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On Axiomatic Characterization of Information-Theoretic Measure of Type $\binom{\alpha,\ \beta}{\gamma,\ \delta}$

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There are two information-theoretic measures (viz., Kullback's relative-information and Kerridge's inaccuracy) associated with a pair of probability distributions of a discrete random variable. These measures have found many applications in Statistics, Economics, Physics etc. Two parameter generalization of these measures has been studied by many authors. A generalized measure of type $\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}$ is characterized in this paper by taking a set of axioms. This measure in particular contains relative-information and inaccuracy and their generalized forms as the limiting cases. Cases of bivariate extensions of this generalized measure and relations between them have been also studied.

1. INTRODUCTION

Associated with a pair of discrete probability distributions $P = (p_1, \ldots, p_n)$, $p_i \geqq 0$ $\sum_{i=1}^{n} p_i = 1$ and $Q = (q_1, \ldots, q_n)$, $q_i > 0$, $\sum_{i=1}^{n} q_i \leqq 1$ an information-theoretic measure of type $\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}$ is given by

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P; Q) = (2^{\alpha-\beta} - 2^{\gamma-\delta})^{-1} \sum_{i=1}^{n} (p_i^\alpha q_i^{\beta-\alpha} - p_i^\gamma q_i^{\delta-\gamma}), \tag{1.1}$$

$\alpha \neq \gamma$ $(\beta \neq \delta)$ whenever $\beta = \delta$ $(\alpha = \gamma)$.

The measure (1.1) has been studied by Sharma and Taneja [13] and Sharma and Gupta [11] by generalizing a functional equation considered by Chaundy and McLeod [1] and by Taneja and Gupta [15] by considering a functional equation which is a generalization of the one considered by Daróczy [2].

For $\gamma = \delta = 1$ (refer Sharma and Autar [9, 10]) the measure (1.1) reduces to

$$I^{(\alpha,\beta)}(P; Q) = (2^{\alpha-\beta} - 1)^{-1} \left(\sum_{i=1}^{n} p_i^\alpha q_i^{\beta-\alpha} - 1\right), \quad \beta \neq \alpha, \quad \alpha > 0. \tag{1.2}$$

The measure (1.2) reduces to (i) Kullback's [7] relative-information when $\beta = 1$ and there is a limiting case $\alpha \to 1$, (ii) Kerridge's [5] inaccuracy when $\alpha = 1$ and there is a limiting case $\beta \to 1$.

Further when $Q \equiv P$ and $\alpha = \gamma = \delta = 1$, (1.1) reduces to

$$(1.3) \quad I^{\beta}(P; Q) = H^{\beta}(P) = (2^{1-\beta} - 1)^{-1} \left(\sum_{i=1}^{n} p_i^{\beta} - 1\right), \quad \beta \neq 1, \quad \beta > 0,$$

which is entropy of type β introduced by Havrda and Charvát [4] and later differently studied by Daróczy [2] and Vajda [16].

In this communication we characterize the measure (1.1) by taking a set of axioms (cf. Havrda and Charvát [4]). We also study bivariate extensions of measure (1.1) and establish relations between them.

2. CHARACTERIZATION THEOREM

Recursivity plays a vital role in characterization of information-theoretic measures. It is well known that the most elegant characterization of Shannon's entropy given by Faddeev (refer Feinstein [3]) so as those given earlier by Shannon [8] and Khinchine [6] use the recursive relation

$$(2.1) \qquad H(p_1, \ldots, p_n) - H(p_1 + p_2, p_3, \ldots, p_n) = p_i H(p_1/p_i, p_2/p_i),$$

where $p_1 + p_2 = p_i > 0$.

Certain changes in this property of additive measures lead to non-additive measures. Havrda and Charvát [4] also used a modification of this property by introducing a parameter β as

$$(2.2) \qquad H^{\beta}(p_1, \ldots, p_n) - H^{\beta}(p_1 + p_2, p_3, \ldots, p_n) = p_i^{\beta} H^{\beta}(p_1/p_i, p_2/p_i), \quad \beta > 0,$$

where $p_1 + p_2 = p_i > 0$.

Further, Taneja [14] generalized (2.2) by taking a general continuous function $f(p_i)$ in place of p_i^{β} and established that such a change does not lead to new measures and the only measures that arise are those studied by Shannon [8] and Havrda and Charvát [4].

Also an axiomatic characterization of (1.2) which is a measure of a pair of probability distributions has been studied by Sharma and Taneja [13] by considering the recursive relation of type (α, β) given by

$$(2.3) \quad I^{(\alpha,\beta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) - I^{(\alpha,\beta)}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) = \\ = p_i^{\alpha} q_i^{\beta-\alpha} I^{(\alpha,\beta)}(p_1/p_i, p_2/p_i; q_1/q_i, q_2/q_i),$$

where $p_1 + p_2 = p_i > 0$, $q_1 + q_2 = q_i > 0$ and α, β are the parameters.

Here we consider a much different form of the recursive relation to characterize (1.1) axiomatically. Precisely, for a pair of probability distributions $P = (p_1, \ldots, p_n)$, $p_i \geqq 0$, $\sum_{i=1}^{n} p_i = 1$ and $Q = (q_1, \ldots, q_n)$, $q_i > 0$, $\sum_{i=1}^{n} q_i \leqq 1$ we consider the following axioms:

(a) $I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_n; q_1, \ldots, q_n)$ is a continuous function of its arguments;

(b) $I_{(\gamma,\delta)}^{(\alpha,\beta)}(1, 0; \frac{1}{2}, \frac{1}{2}) = 1\,;\quad I_{(\gamma,\delta)}^{(\alpha,\beta)}(1, 0; 1, 0) = 0\,;$

(c) $I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_{i-1}, 0, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, 0, q_{i+1}, \ldots, q_n) =$
$= I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)\,,$
for every $i = 1, 2, \ldots, n$;

(d)
$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_{i-1}, v_{i_1}, v_{i_2}, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, w_{i_1}, w_{i_2}, q_{i+1}, \ldots, q_n) =$$
$$= I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_n) +$$
$$+ \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}}\, p_i^{\alpha} q_i^{\beta-\alpha}\, I_{(1,1)}^{(\alpha,\beta)}\left(\frac{v_{i_1}}{p_i}, \frac{v_{i_2}}{p_i}; \frac{w_{i_1}}{q_i}, \frac{w_{i_2}}{q_i}\right) +$$
$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}}\, p_i^{\gamma} q_i^{\delta-\gamma}\, I_{(\gamma,\delta)}^{(1,1)}\left(\frac{v_{i_1}}{p_i}, \frac{v_{i_2}}{p_i}; \frac{w_{i_1}}{q_i}, \frac{w_{i_2}}{q_i}\right),$$

for every $v_{i_1} + v_{i_2} = p_i > 0$, $w_{i_1} + w_{i_2} = q_i > 0$, $i = 1, 2, \ldots, n$ and $A_{\alpha,\beta} = (2^{\alpha-\beta} - 1)$ and $A_{\gamma,\delta} = (2^{\gamma-\delta} - 1)$.

Theorem 2.1. Axioms (a)–(d) determine a measure given by

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) = (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} (p_i^{\alpha} q_i^{\beta-\alpha} - p_i^{\gamma} q_i^{\delta-\gamma})\,, \tag{2.4}$$
$$A_{\alpha,\beta} \neq A_{\gamma,\delta}, \quad \alpha, \gamma > 0\,,$$

where*) $A_{\alpha,\beta} = (2^{\alpha-\beta} - 1)$ and $A_{\gamma,\delta} = (2^{\gamma-\delta} - 1)$.

The proof of the above theorem is based on the following lemmas:

Lemma 1. If $v_k \geqq 0$, $k = 1, 2, \ldots, m$, $\sum_{k=1}^{m} v_k = p_i > 0$, $w_k > 0$, $k = 1, 2, \ldots, m$, $\sum_{k=1}^{m} w_k = q_i > 0$, then

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_{i-1}, v_1, \ldots, v_m, p_{i+1}, \ldots, p_n; q_1, \ldots, q_{i-1}, w_1, \ldots, w_m, \tag{2.5}$$
$$q_{i+1}, \ldots, q_n) = I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) +$$

*) Throughout this paper, we shall adopt the notation $A_{\alpha,\beta}$ for $(2^{\alpha-\beta} - 1)$ and $A_{\gamma,\delta}$ for $(2^{\gamma-\delta} - 1)$.

$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\,p_i^{\alpha}\,q_i^{\beta-\alpha}\,I^{(\alpha,\beta)}_{(1,1)}\left(\frac{v_1}{p_i},\ldots,\frac{v_m}{p_i};\frac{w_1}{q_i},\ldots,\frac{w_m}{q_i}\right)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\,p_i^{\gamma}\,q_i^{\delta-\gamma}\,I^{(1,1)}_{(\gamma,\delta)}\left(\frac{v_1}{p_i},\ldots,\frac{v_m}{p_i};\frac{w_1}{q_i},\ldots,\frac{w_m}{q_i}\right).$$

Proof. We prove the lemma by induction. For $m = 2$ the result holds (cf. axiom (d)). Let us suppose that the result is true for $m = t$. We have (under the notation $\overline{V} = v_2 + \ldots + v_{t+1}$, $\overline{W} = w_2 + \ldots + w_{t+1}$, $v_1 + \overline{V} = p_i > 0$, $w_1 + \overline{W} = q_i > 0$)

$$(2.6)\qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1,\ldots,p_{i-1},v_1,\ldots,v_{t+1},p_{i+1},\ldots,p_n;q_1,\ldots,q_{i-1},$$
$$w_1,\ldots,w_{t+1},q_{i+1},\ldots,q_n)=$$
$$=I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1,\ldots,p_{i-1},v_1,\overline{V},p_{i+1},\ldots,p_n;q_1,\ldots,q_{i-1},w_1,\overline{W},q_{i+1},\ldots,q_n)+$$
$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\,\overline{V}^{\alpha}\overline{W}^{\beta-\alpha}\,I^{(\alpha,\beta)}_{(1,1)}\left(\frac{v_2}{\overline{V}},\ldots,\frac{v_{t+1}}{\overline{V}};\frac{w_2}{\overline{W}},\ldots,\frac{w_{t+1}}{\overline{W}}\right)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\,\overline{V}^{\alpha}\overline{W}^{\delta-\gamma}\,I^{(1,1)}_{(\gamma,\delta)}\left(\frac{v_2}{\overline{V}},\ldots,\frac{v_{t+1}}{\overline{V}};\frac{w_2}{\overline{W}},\ldots,\frac{w_{t+1}}{\overline{W}}\right)=$$
$$=I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1,\ldots,p_n;q_1,\ldots,q_n)+$$
$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\,p_i^{\alpha}q_i^{\beta-\alpha}\,I^{(\alpha,\beta)}_{(1,1)}\left(\frac{v_1}{p_i},\frac{\overline{V}}{p_i};\frac{w_1}{q_i},\frac{\overline{W}}{q_i}\right)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\,p_i^{\gamma}q_i^{\delta-\gamma}\,I^{(1,1)}_{(\gamma,\delta)}\left(\frac{v_1}{p_i},\frac{\overline{V}}{p_i};\frac{w_1}{q_i},\frac{\overline{W}}{q_i}\right)+$$
$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\,\overline{V}^{\alpha}\overline{W}^{\beta-\alpha}\,I^{(\alpha,\beta)}_{(1,1)}\left(\frac{v_2}{\overline{V}},\ldots,\frac{v_{t+1}}{\overline{V}};\frac{w_2}{\overline{W}},\ldots,\frac{w_{t+1}}{\overline{W}}\right)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\,\overline{V}^{\gamma}\overline{W}^{\delta-\gamma}\,I^{(1,1)}_{(\gamma,\delta)}\left(\frac{v_2}{\overline{V}},\ldots,\frac{v_{t+1}}{\overline{V}};\frac{w_2}{\overline{W}},\ldots,\frac{w_{t+1}}{\overline{W}}\right)=$$
$$=I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1,\ldots,p_n;q_1,\ldots,q_n)+$$
$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\,p_i^{\alpha}q_i^{\beta-\alpha}\,I^{(\alpha,\beta)}_{(1,1)}\left(\frac{v_1}{p_i},\frac{\overline{V}}{p_i};\frac{w_1}{q_i},\frac{\overline{W}}{q_i}\right)+$$
$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\,\overline{V}^{\alpha}\overline{W}^{\beta-\alpha}\,I^{(\alpha,\beta)}_{(1,1)}\left(\frac{v_2}{\overline{V}},\ldots,\frac{v_{t+1}}{\overline{V}};\frac{w_2}{\overline{W}},\ldots,\frac{w_{t+1}}{\overline{W}}\right)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\,p_i^{\gamma}q_i^{\delta-\gamma}\,I^{(1,1)}_{(\gamma,\delta)}\left(\frac{v_1}{p_i},\frac{\overline{V}}{p_i};\frac{w_1}{q_i},\frac{\overline{W}}{q_i}\right)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\,\overline{V}^{\gamma}\overline{W}^{\delta-\gamma}\,I^{(1,1)}_{(\gamma,\delta)}\left(\frac{v_2}{\overline{V}},\ldots,\frac{v_{t+1}}{\overline{V}};\frac{w_2}{\overline{W}},\ldots,\frac{w_{t+1}}{\overline{W}}\right).$$

 One more application of the induction premise yields

$$(2.7)\qquad I_{(\gamma,\delta)}^{(\alpha,\beta)}\left(\frac{v_1}{p_i},\ldots,\frac{v_{t+1}}{p_i};\frac{w_1}{q_i},\ldots,\frac{w_{t+1}}{q_i}\right)=$$

$$=I_{(\gamma,\delta)}^{(\alpha,\beta)}\left(\frac{v_1}{p_i},\frac{V}{p_i};\frac{w_1}{q_i},\frac{W}{q_i}\right)+$$

$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\left(\frac{V}{p_i}\right)^{\alpha}\left(\frac{W}{q_i}\right)^{\beta-\alpha}I_{(1,1)}^{(\alpha,\beta)}\left(\frac{v_2}{V},\ldots,\frac{v_{t+1}}{V};\frac{w_2}{W},\ldots,\frac{w_{t+1}}{W}\right)+$$

$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\left(\frac{V}{p_i}\right)^{\gamma}\left(\frac{W}{q_i}\right)^{\delta-\gamma}I_{(\gamma,\delta)}^{(1,1)}\left(\frac{v_2}{V},\ldots,\frac{v_{t+1}}{V};\frac{w_2}{W},\ldots,\frac{w_{t+1}}{W}\right).$$

For $\gamma=\delta=1$ (2.7) gives

$$(2.8)\qquad I_{(1,1)}^{(\alpha,\beta)}\left(\frac{v_1}{p_i},\ldots,\frac{v_{t+1}}{p_i};\frac{w_1}{q_i},\ldots,\frac{w_{t+1}}{q_i}\right)=I_{(1,1)}^{(\alpha,\beta)}\left(\frac{v_1}{p_i},\frac{V}{p_i};\frac{w_1}{q_i},\frac{W}{q_i}\right)+$$

$$+\left(\frac{V}{p_i}\right)^{\alpha}\left(\frac{W}{q_i}\right)^{\beta-\alpha}I_{(1,1)}^{(\alpha,\beta)}\left(\frac{v_2}{V},\ldots,\frac{v_{t+1}}{V};\frac{w_2}{W},\ldots,\frac{w_{t+1}}{W}\right).$$

For $\alpha=\beta=1$ (2.7) gives

$$(2.9)\qquad I_{(\gamma,\delta)}^{(1,1)}\left(\frac{v_1}{p_i},\ldots,\frac{v_{t+1}}{p_i};\frac{w_1}{q_i},\ldots,\frac{w_{t+1}}{q_i}\right)=I_{(\gamma,\delta)}^{(1,1)}\left(\frac{v_1}{p_i},\frac{V}{p_i};\frac{w_1}{q_i},\frac{W}{q_i}\right)+$$

$$+\left(\frac{V}{p_i}\right)^{\gamma}\left(\frac{W}{q_i}\right)^{\delta-\gamma}I_{(\gamma,\delta)}^{(1,1)}\left(\frac{v_2}{V},\ldots,\frac{v_{t+1}}{V};\frac{w_2}{W},\ldots,\frac{w_{t+1}}{W}\right).$$

Expression (2.6) together with (2.8) and (2.9) gives the desired result.

Lemma 2. If $v_{ij}\geqq 0, j=1,2,\ldots,m_i; \sum_{j=1}^{m_i} v_{ij}=p_i>0, i=1,2,\ldots,n, \sum_{i=1}^{n} p_i=1;$ $w_{ij}>0, j=1,2,\ldots,m_i, \sum_{j=1}^{m_i} w_{ij}=q_i>0, i=1,2,\ldots,n, \sum_{i=1}^{n} q_i\leqq 1,$ then

$$(2.10)\qquad I_{(\gamma,\delta)}^{(\alpha,\beta)}(v_{11},\ldots,v_{1m_1},\ldots,v_{n1},\ldots,v_{nm_n};w_{11},\ldots,w_{1m_1},\ldots,w_{n1},\ldots,w_{nm_n})=$$

$$=I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1,\ldots,p_n;q_1,\ldots,q_n)+$$

$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\sum_{i=1}^{n}p_i^{\alpha}q_i^{\beta-\alpha}I_{(1,1)}^{(\alpha,\beta)}\left(\frac{v_{i1}}{p_i},\ldots,\frac{v_{im_i}}{p_i};\frac{w_{i1}}{q_i},\ldots,\frac{w_{im_i}}{q_i}\right)+$$

$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\sum_{i=1}^{n}p_i^{\gamma}q_i^{\delta-\gamma}I_{(\gamma,\delta)}^{(1,1)}\left(\frac{v_{i1}}{p_i},\ldots,\frac{v_{im_i}}{p_i};\frac{w_{i1}}{q_i},\ldots,\frac{w_{im_i}}{q_i}\right).$$

The proof of this lemma directly follows from the Lemma 1.

Lemma 3. If $F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(1/m, \ldots, 1/m; 1/r, \ldots, 1/r)$ $1 \leqq m \leqq r$, then

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) = \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} F^{(\alpha,\beta)}_{(1,1)}(m; r) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} F^{(1,1)}_{(\gamma,\delta)}(m; r), \tag{2.11}$$

where

$$F^{(\alpha,\beta)}_{(1,1)}(m; r) = A^{-1}_{\alpha,\beta}(m^{1-\alpha}r^{\alpha-\beta} - 1), \quad A_{\alpha,\beta} \neq 0 \tag{2.12}$$

and

$$F^{(1,1)}_{(\gamma,\delta)}(m; r) = A^{-1}_{\gamma,\delta}(m^{1-\gamma}r^{\gamma-\delta} - 1), \quad A_{\gamma,\delta} \neq 0 . \tag{2.13}$$

Proof. Replacing in Lemma 2 m_i by m, $v_{ij} = 1/mn$, $w_{ij} = 1/rs$, $q_i = 1/s$, $i = 1, \ldots$ $\ldots, n$, $j = 1, \ldots, m$ where m, n, r and s are positive integers such that $1 \leqq m \leqq$ $\leqq r$, $1 \leqq n \leqq s$ we obtain

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(mn; rs) = F^{(\alpha,\beta)}_{(\gamma,\delta)}(n; s) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} n^{1-\alpha}s^{\alpha-\beta} F^{(\alpha,\beta)}_{(1,1)}(m; r) + \\ + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} n^{1-\gamma}s^{\gamma-\delta} F^{(1,1)}_{(\gamma,\delta)}(m; r) . \tag{2.14}$$

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(mn; rs) = F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} m^{1-\alpha}r^{\alpha-\beta} F^{(\alpha,\beta)}_{(1,1)}(n; s) + \\ + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} m^{1-\gamma}r^{\gamma-\delta} F^{(1,1)}_{(\gamma,\delta)}(n; s) . \tag{2.15}$$

Putting $n = s = 1$ in (2.14) and using $F^{(\alpha,\beta)}_{(\gamma,\delta)}(1; 1) = 0$ (for all $\alpha, \beta, \gamma, \delta > 0$) we get

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) = \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} F^{(\alpha,\beta)}_{(1,1)}(m; r) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} F^{(1,1)}_{(\gamma,\delta)}(m; r)$$

which is (2.11)

Equating (2.14) and (2.15) we get

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(n; s) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} n^{1-\alpha}s^{\alpha-\beta} F^{(\alpha,\beta)}_{(1,1)}(m; r) + \\ + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} n^{1-\gamma}s^{\gamma-\delta} F^{(1,1)}_{(\gamma,\delta)}(m; r) = \\ = F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} m^{1-\alpha}r^{\alpha-\beta} F^{(\alpha,\beta)}_{(1,1)}(n; s) + \\ + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} m^{1-\gamma}r^{\gamma-\delta} F^{(1,1)}_{(\gamma,\delta)}(n; s) . \tag{2.16}$$

Expression (2.16) together with (2.11) gives

$$(2.17)\qquad A_{\alpha,\beta}\big[\big(1 - m^{1-\alpha}r^{\alpha-\beta}\big)\, F^{(\alpha,\beta)}_{(1,1)}(n; s) + \big(n^{1-\alpha}s^{\alpha-\beta} - 1\big)\, F^{(\alpha,\beta)}_{(1,1)}(m; r)\big] = \\ = A_{\gamma,\delta}\big[\big(1 - m^{1-\gamma}r^{\gamma-\delta}\big)\, F^{(1,1)}_{(\gamma,\delta)}(n; s) + \big(n^{1-\gamma}s^{\gamma-\delta} - 1\big)\, F^{(1,1)}_{(\gamma,\delta)}(m; r)\big]\,.$$

Putting $n = 1$, $s = 2$ in (2.17) and using

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(1; 2) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(1, 0; \tfrac{1}{2}, \tfrac{1}{2}) = 1 \quad \text{for all} \quad \alpha, \beta, \gamma, \delta > 0$$

we get

$$A_{\alpha,\beta}\big[\big(1 - m^{1-\alpha}r^{\alpha-\beta}\big) + \big(2^{\alpha-\beta} - 1\big)\, F^{(\alpha,\beta)}_{(1,1)}(m; r)\big] = \\ = A_{\gamma,\delta}\big[\big(1 - m^{1-\gamma}r^{\gamma-\delta}\big) + \big(2^{\gamma-\delta} - 1\big)\, F^{(1,1)}_{(\gamma,\delta)}(m; r)\big] = C \quad \text{(say)}\,.$$

For $m = 1$, $r = 1$ we get $C = 0$. Thus we have

$$F^{(\alpha,\beta)}_{(1,1)}(m; r) = A^{-1}_{\alpha,\beta}\big(m^{1-\alpha}r^{\alpha-\beta} - 1\big)\,, \quad A_{\alpha,\beta} \neq 0$$

and

$$F^{(1,1)}_{(\gamma,\delta)}(m; r) = A^{-1}_{\gamma,\delta}\big(m^{1-\gamma}r^{\gamma-\delta} - 1\big)\,, \quad A_{\gamma,\delta} \neq 0$$

which are (2.12) and (2.13) respectively. This completes the proof of the lemma.

Now (2.11) together with (2.12) gives

$$(2.18)\qquad F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) = \big(A_{\alpha,\beta} - A_{\gamma,\delta}\big)^{-1}\big(m^{1-\alpha}r^{\alpha-\beta} - m^{1-\gamma}r^{\gamma-\delta}\big)\,.$$

Proof of the Theorem. We prove the theorem for rational values of p's and q's and then the continuity axiom (a) extends the result for all real values. Therefore let m, a_i and b_i be positive integers such that $a_i \leqq b_i$ for every $i = 1, 2, \ldots, n$ and if we put $p_i = a_i/m$ $q_i = b_i/r$, $i = 1, 2, \ldots, n$ where $\sum_{i=1}^{n} a_i = m$, $\sum_{i=1}^{n} b_i \leqq r$, then the application of the Lemma 2 gives

$$I^{(\alpha,\beta)}_{(\gamma,\delta)}\Big(\underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{a_1}, \ldots, \underbrace{\frac{1}{m}, \ldots, \frac{1}{m}}_{a_n}; \underbrace{\frac{1}{r}, \ldots, \frac{1}{r}}_{b_1}, \ldots, \underbrace{\frac{1}{r}, \ldots, \frac{1}{r}}_{b_n}\Big) = \\ = I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) + \\ + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)}\Big(\frac{1}{a_i}, \ldots, \frac{1}{a_i}; \frac{1}{b_i}, \ldots, \frac{1}{b_i}\Big) + \\ + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \sum_{i=1}^{n} p_i^{\gamma} q_i^{\delta-\gamma} I^{(1,1)}_{(\gamma,\delta)}\Big(\frac{1}{a_i}, \ldots, \frac{1}{a_i}; \frac{1}{b_i}, \ldots, \frac{1}{b_i}\Big)$$

or

$$F^{(\alpha,\beta)}_{(\gamma,\delta)}(m; r) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) +$$

$$+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\sum_{i=1}^{n}p_i^{\alpha}q_i^{\beta-\alpha}F_{(1,1)}^{(\alpha,\beta)}(a_i;b_i)+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\sum_{i=1}^{n}p_i^{\gamma}q_i^{\delta-\gamma}F_{(\gamma,\delta)}^{(1,1)}(a_i;b_i)$$

i.e.

(2.19)
$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1,\ldots,p_n;q_1,\ldots,q_n)=$$
$$=F_{(\gamma,\delta)}^{(\alpha,\beta)}(m;r)-\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}\sum_{i=1}^{n}p_i^{\alpha}q_i^{\beta-\alpha}F_{(1,1)}^{(\alpha,\beta)}(a_i;b_i)-$$
$$-\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{b,\beta}}\sum_{i=1}^{n}p_i^{\gamma}q_i^{\delta-\gamma}F_{(\gamma,\delta)}^{(1,1)}(a_i;b_i)\,.$$

Expression (2.19) together with (2.12), (2.13) and (2.18) gives (2.4).

3. BIVARIATE DISTRIBUTIONS

Let $X=(x_1,\ldots,x_n)$ and $Y=(y_1,\ldots,y_m)$ be two discrete random variables. The information-theoretic measure of type $\begin{pmatrix}\alpha,\beta\\ \gamma,\delta\end{pmatrix}$ for two probability distributions $P(X)$ and $Q(X)$ of X is given by

(3.1)
$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P(X);Q(X))=I_{(\gamma,\delta)}^{(\alpha,\beta)}(p(x_1),\ldots,p(x_n);q(x_1),\ldots,q(x_n))=$$
$$=(A_{\alpha,\beta}-A_{\gamma,\delta})^{-1}\sum_{i=1}^{n}(p^{\alpha}(x_i)\,q^{\beta-\alpha}(x_i)-p^{\gamma}(x_i)\,q^{\delta-\gamma}(x_i))\,,$$

where $p(x_i)=P(X=x_i)$, $q(x_i)=Q(X=x_i)$, $i=1,2,\ldots,n$ and $\sum_{i=1}^{n}p(x_i)=\sum_{i=1}^{n}q(x_i)=1$.

An analogous formula holds for the random variable Y.

Next if $p(x_i,y_j)=P(X=x_i,Y=y_j)$ and $q(x_i,y_j)=Q(X=x_i,Y=y_j)$ are the joint probabilities of (x_i,y_j) then the joint information-theoretic measure of type $\begin{pmatrix}\alpha,\beta\\ \gamma,\delta\end{pmatrix}$ is given by

(3.2)
$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P(X,Y)\,;\,Q(X,Y))=$$
$$=I_{(\gamma,\delta)}^{(\alpha,\beta)}(p(x_1,y_1),\ldots,p(x_1,y_m),\ldots,p(x_n,y_1),\ldots,p(x_n,y_m);$$
$$q(x_1,y_1),\ldots,q(x_1,y_m),\ldots,q(x_n,y_1),\ldots,q(x_n,y_m)=$$
$$=(A_{\alpha,\beta}-A_{\gamma,\delta})^{-1}\sum_{i=1}^{n}\sum_{j=1}^{m}(p^{\alpha}(x_i,y_j)\,q^{\beta-\alpha}(x_i,y_j)-p^{\gamma}(x_i,y_j)\,q^{\delta-\gamma}(x_i,y_j))\,,$$

where $\sum_{i=1}^{n}\sum_{j=1}^{m}p(x_i,y_j)=\sum_{i=1}^{n}\sum_{j=1}^{m}q(x_i,y_j)=1$.

Further $P(Y/x_i) = \{p(y_1/x_i), \ldots, p(y_m/x_i)\}$ and $Q(Y/x_i) = \{q(y_1/x_i), \ldots, q(y_m/ / x_i)\}$ are complete distributions of Y given $X = x_i$, therefore the conditional information-theoretic measure of type $\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}$ of Y given X may be defined as follows:

$$(3.3) \qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y/X)\,;\; Q(Y/X)) =$$

$$= \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \sum_{i=1}^{n} p^{\alpha}(x_i)\, q^{\beta-\alpha}(x_i)\, I^{(\alpha,\beta)}_{(1,1)}(P(Y/x_i)\,;\; Q(Y/x_i)) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \sum_{i=1}^{n} p^{\gamma}(x_i)\, q^{\delta-\gamma}(x_i)\, I^{(1,1)}_{(\gamma,\delta)}(P(Y/x_i);\; Q(Y/x_i)),$$

where

$$(3.4) \qquad I^{(\alpha,\beta)}_{(1,1)}(P(Y/x_i);\; Q(Y/x_i)) = A^{-1}_{\alpha,\beta}\, [\sum_{j=1}^{m} p^{\alpha}(y_j/x_i)\, q^{\beta-\alpha}(y_j/x_i) - 1]$$

and

$$(3.5) \qquad I^{(1,1)}_{(\gamma,\delta)}(P(Y/x_i);\; Q(Y/x_i)) = A^{-1}_{\gamma,\delta}\, [\sum_{j=1}^{m} p^{\gamma}(y_j/x_i)\, q^{\delta-\gamma}(y_j/x_i) - 1],$$

$i = 1, 2, \ldots, n$.

Similarly we can obtain the information-theoretic measure of type $\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}$ of X given Y.

The interdependence and relationships among these bivariate measures are studied in the following theorems:

Theorem 3.1. If X and Y are two discrete random variables then

$$(3.6) \qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X, Y);\; Q(X, Y)) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X);\; Q(X)) + I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y/X);\; Q(Y/X)) =$$
$$= I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y);\; Q(Y)) + I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X/Y);\; Q(X/Y)),$$

where $p(x_i, y_j) = p(x_i)\, p(y_j/x_i) = p(y_j)\, p(x_i/y_j)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Proof. From the definitions given above we have

$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X);\; Q(X)) + I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y/X);\; Q(Y/X)) =$$

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} (p^{\alpha}(x_i)\, q^{\beta-\alpha}(x_i) - p^{\gamma}(x_i)\, q^{\delta-\gamma}(x_i)) +$$

$$+ (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} p^{\alpha}(x_i)\, q^{\beta-\alpha}(x_i)\, [\sum_{j=1}^{m} p^{\alpha}(y_j/x_i)\, q^{\beta-\alpha}(y_j/x_i) - 1] +$$

$$+ (A_{\gamma,\delta} - A_{\alpha,\beta})^{-1} \sum_{i=1}^{n} p^{\gamma}(x_i)\, q^{\delta-\gamma}(x_i)\, [\sum_{j=1}^{m} p^{\gamma}(y_j/x_i)\, q^{\delta-\gamma}(y_j/x_i) - 1] =$$

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} [p^{\alpha}(x_i)\, p^{\alpha}(y_j/x_i)\, q^{\beta-\alpha}(x_i)\, q^{\beta-\alpha}(y_j/x_i) -$$

$$- p^{\gamma}(x_i)\, p^{\gamma}(y_j/x_i)\, q^{\delta-\gamma}(x_i)\, q^{\delta-\gamma}(y_j/x_i)] =$$

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} [p^{\alpha}(x_i, y_j)\, q^{\beta-\alpha}(x_i, y_j) - p^{\gamma}(x_i, y_j)\, q^{\delta-\gamma}(x_i, y_j)] =$$

$$= I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X, Y); Q(X, Y)) .$$

Theorem 3.2. If X and Y are statistically independent discrete random variables then

(a)

$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X, Y); Q(X, Y)) = G^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X); Q(X))\, I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y); Q(Y)) + G^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y); Q(Y))\, I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X); Q(X)) , \tag{3.7}$$

where

$$G^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X); Q(X)) = \tfrac{1}{2} \sum_{i=1}^{n} (p^{\alpha}(x_i)\, q^{\beta-\alpha}(x_i) + p^{\gamma}(x_i)\, q^{\delta-\gamma}(x_i)) , \tag{3.8}$$

(b)

$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X, Y); Q(X, Y)) \leqq I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X); Q(X)) + I^{(\alpha,\beta)}_{(\gamma,\delta)}(P(Y); Q(Y)) , \tag{3.9}$$

for all $\alpha, \gamma \geqq 1$, $\beta - \alpha \geqq 1$ and $\delta - \gamma \geqq 1$.

Proof. The proof of the part (a) follows by simple computation.

(b) For $\alpha, \gamma \geqq 1$, $\beta - \alpha \geqq 1$ and $\delta - \gamma \geqq 1$ we have from (3.8)

$$G^{(\alpha,\beta)}_{(\gamma,\delta)}(P(X); Q(X)) \leqq 1 . \tag{3.10}$$

Expression (3.7) together with (3.10) gives (3.9).

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