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# BILLINGSLEY-TYPE TIGHTNESS CRITERIA FOR MULTIPARAMETER STOCHASTIC PROCESSES

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This paper gives an extension of the tightness criterion from processes on $C(0, 1)$ or $D(0, 1)$ (see Billingsley [2], Theorems 12.3 and 15.6) to processes on $C_k(0, 1)$ or $D_k(0, 1)$ for $k > 1$. The proposed criteria coincide with those of Billingsley if $k = 1$. Theorem 2 provides a generalization of the criterion for processes on $D_k(0, 1)$ proved by Bickel and Wichura [1], in the sense that our criterion is not restricted to processes vanishing along the lower boundary of $\langle 0, 1\rangle^k$.

## 0. INTRODUCTION

Throughout this paper we shall speak on the stochastic process $(X(t), t \in \langle 0, 1\rangle^k)$ if there is a given nonempty set $\Omega$ and $P$: $\exp \Omega \to \langle 0, +\infty\rangle$ with the following properties:

(1) if $A \subset B \subset \Omega$ then $P(A) \leqq P(B)$,

(2) if $A_n \subset \Omega$, $n \in \mathbb{N}$ then $P(\bigcup_{n=1}^{+\infty} A_n) \leqq \sum_{n=1}^{+\infty} P(A_n)$ and $X(t)$: $\Omega \to \mathbb{R}$ for every $t \in \langle 0, 1\rangle^k$.

We shall use the following notation:

(3) $\Phi_k$ is the set of all permutations of coordinates at $\langle 0, 1\rangle^k$,

(4) $\Psi_k$ is the set of permutations that reverse only the $j$th and the last coordinates at $\langle 0, 1\rangle^k$ for some $j$ ($j = 1, 2, \ldots, k$),

(5) $\Delta X(\varphi, d, j)(A) = \sum_{i=1}^{d} \sum_{\delta_i = a_i, b_i} (-1)^{\Sigma_p I(\delta_p = a_p)} X \circ \varphi(\delta_1, \ldots, \delta_d, \underbrace{0, \ldots, 0}_{k-d-j}, \underbrace{1, \ldots, 1}_{j})$ is the increment of $X \circ \varphi(., \ldots, ., 0, \ldots, 0, 1, \ldots, 1)$ around $A$, where $\varphi \in \Phi_k$, $d = 1, \ldots, k, j = 0, \ldots, k - d$, $A = \bigtimes_{i=1}^{d} \langle a_i, b_i) \subset \langle 0, 1)^d$,

(6) $\|X\| = \sup \{|X(t)| \mid t \in \langle 0, 1\rangle^k\}$.

We shall distinguish two different continuity moduls of a function.

## 1. UNIFORM CONTINUITY MODUL

The usual continuity modul is defined as

$$c(x, \varepsilon, k) = \max_{\varphi \in \Psi_k} \tilde{c}(x \circ \varphi, \varepsilon, k) \quad \text{for} \quad \varepsilon > 0 \quad \text{and} \quad x \in \mathbb{R}^{\langle 0,1 \rangle^k}$$

where

$$\tilde{c}(x, \varepsilon, k) = \sup \{ |x(t) - x(s)| \, | 0 \leqq t_k < s_k \leqq 1, \ s_k - t_k < \varepsilon \text{ and } 0 \leqq t_i = s_i \leqq 1 \text{ for } i = 1, \ldots, k - 1 \} .$$

The following theorem gives an extension of Theorem 12.3 of Billingsley [2].

**Theorem 1.** Let $(X(t) \ t \in \langle 0, 1 \rangle^k)$ be a stochastic process right-continuous at every coordinate. Let there exist $\alpha, \beta > 0$ and bounded measures $\mu_{\varphi,d}$ on $\mathscr{B}(\langle 0, 1 \rangle^d)$, depending on $\varphi \in \Phi_k$ and $d = 1, \ldots, k$, such that:

(7) the measures $\mu_{\varphi,d}$ have continuous marginals,

(8) $P(|\Delta X(\varphi, d, 0)(A)| > y) \leqq y^{-\alpha} \mu_{\varphi,d}(A)^{1+\beta}$ for every $\varphi \in \Phi_k$, $d = 1, \ldots, k$, $A = \bigtimes_{i=1}^{d} \langle a_i, b_i \rangle \subset \langle 0, 1 \rangle^d$ and $y > 0$.

Then

(9) $P(\|X\| > y) \leqq P(|X(\underbrace{0, \ldots, 0}_{k})| > y/2) + 2^\alpha k^\alpha Q y^{-\alpha}$, and

(10) $P(c(X, \varepsilon, k) > y) \leqq Q y^{-\alpha} R(\varepsilon)$ for every $\varepsilon, y > 0$, where $0 \leqq R(\varepsilon) \leqq 1$, $\lim_{\varepsilon \to 0+} R(\varepsilon) = 0$ and $Q$, $R$ depend only on $\alpha$, $\beta$, $k$ and the measures $\mu_{\varphi,d}$.

Proof. Let $\mu_d$ be a bounded measure on $\mathscr{B}(\langle 0, 1 \rangle^d)$ with continuous and increasing marginals, $d = 1, \ldots, k$, possessing the following properties:

(11) $$\mu_{\varphi,d}(A) \leqq \mu_d(A) \quad \text{for every} \quad \varphi \in \Phi_k , \quad A \in \mathscr{B}(\langle 0, 1 \rangle^d) ,$$

(12) $$\mu_{d+1}(\langle 0, 1 \rangle \times A) \leqq \mu_d \quad \text{for} \quad d = 1, \ldots, k - 1 , \quad A \in \mathscr{B}(\langle 0, 1 \rangle^d) .$$

There exists a positive integer $n(\varepsilon)$ to every $\varepsilon > 0$ such that

(13) $$\mu_1(\langle 0, 1 \rangle) \, 2^{-n(\varepsilon)} < \sup \{ \mu_1(\langle a, b \rangle) \mid b - a \leqq \varepsilon \} \leqq \mu_1(\langle 0, 1 \rangle) \, 2^{1-n(\varepsilon)} .$$

We shall prove that

(14) $$P(\|X\| > y) \leqq P(|X(0)| > y/2) + 2^\alpha k^\alpha \, Q(k) \, \mu_1(\langle 0, 1 \rangle)^{1+2\gamma} \, y^{-\alpha}$$

and

(15) $$P(c(X, \varepsilon, k) > y) \leqq Q(k) \, \mu_1(\langle 0, 1 \rangle)^{1+2\gamma} \, y^{-\alpha} \, 2^{\gamma(1-n(\varepsilon))} ,$$

where $\gamma = \frac{1}{2}\beta$.

Put $D = 2 \sum_{p=1}^{+\infty} p^{-2}$ .

I. Let first the process $X$ satisfy

(16) $$P(\|X \circ \varphi(\cdot, a) - X \circ \varphi(\cdot, b)\| > y) \leqq W . \, \mu_1(\langle a, b \rangle)^{1+2\gamma} . \, y^{-\alpha}$$
$$\text{for every} \quad \varphi \in \Phi , \quad 0 \leqq a < b \leqq 1 , \quad y > 0 .$$

There exist the points $c_{ip} \in \langle 0, 1\rangle$, $i = 0, \ldots, 2^{p-1}$, $p \in \mathbb{N}$ such that

(17) $\mu_1(\langle 0, c_{ip}\rangle) = i\, 2^{1-p} \mu_1(\langle 0, 1\rangle)$ according to continuity of $\mu_1$.

Let $A^* \subset \Omega$ be a random event

(18) $$A^* = [|X \circ \varphi(t, c_{ip}) - X \circ \varphi(t, c_{i+1,p})| \leqq D^{-1} P^{-2} y, \quad \text{for every } t \in \langle 0, 1\rangle^{k-1}, \quad i = 0, \ldots, 2^{p-1} - 1, \quad p = n(\varepsilon), n(\varepsilon) + 1, \ldots]$$

for some $\varepsilon > 0$ and $\varphi \in \Phi_k$.

If $0 \leqq a < b \leqq 1$, $b - a \leqq \varepsilon$ then we can put

$$a_p = \begin{cases} 0 & \text{if } a = 0 \\ c_{ip} & \text{if } c_{ip} \geqq a > c_{i-1,p} \end{cases}$$

$$b_p = \begin{cases} 0 & \text{if } b = 0 \\ c_{ip} & \text{if } c_{ip} \geqq b > c_{i-1,p} . \end{cases}$$

Then, due to the right-continuity of $X \circ \varphi$ in the last coordinate,

$$|X \circ \varphi(t, a) - X \circ \varphi(t, b)| \leqq |X \circ \varphi(t, a_{n(\varepsilon)}) - X \circ \varphi(t, b_{n(\varepsilon)})| + \\ + \sum_{p=n(\varepsilon)}^{+\infty} (|X \circ \varphi(t, a_{p+1}) - X \circ \varphi(t, a_p)| + |X \circ \varphi(t, b_{p+1}) - X \circ \varphi(t, b_p)|) \leqq \\ \leqq D^{-1} n(\varepsilon)^{-2} y + 2 \sum_{p=n(\varepsilon)+1}^{+\infty} D^{-1} P^{-2} y < 2D^{-1} y \sum_{p=1}^{+\infty} P^{-2} = y .$$

This yields

$$P(\tilde{c}(X \circ \varphi. \varepsilon, k) > y) \leqq P(\Omega - A^*) \leqq \\ \leqq \sum_{p=n(\varepsilon)}^{+\infty} \sum_{i=0}^{2^{p-1}-1} P(\sup \{|X \circ \varphi(t, c_{ip}) - X \circ \varphi(t, c_{i+1,p})| \mid t \in \langle 0, 1\rangle^{k-1}\} > \\ > D^{-1} p^{-2} y) \leqq \sum_{p=n(\varepsilon)}^{+\infty} \sum_{i=0}^{2^{p-1}-1} W D^{\alpha} p^{2\alpha} y^{-\alpha} \mu_1(\langle c_{ip}, c_{i+1,p}))^{1+2\gamma} = \\ = W D^{\alpha} y^{-\alpha} \mu_1(\langle 0, 1\rangle)^{1+2\gamma} \sum_{p=n(\varepsilon)}^{+\infty} p^{2\alpha} \,.\, 2^{(1-p)2\gamma} .$$

Then

(19) $$P(c(X, \varepsilon, k) > y) \leqq k W D^{\alpha} (\sum_{p=1}^{+\infty} p^{2\alpha}\, 2^{(1-p)\gamma})\, \mu_1(\langle 0, 1\rangle)^{1+2\gamma}\, y^{-\alpha}\, 2^{\gamma(1-n(\varepsilon))}$$

and

$$P(\|X\| > y) \leqq P(|X(0)| > y/2) + P(c(X, 1, k) > y/(2k)) \leqq \\ \leqq P(|X(0)| > y/2) + k W D^{\alpha} (\sum_{p=1}^{+\infty} P^{2\alpha}\, 2^{(1-p)\gamma})\, \mu_1(\langle 0, 1\rangle)^{1+2\gamma} (2k)^{\alpha}\, y^{-\alpha} .$$

II. (14 and (15) will be now proved by induction over $k$.

i) Let $k = 1$. Then (16) holds with $W = 1$ and by (19)

$$P(\|X\| > y) \leqq P(|X(0)| > y/2) + 2^{\alpha} D^{\alpha} (\sum_{p=1}^{+\infty} P^{2\alpha}\, 2^{(1-p)\gamma})\, \mu_1(\langle 0, 1\rangle)^{1+2\gamma}\, y^{-\alpha}$$

and

$$P(c(X, \varepsilon, k) > y) \leqq D^\alpha(\sum_{p=1}^{+\infty} P^{2\alpha}\, 2^{(1-p)\gamma})\, \mu_1(\langle 0, 1\rangle)^{1+2\gamma}\, y^{-\alpha}\, 2^{\gamma(1-n(\varepsilon))}$$

which coincide with (14) and (15) with $Q(1) = D^\alpha \sum_{p=1}^{+\infty} P^{2\alpha}\, 2^{(1-p)\gamma}$.

ii) Let (14) and (15) hold for $k$ and let $(X(t),\ t \in \langle 0, 1\rangle^{k+1})$ be a process satisfying our assumptions. Then it satisfies (16). Actually, let $\varphi \in \Phi_k$, $0 \leqq a < b \leqq 1$ and put $Y(t) = X \circ \varphi(t, a) - X \circ \varphi(t, b)$.

Then

$$P(|\Delta Y(\psi, d, 0)(A)| > y) = P(|\Delta X(\varphi \circ \tilde{\psi}, d + 1, 0)(A \times \langle a, b\rangle)| > y) \leqq$$
$$\leqq y^{-\alpha}\, \mu_{d+1}(A \times \langle a, b\rangle)^{1+2\gamma}$$

for every $\psi \in \Phi_k$, $\tilde{\psi}(t, u) = (\psi(t), u)$, $A = \bigtimes_{i=1}^{d} \langle a_i, b_i\rangle$, $y > 0$.

Moreover

$$P(\sup\{|X \circ \varphi(t, a) - X \circ \varphi(t, b)|\ |t \in \langle 0, 1\rangle^k\} > y) =$$
$$= P(\|Y\| > y) \leqq P(|Y(0)| > y/2) + 2^\alpha k^\alpha\, Q(k)\, \mu_2(\langle 0, 1\rangle \times \langle a, b\rangle)^{1+2\gamma}\, y^{-d} \leqq$$
$$\leqq 2^\alpha y^{-\alpha}\, \mu_1(\langle a, b\rangle)^{1+2\gamma} + 2^\alpha k^\alpha\, Q(k)\, \mu_1(\langle a, b\rangle)^{1+2\gamma}\, y^{-\alpha} =$$
$$= 2^\alpha(1 + k^\alpha\, Q(k))\, \mu_1(\langle a, b\rangle)^{1+2\gamma}\, y^{-\alpha}.$$

Then (14) and (15) hold with $Q(k + 1) = (k + 1)\, 2^\alpha(1 + k^\alpha\, Q(k))\, Q(1)$ and that we wished to prove.

III. Construct the measure $\hat{\mu}_k = \sum_{\varphi \in \Phi_k} \mu_{\varphi k}$,

$$\hat{\mu}_d = \mu_{d+1}(\langle 0, 1\rangle \times \cdot) = \sum_{\varphi \in \Phi_k} \mu_{\varphi d}.$$

Then $\mu_d = \hat{\mu}_d + \lambda_d$, with $\lambda_d$ being the Lebesgue measure on $\langle 0, 1\rangle^d$, are the desired measures for steps I, II. □

## 2. SKOROCHOD CONTINUITY MODUL

The Skorochod continuity modul is defined as

$$s(x, \varepsilon, k) = \max_{\varphi \in \Psi_k} \tilde{s}(x \circ \varphi, \varepsilon, k) \quad \text{for} \quad \varepsilon > 0 \quad \text{and} \quad x \in \mathbb{R}^{\langle 0,1\rangle^k},$$

where

$$\tilde{s}(x, \varepsilon, k) = \sup\{\min\{|x(t) - x(s)|, |x(s) - x(u)|\}\ |0 \leqq t_k < s_k < u_k \leqq 1,$$
$$u_k - t_k < \varepsilon \quad \text{and} \quad 0 \leqq t_i = s_i = u_i \leqq 1 \quad \text{for} \quad i = 1, \ldots, k - 1\}.$$

The following theorem is an extension of Theorem 15.6 in Billingsley [2].

**Theorem 2.** Let $(X(t),\ t \in \langle 0, 1 \rangle^k)$ be a stochastic process right-continuous at every coordinate. Let there exist $\alpha, \beta > 0$ and bounded measures $\mu_{\varphi,d,j}$ on $\mathscr{B}(\langle 0, 1 \rangle^d)$, depending on $\varphi \in \Phi_k$, $d = 1, \ldots, k$, $j = 0, \ldots, k - d$, such that

(20) the measures $\mu_{\varphi,d,j}$ have continuous marginals,

(21) $P(|\Delta X(\varphi, d, j)(A)| > y, |\Delta X(\varphi, d, j)(B)| > y) \leqq y^{-\alpha} \mu_{\varphi,d,j}(A \cup B)^{1+\beta}$

for every $\varphi \in \Phi_k$, $d = 1, \ldots, k$, $j = 0, \ldots, d - k$, $A = \bigtimes_{i=1}^{d} \langle a_i, b_i \rangle$, $B = \bigtimes_{i=1}^{d} \langle h_i, g_i \rangle$, $A, B \subset \langle 0, 1 \rangle^d$, $A \cap B = \emptyset$, clo $A \cap$ clo $B \neq \emptyset$ and $y > 0$.

Then

(22) $P(s(X, \varepsilon, k) > y) \leqq Q y^{-\alpha} R(\varepsilon)$, and

(23) $P(\|X\| > y) \leqq P(\max\{|X(\delta)|\ \delta_i = 0, 1\} > y/2) + (2k)^\alpha Q y^{-\alpha}$ where $0 \leqq R(\varepsilon) \leqq 1$, $\lim_{\varepsilon \to 0+} R(\varepsilon) = 0$ and $Q$, $R$ depend only on $\alpha$, $\beta$, $k$ and on measures $\mu_{\varphi,d,j}$.

Theorem 2 will be proved in several steps. First, define for $x, y \in \mathbb{R}^{\langle 0,1 \rangle^k}$

$$s(x, y, \varepsilon, k) = \max_{\varphi \in \Psi_k} \tilde{s}(x \circ \varphi, y \circ \varphi, \varepsilon, k, 0, 1),$$

where

$$\tilde{s}(x, y, \varepsilon, k, a, b) = \sup\{\min\{|x(t) - x(s)|, |y(s) - y(u)|\}\ |a \leqq t_k < s_k < u_k \leqq b,$$
$$u_k - t_k < \varepsilon \text{ and } 0 \leqq t_i = s_i = u_i \leqq 1 \text{ for } i = 1, \ldots, k - 1\}.$$

Moreover, denote

$$\|x, y\| = \sup\{\min\{|x(t)|, |y(t)|\}\ |t \in \langle, 1 \rangle^k\}.$$

We shall start with the following lemma.

**Lemma 1.** Let $(X(t), t \in \langle 0, 1 \rangle^k)$, $(Y(t), t \in \langle 0, 1 \rangle^k)$ be two stochastic processes. Let there exist $\alpha, \gamma > 0$ and bounded measures $\mu_d$ on $\mathscr{B}(\langle 0, 1 \rangle^d)$ such that:

(24) the measures $\mu_d$ have continuous and increasing marginals and

$\mu_d(A) \geqq \mu_{d+1}(\langle 0, 1 \rangle \times A)$ for $A \in \mathscr{B}(\langle 0, 1 \rangle^d)$,

(25) $P(|\Delta X(\varphi, d, j)(A)| > y, |\Delta X(\varphi, d, j)(B)| > y) \leqq y^{-\alpha} \mu_d(A \cup B)^{1+2\gamma}{}_p$

$P(|\Delta Y(\varphi, d, j)(A)| > y, |\Delta Y(\varphi, d, j)(B)| > y) \leqq y^{-\alpha} \mu_d(A \cup B)^{1+2\gamma}$

$P(|\Delta X(\varphi, d, j)(A)| > y, |\Delta Y(\varphi, d, j)(B)| > y) \leqq y^{-\alpha} \mu_d(A \cup B)^{1+2\gamma}$

$P(|\Delta X(\varphi, d, j)(A)| > y, |\Delta Y(\psi, d, j)(A)| > y) \leqq y^{-\alpha} \mu_d(A)^{1+2\gamma}$

for every $\varphi \in \Phi_k$, $d = 1, \ldots, k$, $j = 0, \ldots, k - d$, $A = \bigtimes_{i=1}^{d} \langle a_i, b_i \rangle$, $B = \bigtimes_{i=1}^{d} \langle h_i, g_i \rangle$, $A, B \subset \langle 0, 1 \rangle^d$, $A \cap B = \emptyset$, clo $A \cap$ clo $B \neq \emptyset$ and every $y > 0$.

Then

(26) $P(\tilde{s}(X, Y, \varepsilon, k, a, b) > y) \leqq Q(k)\, \mu_1(\langle a, b\rangle)^{1+2\gamma}\, y^{-\alpha}\, 2^{(1-n(\varepsilon))\gamma}$,

$$P(s(X, Y, \varepsilon, k) > y) \leqq k\, Q(k)\, \mu_1(\{0, 1\rangle)^{1+2\gamma}\, y^{-\alpha}\, 2^{(1-n(\varepsilon))\gamma}$$

and

(27) $P(s(X, \varepsilon, k) > y) \leqq k\, Q(k)\, \mu_1(\langle 0, 1\rangle)^{1+2\gamma}\, y^{-\alpha}\, 2^{(1-n(\varepsilon))\gamma}$,

where $Q(k)$ depends of $\alpha, \gamma, k$ only and $n(\varepsilon)$ is given by (13).

Proof. Put $D = 4\sum_{p=1}^{+\infty} P^{-2}$.

I. Let the processes $X$, $Y$ satisfy

(28) $$P(\|Z \circ \varphi(\cdot, a) - Z \circ \varphi(\cdot, b), V \circ \varphi(\cdot, b) - V \circ \varphi(\cdot, c)\| > y) \leqq \leqq Wy^{-\alpha}\, \mu_1(\langle a, c\rangle)^{1+2\gamma}$$

for every $Z, V \in \{X, Y\}$, and

$$P(\|X \circ \varphi(\cdot, a) - X \circ \varphi(\cdot, c), Y \circ \varphi(\cdot, a) - Y \circ \varphi(\cdot, c)\| > y) \leqq \leqq Wy^{-\alpha}\, \mu_1(\langle a, c\rangle)^{1+2\gamma}$$

for every $\varphi \in \Psi_k$, $0 \leqq a < b < c \leqq 1$ and $y > 0$.

Fix $\tilde{\alpha}, \tilde{\beta}$, $0 \leqq \tilde{\alpha} < \tilde{\beta} \leqq 1$ and find the points $c_{ip}$ such that

$$i\, 2^{1-p}\, \mu_1(\langle \tilde{\alpha}, \tilde{\beta}\rangle) = \mu_1(\langle \tilde{\alpha}, c_{ip}\rangle)\,.$$

Let $A^{**} \subset \Omega$ denote the rabnom event

(29) $$A^{**} = [\|Z \circ \varphi(\cdot, c_{ip}) - Z \circ \varphi(\cdot, c_{i+1,p}), V \circ \varphi(\cdot, c_{i+1,p}) - - V \circ \varphi(\cdot, c_{i+2,p})\| \leqq D^{-1}p^{-2}y \quad \text{for} \quad Z, V \in \{X, Y\}$$

and

$$\|X \circ \varphi(\cdot, c_{in(\varepsilon)}) - X \circ \varphi(\cdot, c_{i+1,n(\varepsilon)}), Y \circ \varphi(\cdot, c_{in(\varepsilon)}) - Y \circ \varphi(\cdot, c_{i+1,n(\varepsilon)})\| \leqq$$
$$\leqq D^{-1}\, n(\varepsilon)^{-2}\, y \ \text{for} \ i = 0, 1, \ldots, 2^{p-1} - 2 \quad \text{and} \quad p = n(\varepsilon) + 1, n(\varepsilon) + 2, \ldots]$$

for some $\varepsilon > 0$, $\varphi \in \Psi_k$.

If $\tilde{\alpha} \leqq a = c_{iq} < b = c_{jq} < c = c_{lq} \leqq \tilde{\beta}$ ans $c - a < \varepsilon$ then it is possible to find the sequences $a_p = a_{i_p p} \leqq b_p = c_{l_p p}$; $h_p = c_{r_p p} \leqq c_p = c_{j_p p}$ such that

$$a_p = a\,, \quad b_p = h_p = b\,, \quad c_p = c \quad \text{for} \quad p \geqq q\,,$$

$$|i_{p+1} - 2i_p| \leqq 1\,, \quad |l_{p+1} - 2l_p| \leqq 1\,, \quad |r_{p+1} - 2r_p| \leqq 1\,, \quad |j_{p+1} - 2j_p| \leqq 1$$

and

$$|X \circ \varphi(t, a_{p+1}) - X \circ \varphi(t, a_p)| \leqq D^{-1}(p+1)^{-2}\, y\,,$$
$$|X \circ \varphi(t, b_{p+1}) - X \circ \varphi(t, b_p)| \leqq D^{-1}(p+1)^{-2}\, y\,,$$
$$|Y \circ \varphi(t, h_{p+1}) - Y \circ \varphi(t, h_p)| \leqq D^{-1}(p+1)^{-2}\, y\,,$$
$$|Y \circ \varphi(t, c_{p+1}) - Y \circ \varphi(t, c_p)| \leqq D^{-1}(p+1)^{-2}\, y\,.$$

Then, for $\omega \in A^{**}$

$$\min\{|X\circ\varphi(t,a) - X\circ\varphi(t,b)|, |Y\circ\varphi(t,b) - Y\circ\varphi(t,c)|\} \leqq$$

$$\leqq \sum_{p=n(\varepsilon)}^{+\infty} (|X\circ\varphi(t,a_{p+1}) - X\circ\varphi(t,a_p)| + |X\circ\varphi(t,b_{p+1}) - X\circ\varphi(t,b_p)| +$$

$$+ |Y\circ\varphi(t,h_{p+1}) - Y\circ\psi(t,h_p)| + |Y\circ\varphi(t,c_{p+1}) - Y\circ\varphi(t,c_p)|) +$$

$$+ \sum_{i=i_{n(\varepsilon)}}^{l_{n(\varepsilon)}-1} \sum_{j=r_{n(\varepsilon)}}^{j_{n(\varepsilon)}-1} \min\{|X\circ\varphi(t,c_{in(\varepsilon)}) - X\circ\varphi(t,c_{i+1,n(\varepsilon)})|,$$

$$|Y\circ\varphi(t,c_{j+1,n(\varepsilon)}) - Y\circ\varphi(t,c_{j,n(\varepsilon)})|\} \leqq 4D^{-1}y\sum_{p=1}^{+\infty} P^{-2} = y.$$

This yields

$$P(\tilde{s}(X,Y,\varepsilon,k,\tilde{\alpha},\tilde{\beta}) > y) \leqq P(\Omega - A^{**}) \leqq \sum_{Z,V\in\{X,Y\}} \sum_{p=n(\varepsilon)+1}^{+\infty} \sum_{i=0}^{2^{p-1}-2}$$

$$P(\|Z\circ\varphi(\cdot,c_{i,p}) - Z\circ\varphi(\cdot,c_{i+1,p}), V\circ\varphi(\cdot,c_{i+1,p}) - V\circ\varphi(\cdot,c_{i+2,p})\| >$$

$$> D^{-1}P^{-2}y) + \sum_{i=0}^{2^{p-1}-1} P(\|X\circ\varphi(\cdot,c_{in(\varepsilon)}) - X\circ\varphi(\cdot,c_{i+1,n(\varepsilon)}),$$

$$Y\circ\varphi(\cdot,c_{in(\varepsilon)}) - Y\circ\varphi(\cdot,c_{i+1,n(\varepsilon)})\| > D^{-1}\,n(\varepsilon)^{-2}\,y) \leqq$$

$$\leqq 4\sum_{p=n(\varepsilon)}^{+\infty} \sum_{i=0}^{2^{p-1}-1} WD^{\alpha}P^{2\alpha}y^{-\alpha}\,2^{(2-p)(1+2\gamma)}\,\mu_1(\langle\tilde{\alpha},\tilde{\beta}\rangle)^{1+2\gamma} =$$

$$= 2^{2\gamma}\,8WD^{\alpha}\sum_{p=n(\varepsilon)}^{+\infty} P^{2\alpha}\,2^{(1-p)2\gamma}\,\mu_1(\langle\tilde{\alpha},\tilde{\beta}\rangle)^{1+2\gamma}\,y^{-\alpha}.$$

Then

$$P(\tilde{s}(X,Y,\varepsilon,k,\tilde{\alpha},\tilde{\beta}) > y) \leqq 8\cdot4^{\gamma}WD^{\alpha}\Big(\sum_{p=1}^{+\infty} P^{2\alpha}\,2^{(1-p)\gamma}\Big)\,\mu_1(\langle\tilde{\alpha},\tilde{\beta}\rangle)^{1+2\gamma}\,.\,y^{-\alpha}\,2^{(1-n(\varepsilon))\gamma}, \tag{30}$$

$$P(s,X,Y,\varepsilon,k) > y) \leqq k\,8\cdot4^{\gamma}WD^{\alpha}\Big(\sum_{p=1}^{+\infty} P^{2\alpha}\,2^{(1-p)\gamma}\Big)\,\mu_1(\langle 0,1\rangle)^{1+2\gamma}\,.\,y^{-\alpha}\,2^{(1-n(\varepsilon))\gamma}.$$

Quite analogously we prove

$$P(s,(X,\varepsilon,k) > y) \leqq k\,8\cdot4^{\gamma}WD^{\alpha}\Big(\sum_{p=1}^{+\infty} P^{2\alpha}\,2^{(1-p)\gamma}\Big)\,\mu_1(\langle 0,1\rangle)^{1+2\gamma}\,.\,y^{-\alpha}\,2^{(1-n(\varepsilon))\gamma}, \tag{31}$$

because there exist sequences $a_p$, $b_p$, $h_p$, $c_p$ such that $b_p = h_p$.

II. Using the induction over $k$, we shall prove (26).

i) Let $k = 1$ then (28) holds with $W = 1$, and by (30)

$$Q(1) = 8\,.\,4^{\gamma}D^{\alpha}\Big(\sum_{p=1}^{+\infty} P^{2\alpha}\,2^{(1-p)\gamma}\Big).$$

ii) Let the lemma hold for $k$ and we have a pair of stochastic processes $X$, $Y$ satisfying our assumptions for $k + 1$. We shall prove that it satisfies (28). Take $\varphi \in \Psi_k$,

$0 \leqq a < b < c \leqq 1$, $Z, V \in \{X, Y\}$ and put

$$Z_0(t, u) = Z \circ \varphi(t, 0, u)\,, \quad Z_1(t, u) = Z \circ \varphi(t, 1, u)\,,$$
$$V_0(t, u) = V \circ \varphi(t, 0, u)\,, \quad V_1(t, u) = V \circ \varphi(t, 1, u)\,,$$
$$Z_2(t, u) = Z \circ \varphi(t, u, a) - Z \circ \varphi(t, u, b)$$
$$V_2(t, u) = V \circ \varphi(t, u, b) - V \circ \varphi(t, u, c)\,.$$

Then it is possible to write

$$\|Z \circ \varphi(\cdot, a) - Z \circ \varphi(\cdot, b), V \circ \varphi(\cdot, b) - V \circ \varphi(\cdot, c)\| \leqq$$
$$\leqq \|Z_0(\cdot, a) - Z_0(\cdot, b), V_0(\cdot, b) - V_0(\cdot, c)\| + \|Z_1(\cdot, a) - Z_1(\cdot, b), V_1(\cdot. b) - - V_1(\cdot, c)\| +$$
$$+ s(Z_2, 1, k) + s(V_2, 1, k) + s(Z_2, V_2, 1, k) + s(V_2, Z_2, 1, k) \leqq$$
$$\leqq \tilde{s}(Z_0, V_0, 1, k, a, c) + \tilde{s}(Z_1, V_1, k, a, c) + s(Z_2, 1, k) + s(V_2, 1, k) +$$
$$+ s(Z_2, V_2, 1, k) + s(V_2, Z_2, 1, k)$$

$\{Z_0, V_0\}$, $\{Z_1, V_1\}$, $\{Z_2, V_2\}$ clearly satisfy the conditions of the lemma. Then, by the induction hypothesis,

$$P(\tilde{s}(Z_0, V_0, 1, k, a, c) > y) \leqq Q(k)\, \mu_1(\langle a, c\rangle)^{1+2\gamma}\, y^{-\alpha}\,,$$
$$P(\tilde{s}(Z_1, V_1, 1, k, a, c) > y) \leqq Q(k)\, \mu_1(\langle a, c\rangle)^{1+2\gamma}\, y^{-\alpha}\,,$$
$$P(s(Z_2, 1, k) > y) \leqq k\, Q(k)\, \mu_2(\langle 0, 1\rangle \times \langle a, b\rangle)^{1+2\gamma}\, y^{-\alpha}\,,$$
$$P(s(V_2, 1, k) > y) \leqq k\, Q(k)\, \mu_2(\langle 0, 1\rangle \times \langle b, c\rangle)^{1+2\gamma}\, y^{-\alpha}$$
$$P(s(Z_2, V_2, 1, k) > y) \leqq k\, Q(k)\, \mu_2(\langle 0, 1\rangle \times \langle a, c\rangle^{1+2\gamma}\, y^{-\alpha}\,,$$
$$P(s(V_2, Z_2, 1, k) > y) \leqq k\, Q(k)\, \mu_2(\langle 0, 1\rangle \times \langle a, c\rangle)^{1+2\gamma}\, y^{-\alpha}\,.$$

This yields

$$P(\|Z \circ \psi(\cdot, a) - Z \circ \varphi(\cdot, b), V \circ \varphi(\cdot, b) - V \circ \varphi(\cdot, c)\| > y) \leqq$$
$$\leqq 6^{1+\alpha} k\, Q(k)\, \mu_1(\langle a, c\rangle)^{1+2\gamma}\, y^{-\alpha}$$

and analogously

$$P(\|X \circ \varphi(\cdot, a) - X \circ \varphi(\cdot, c), Y \circ \varphi(\cdot, a) - Y \circ \varphi(\cdot, c)\| > y) \leqq$$
$$\leqq 6^{1+\alpha} k\, Q(k)\, \mu_1(\langle a, c\rangle)^{1+2\gamma}\, y^{-\alpha}\,.$$

Hence (28) holds with $W = 6^{1+\alpha} k\, Q(k)$ and by (30) and (31) $Q(k + 1) = 6^{1+\alpha}\, Q(k)\,.\,k\,.\,Q(1)$. This completes the induction procedure and hence also the proof of the lemma. □

Proof of Theorem 2. Put $\hat{\mu}_k = \sum_{\varphi \in \Phi_k} \mu_{\varphi,k,0}$ and $\hat{\mu}_d = \sum_{\varphi \in \Phi_k} \sum_{j=0}^{k-d} \mu_{\varphi,d,j} + \hat{\mu}_{d+1}(\langle 0, 1\rangle \times .)$ for $d = k - 1, k - 2, \ldots, 1$. Then the measures $\mu_d = \hat{\mu}_d + \lambda_d$, with $\lambda_d$ being the Lebesgue measure on $\langle 0, 1\rangle^d$, satisfy (24). Then (22) follows by applying Lemma 1 to the pair $X$, $Y$ with $Y \equiv 0$.

Noticing that

$$\|X\| \leqq \max \{|X(\delta)| \mid \delta_i = 0, 1\} + k\, s(X, 1, k)$$

we arrive at (23).

This completes the proof of Theorem 2. □

**Remark.** Let $X$ be a random element of $C_k(0, 1)$; then the relation (10) provides a tightness criterion for $X$; analogously (22) represents a tightness criterion for a random element of $D_k(0, 1)$. Hence, Theorems 1 and 2 enable to verify the convergence in distribution of a sequence $\{X_n\}$ of random elements of $C_k(0, 1)$ and $D_k(0, 1)$, respectively, and that under a more general setup than in Bickel and Wichura [1]. For definition of weak convergence of measures on $D_k(0, 1)$, see Straf [4] and Neuhaus [3].

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