n-linear functional on Θ^n. For example if Θ is finite-dimensional and parametrized:

$$\Theta = \{\vartheta : \vartheta(x) = \sum_{i=1}^m \alpha_i f_i(x)\}\,,$$

h may be expressed as

$$h(\vartheta) = \sum_{i_1,\ldots,i_n=1}^m \beta_{i_1,\ldots,i_n} \alpha_{i_1} \ldots \alpha_{i_n}\,; \quad (\vartheta(x) = \sum_1^m \alpha_i f_i(x))\,.$$

Let us denote by Q the linear space of all polynomials of the random variables $\{Y_\xi(B);\ B \in \mathscr{B}\}$ and of the L^2-limits of such polynomials. Under the hypothesis

$\vartheta = 0$, Q is a Hilbert spaee with the inner product

$$\langle Y_1, Y_2 \rangle_Q = \mathrm{E}_0(Y_1 Y_2)\,; \quad (Y_1, Y_2 \in Q)\,.$$

Unbiassed estimates for h are elements of Q, however for a non-linear h, the variance of the best (minimal variance) unbiassed estimate depends on ϑ, which complicates the construction of a priori designs. Hilbert-space methods can be used to order designs in a way which does not depend on ϑ but which respects the variance of the best estimate. Further, the Hilbert-space methods allow to reformulate the design for nonlinear estimation in terms of the problem of design of linear estimation (in another regression experiment).

The main steps of the used construction and the results are presented here in Propositions 2–5. We omit the proofs, since Propositions 2, 3 and 5 are the results of the theory of the Wiener chaos (cf. [13], Chap. 5) and Proposition 4 and its corollary, Proposition 6, are proved in [17].

We note that the mapping $Y \in H \to f_Y \in L^2(\mathscr{X}, \mathscr{B}, \xi)$ is that mentioned in the proof of Proposition 1.

Proposition 2. Denote by $[L^2(\xi)]^{\odot i}$ the i-th symmetric tensor power of the Hilbert space $L^2(\xi) = L^2(\mathscr{X}, \mathscr{B}, \xi)$. Under the hypothesis $\vartheta = 0$ there is an isomorphism Ψ of $\sum_{i=0}^{\infty} [L^2(\xi)]^{\odot i}$ onto Q defined by

$$\Psi\left[\sum_{i=0}^{\infty} \frac{f_Y^{\otimes i}}{\sqrt{i\,!}}\right] = \mathrm{e}^{Y - \mathrm{E}_0 Y^2}\,; \quad (Y \in Q)\,.$$

Thus $\mathrm{D}_0(Y) = \mathrm{E}_0(Y^2) = \|f_Y\|^2$, $\|\ \|$ being the norm in $\sum_{i=0}^{\infty} [L^2(\xi)]^{\odot i}$.

Proposition 3. For every $l \in \sum_{i=0}^{\infty} [L^2(\xi)]^{\odot i}$ and every $\vartheta \in \Theta$ we have

$$\mathrm{E}_\vartheta\, \Psi(l) = \left\langle l, \sum_{i=0}^{\infty} \frac{\vartheta^{\otimes i}}{\sqrt{i\,!}} \right\rangle$$

$\langle\ ,\ \rangle$ being the inner product in $\sum_{i=0}^{\infty} [L^2(\xi)]^{\odot i}$.

Proposition 4. For every $l \in [L^2(\xi)]^{\odot n}$ and for every $\vartheta \in \Theta$ the following inequality is valid

$$\mathrm{D}_0[\Psi(l)] \leqq \mathrm{D}_\vartheta[\Psi(l)] \leqq \mathrm{D}_0[\Psi(l)] \sum_{s=0}^{n-1} \binom{n}{s} \frac{[\int \vartheta^2\, \mathrm{d}\xi]^s}{s!}\,,$$

$\mathrm{D}_\vartheta[\Psi(l)]$ being the variance of the random variable $\Psi(l) \in Q$ under the hypothesis that the response function is ϑ.

Proposition 5. Let h be a homogeneous polynomial of degree n, defined on Θ. Let $\Theta^{\odot n}$ be the linear space of functions on X^n which is spanned by the set

$$\{\vartheta^{\otimes n} : \vartheta^{\otimes n}(x_1, \ldots, x_n) = \vartheta(x_1) \ldots \vartheta(x_n);\ (\vartheta \in \Theta)\}.$$

Then there is a unique linear functional g on $\Theta^{\odot n}$ such that

$$g(\vartheta^{\otimes n}) = \sqrt{(n!)}\, h(\vartheta)\,;\quad (\vartheta \in \Theta).$$

Proposition 6. Let ξ be a design of the experiment and let h, g be as in Proposition 5. Then:

1) h is estimable (without bias) under the design ξ iff g is estimable under the design ξ^n on $\mathscr{X}^n$.

2) With every unbiassed estimate Y of h we may associate an unbiassed estimate Z of g so that for every $\vartheta \in \Theta$

$$\mathrm{D}(Z) \leqq \mathrm{D}_\vartheta(Y) \leqq \mathrm{D}(Z) \sum_{s=0}^{n-1} \binom{n}{s} \frac{[\int \vartheta^2\, \mathrm{d}\xi]^s}{s!}.$$

3) If Z^* is the best unbiassed estimate for g (under the design ξ^n), then for every $\vartheta \in \Theta$

$$\mathrm{D}(Z^*) \leqq \min_Y \mathrm{D}_\vartheta(Y) \leqq \mathrm{D}(Z^*) \sum_{s=0}^{n-1} \binom{n}{s} \frac{[\int \vartheta^2\, \mathrm{d}\xi]^s}{s!},$$

where the minimum is taken over the set of all unbiassed estimates for h.

The last proposition asserts that if estimating a homogeneous polynomial h, an adequate criterion of optimality of the design is the variance of the best estimate of the associated linear fuctional g. However, we have to look for an optimum design not among all designs on $\mathscr{X}^n$ but only among those which are of the form ξ^n for some ξ on $\mathscr{X}$.

5. AMBIGUITY OF OPTIMUM DESIGNS FOR THE INTERPOLATION OF INFINITE-DIMENSIONAL POLYNOMIALS ON $\langle -1,1 \rangle$.

Hilbert-space methods may be used to complement the theoretical study [10] published recently.

Let us consider a regression experiment on the set $\mathscr{X} = \langle -1,1 \rangle$, in which a response function may be any polynomial on $\langle -1,1 \rangle$ (without restrictions on the degree of the polynomial). Can we propose designs for the interpolation of the response function is such a case? The answer is negative, as it is easily seen, but the approximative approach to the problem has some interesting features.

The set Θ_∞ of all polynomials is dense in $L^2(\mathscr{X}, \mathscr{B}, \xi)$ for every design ξ. Thus, according to Proposition 1, a linear functional g is estimable under the design ξ iff there is a unique $l \in L^2(\xi)$ such that $g(.) = \int l \,.\, \mathrm{d}\xi$; hence $\mathrm{var}_\xi\, g = \int l^2 \,\mathrm{d}\xi$. For example the evaluation functional (in the point x):

$$g_x : \vartheta \in \Theta_\infty \mapsto \vartheta(x) \in R$$

is estimable iff $\xi(x) > 0$; then $\mathrm{var}_\xi\, g = 1/\xi(x)$. It follows that there is no design on $\langle -1,1 \rangle$ which allows an unbiassed estimation of the whole response function $\vartheta \in \Theta_\infty$.

Thus we proceed by approximations of ϑ. Let us denote $I(n, x) \equiv \langle -1,1 \rangle \cap \cap (x - 1/n, x + 1/n)$. Consider a design ξ which is absolutely continuous with respect to the Lebesgue measure on $\langle -1,1 \rangle$ and let us denote

$$g_x(\vartheta \mid n, \xi) = \frac{\int_{I(n,x)} \vartheta \,\mathrm{d}\xi}{\xi[I(n, x)]}\,; \quad (\vartheta \in \Theta,\ n = 1, 2, \ldots, x \in \langle -1,1 \rangle)\,.$$

Let $Z_n^\xi(x)$ stand for the best unbiassed estimate for $g_x(. \mid n, \xi)$ under the design ξ. $\{Z_n^\xi(x); x \in \mathscr{X}\}$ is a second order random process with the mean $\mathrm{E}_\vartheta(Z_n^\xi(x)) = g_x(\vartheta \mid n, \xi)$ and with the covariance function $\mathrm{E}_\vartheta[(Z_n^\xi(x_1) - \mathrm{E}_\vartheta(Z_n^\xi(x_1)))\,(Z_n(x_2) - \mathrm{E}_\vartheta(Z_n(x_2)))] = \xi[I(n, x_1) \cap I(n, x_2)]/\xi[I(n, x_1)]\, \xi[I(n, x_2)]$. Both, the mean and the covariance function, are continuous (cf. [16], Lemma 1) and the random process $\{Z_n^\xi(x); x \in \mathscr{X}\}$ is a biassed but efficient estimate for the response function ϑ, with the bias being evaluated by

$$\sup_{x \in \langle -1,1 \rangle} |\mathrm{E}_\vartheta\, Z_n^\xi(x) - \vartheta(x)| \leqq \sup_{\substack{x_1, x_2 \in \langle -1,1 \rangle \\ |x_1 - x_2| < 1/n}} |\vartheta(x_1) - \vartheta(x_2)|\,.$$

Hence with $n \to \infty$ the bias tends to zero with a rate which is given by the modulus of continuity of the response function ϑ:

$$\lim_{n \to \infty} n \,.\, \sup_{x \in \langle -1,1 \rangle} |\mathrm{E}_\vartheta\, Z_n^\xi(x) - \vartheta(x)| \leqq \sup_{x_1, x_2 \in \langle -1,1 \rangle} \frac{|\vartheta(x_1) - \vartheta(x_2)|}{|x_1 - x_2|}\,. \tag{6}$$

The sequence of estimates $\{Z_n^\xi(x);\ x \in \langle -1,1 \rangle\};\ n = 1, 2, \ldots$ is the best under the design ξ in the sense given in the following proposition.

Proposition 7. Let $\{S_n(x) : x \in \langle -1,1 \rangle\!:\ (n = 1, 2, \ldots)\}$ be a set of elements of $L^2\{Y_\xi(B);\ B \in \mathscr{B}\}$ (. i.e. a set of linear estimates under the design ξ) with the property

$$|\mathrm{E}_\vartheta\, S_n(x) - \vartheta(x)| = \sup_{x_1, x_2 \in I(n,x)} |\vartheta(x_1) - \vartheta(x_2)|\,; \quad (\vartheta \in \Theta_\infty, x \in \langle -1,1 \rangle,\ n = 1, 2, \ldots)\,. \tag{7}$$

Then

$$\liminf_{n\to\infty} \frac{\mathrm{D}_\vartheta[S_n(x)]}{\mathrm{D}_\vartheta[Z_n^\xi(x)]} \geqq 1\,; \quad (x \in \langle -1,1\rangle,\ \vartheta \in \Theta) \tag{8}$$

Proof. $S_n(x) \in L^2\{X_\xi(B);\ B \in \mathscr{B}\}$ implies that the functional

$$h_x(.\mid n) : \vartheta \in \Theta_\infty \mapsto \mathrm{E}_\vartheta[S_n(x)] \in R$$

is linear and is estimable under the design ξ. Thus there is a unique $\varphi_x(n) \in L^2(\xi)$ such that

$$h_x(\vartheta \mid n) = \int_{-1}^{1} \varphi_x(n)\, \vartheta\, \mathrm{d}\xi\,; \quad (\vartheta \in \Theta_\infty)\,.$$

Let us suppose that there are $B \in \mathscr{B}$ and $c > 0$ such that $B \cap \overline{I(n,x)} = \emptyset$, $\xi(B) > 0$, and $\varphi_x(n) > c$ on B. Then we have a compact set $C \subset B$ with $\xi(C) > 0$ and a function f, continuous on $\langle -1,1\rangle$ such that $f = 0$ on $\overline{I(n,x)}$, $f = 1$ on C. Hence for every $\varepsilon > 0$ there is a polynomial $\vartheta_\varepsilon \in \Theta_\infty$ such that $-\varepsilon \leqq \vartheta_\varepsilon \leqq \varepsilon$ on C and $1 - \varepsilon \leqq \vartheta_\varepsilon \leqq \leqq 1 + \varepsilon$ on $I(n,x)$. Thus for ε sufficiently small

$$\begin{aligned}\mathrm{E}_{\vartheta\varepsilon}(S_n(x)) - \vartheta_\varepsilon(x) &= \int \varphi_x(n)\, \vartheta_\varepsilon\, \mathrm{d}\xi - \vartheta_\varepsilon(x) \geqq \int_C \varphi_x(n)\, \vartheta_\varepsilon\, \mathrm{d}\xi - \vartheta_\varepsilon(x) \geqq \\ &\geqq (1-\varepsilon)\, c\, \xi(C) - \varepsilon > 2\varepsilon \geqq \sup_{x_1,x_2 \in I(n,x)} |\vartheta_\varepsilon(x_1) - \vartheta_\varepsilon(x_2)|\,.\end{aligned}$$

This violates the assumption (7) and consequently outside of $I(n,x)$, $\varphi_x(n) = 0$ a.e. $[\xi]$. Reasoning similarly for $\varphi_x(n) < -c$; $(c > 0)$ we obtain: $\xi(\{\varphi_x(n) \neq 0\} - - I(n,x)) = 0$ It follows that

$$\left[\int_{-1}^{1} \varphi_x(n)\, \mathrm{d}\xi\right]^2 \leqq \xi[I(n,x)] \int_{-1}^{1} (\varphi_x(n))^2\, \mathrm{d}\xi = \frac{\mathrm{D}_\vartheta(S_n(x))}{\mathrm{D}_\vartheta(Z_n^\xi(x))}\,.$$

On the other hand the function identically equal to 1 is an element of Θ_∞. Thus, according to (7), $\lim_{n\to\infty} \int_{-1}^{1} \varphi_x(n)\, \mathrm{d}\xi = 1$. This together with (7) imply (8). □

We see that it seems to be quite reasonable to consider the optimality of a design ξ according to the properties of $\{Z_n^\xi(x);\ x \in \langle -1,1\rangle\}$ for a large n. So the G-efficiency of two designs ξ, η may be compared by the expression

$$\frac{\sup_{x\in\langle -1,1\rangle} \mathrm{D}_\vartheta(Z_n^\xi(x))}{\sup_{x\in\langle -1,1\rangle} D_\vartheta(Z_n^\eta(x))} = \frac{\inf_{x\in\langle -1,1\rangle} \eta[I(n,x)]}{\inf_{x\in\langle -1,1\rangle} \xi[I(n,x)]}$$

for n tending to infinity. Yet intuitively we may expect that, unless perhaps in the neighbourhood of the points $-1,1$, the normed Lebesgue measure on $\langle -1,1\rangle$, λ, will be G-optimal. Indeed, it may be proved (cf. [16], Theorem 3) that for every $\xi \ll \lambda$

$$\lim_{k\to\infty} \liminf_{n\to\infty} \frac{\inf\{\lambda[I(n, x)] : -1 + 1/k \leqq x \leqq 1 - 1/k\}}{\inf\{\xi[I(n, x)] : -1 + 1/k \leqq x \leqq 1 - 1/k\}} \geqq 1 .$$

However this choice of the optimal design depends on the needed rate of convergence of the bias to zero. Suppose then that τ is a homeomorphism of $\langle -1,1\rangle$ into $(-\infty, \infty)$ and instead of (7) let us require

$$|\mathrm{E}_\vartheta(S_n(x)) - \vartheta(x)| \leqq \sup_{\substack{|\tau(x_1)-\tau(x)|<1/n \\ |\tau(x_2)-\tau(x)|<1/n}} |\vartheta(x_1) - \vartheta(x_2)| ;$$

$$(\vartheta \in \Theta_\infty, x \in \langle -1,1\rangle, n = 1, 2, \ldots) .$$

Then the design $\varkappa$, the image by τ^{-1} of the normed Lebesgue measure on $\tau(\langle -1,1\rangle)$, is the optimal design (cf. [16], Theorem 3). Especially if we require

$$|\mathrm{E}_\vartheta (S_n(x)) - \vartheta(x)| \leqq \sup_{\substack{|\arccos x_1 - \arccos x|<1/n \\ |\arccos x_2 - \arccos x|<1/n}} |\vartheta(x_1) - \vartheta(x_2)| ,$$

then $\mathrm{d}\varkappa/\mathrm{d}\lambda \sim (1 - x^2)^{-1/2}$, which is the optimum design considered in [5] and in [10].

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