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A NON-ORTHOGONAL PROJECTION MODEL FOR EVALUATING PHOTOGRAPHS RECORDING A POPULATION OF SPHERES LYING ON A FLAT PLATE

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A population of spherical particles lying on a flat plate is photographed. The moment relationship converting the moments of the image size distribution into the moments of the particle size distribution is derived. The approximate model presented is based on a non-orthogonal projection following from the use of the linear optics of a pinhole camera. The technical parameters of the optical system used are taken into account.

1. INTRODUCTION

In various technical branches (macromolecular chemistry, powder metallurgy, spray drying etc.) we usually need to evaluate photographs recording spherical particles lying on a flat plate. In practice for processing such a photograph an automatic image analyzer is used and the measured diameters of particular particles are taken as their actual diameters. Such an approach presupposes that the photograph records spheres orthogonally projected. Let us recall the main feature of this projection: two figures in distinct planes are derived from each other if corresponding points can be joined by parallel lines [1]. However, for the imperfections of optical systems usually used in working laboratories in preparing the photograph of the scene described above this quoted condition is not fulfilled. In general, a photograph of objects in space represents a realization of a non-orthogonal projection. This type of projection given by the linear optics of a pinhole camera has been proposed in [2] in connection with the reconstruction of a three-dimensional image from a two-dimensional photography and has been used in [3] for derivation of correction coefficients for spheres being part of a spatial process and being photographed.

The basic idea of this introduced model can be applied in a modified form even to a more precise processing of photographs recording a population of opaque spherical particles lying on a flat plate and for estimating the moments of corresponding size distribution. The present paper offers a solution of this problem.

The model assumptions are formulated in Section 2. The deterministic geometrical relationships holding for a particular spherical particle, of a given size and distance from the optical axis, recorded by using linear optics of a pinhole camera are explored in Section 3. The model – see Section 4 – taking into account the randomness of the size and the position of the spheres on the plate enables to solve the moment relationship converting the moments of the image size distribution into the moments of the particle size distribution. The results derived are discussed in Section 5 and the procedure for the application of the model is suggested in Section 6.

2. THE MODEL ASSUMPTIONS

The model has been constructed under the following assumptions:

a) The opaque spherical particles are distributed on a flat plate homogeneously i.e. their centres orthogonally projected form a two-dimensional Poisson field (for a fixation of the position of particles the plate surface is usually covered by a very thin film of an adhesive material).

b) The density of particles per unit area of the flat plate is only so high as to prevent to develop the masking of small particles by larger ones in the non-orthogonal projection.

c) The particles have diameters which are independently and identically distributed with the unknown probability density function (pdf) $g(y)$, $0 < y < \infty$.

d) The particle size and the particle centre position on the flat plate are independent.

e) Only spheres are brought into calculation the centres of which are situated inside an incomplete right circular cone C_a. Its axis is identical with the optical axis and its circular base, in the plate on which the spheres are lying, coincides with the field of view. Therefore on the automatic image analyzer only a circular mask is displayed.

f) The optics used are those of a pinhole camera.

A simplified scheme of the linear optics used is presented in Figure 1; it shows the normal to planes $\varrho(\cdot)$, mutually parallel and parallel to the $\dot{x}\dot{y}$-plane: $\varrho(0)$ – the plane of the pinhole, $\varrho(b)$ – the image plane represented by a photographical plate or a film strip, $\varrho(-b)$ – the plane of focussing, $\varrho(-a)$ and $\varrho(-(a + c))$ two planes bounding the projection field having the depth c. In comparison with the model introduced by Horálek and Coleman [3] we extend the projection field in the direction of the camera pinhole, retaining the plane of focussing $\varrho(-b)$. However, this simplified scheme cannot grasp the actual analyzed situation in a satisfactory way: in the space photographically recorded the geometrical locus of points having the same distance of the camera pinhole is represented namely by a lateral surface of a spherical sector.

As a result of this fact:

i) $\varrho(-b)$ even $\varrho(-a)$ and $\varrho(-(a + c))$ are not planes parallel to $\varrho(0)$ or $\varrho(b)$ but spherical surfaces (see Fig. 2) and therefore the corresponding three-dimensional projection field (better said, projection space) takes the shape of the incomplete right circular cone C_a – see point e); from the spherical surface $\varrho(-(a + c))$ we record only a circle of a radius identical with the radius of the circular base of the cone C_a;

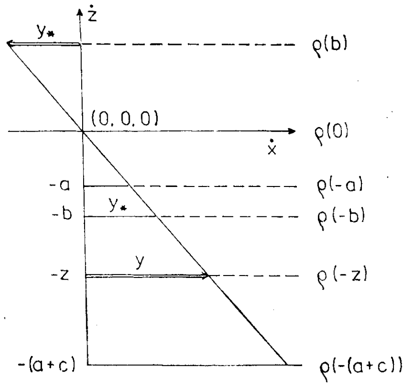

Fig. 1. The pinhole optics with the camera pinhole in the origin (0, 0, 0), the plane focussing in $\varrho(-b)$, the image plane in $\varrho(b)$ and the depth c of projection field bounded by the planes $\varrho(-a)$ and $\varrho(-(a + c))$. The object of size y in the plane $\varrho(-z)$ in front of the pinhole and its image of size y in the image plane $\varrho(b)$.

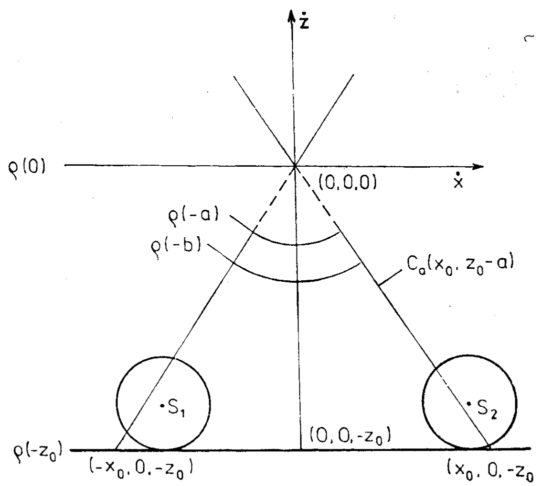

Fig. 2. The product of the $\dot{x}\dot{z}$-plane and of the incomplete cone $C_a(x_0, z_0 - a)$ and two spheres lying on the plane $\varrho(-z_0)$ and having the $\dot{x}$-coordinates of corresponding centres $S_1 \in C_a(x_0, z_0 - a)$ and $S_2 \notin C_a(x_0, z_0 - a)$ on the straight line $(y = 0,\ z = -z_0)$.

ii) the distances r_s of the camera pinhole to the centre of a particle located near periphery of a circular field of view recorded and of a particle located in the central part of this field mutually markedly differ. That influences (see section 5) the choice of b and was the reason for the extension of the projection field from $\varrho(-b)$ to $\varrho(-a)$.

The described geometrical approach of the linear optics used and the presence of unavoidable fluctuation of the size of particles all over the field of view are fully projected in the construction of the model (see Sections 3 and 4).

3. THE GEOMETRICAL RELATIONSHIPS

Consider the three-dimensional euclidean space E_3 with coordinate axes $\dot{x}$, $\dot{y}$, $\dot{z}$. In the origin of this system we shall locate the pinhole of the camera.

The projection space C_a mentioned in the model assumptions is defined by

$$C_a = C_a(x_0, z_0 - a) = C(x_0, z_0) \cap \overline{D(a)}, \tag{1}$$

where $\overline{D(a)}$ is the complement of the sphere $D(a)$ with radius a and centred in the origin; the cone C has its vertex in the origin of the coordinate system, its symmetry axis identical with the optical axis coinciding with the $\dot{z}$-axis and its circular base of radius x_0 in the plane $\varrho(-z_0)$ defined by $\dot{z} = -z_0$ (see Fig. 2). Therefore, the incomplete right circular cone C_a (see Fig. 2) is represented by all points of a complete right circular cone $C(x_0, z_0)$ excluding those belonging to the sphere $D(a)$.

The position of the centre $S(\varphi, r, z)$ of a particular sphere lying on the plane $\dot{x}\dot{y}$ can be identified by cylindrical coordinates where φ and r are polar coordinates of the orthogonal projection of the centre S into the plane $\dot{x}\dot{y}$ and z is the orientated distance of the point S from the plane $\dot{x}\dot{y}$. However, from the point of view of the problem solved and model assumptions introduced in the foregoing section, it appears as sufficient to define the position of the sphere centre S only by its distance r from the $\dot{z}$-axis and its orientated distance z, introduced above. For example, the centre $S(r, -z_0 + (y/2))$ belongs to the sphere lying on the plane $\varrho(-z_0)$, having the diameter y and the distance r of the centre S to the $\dot{z}$-axis.

Let $S(r, -z_0 + (y/2)) \in C_a(x_0, z_0 - a)$ be the centre of a particular sphere recorded on the photograph (see Fig. 3). Consider now another complete right circular cone $C_S(y_t/2; r_t)$ with its symmetry axis in the straight line passing through the point $S(r, -z_0 + (y/2))$ and through its vertex in the origin. The lateral area of the cone C_S is formed by a rotation of a tangent, passing through the origin, to the sphere with the centre S. The points of contact of the cone C_S and the sphere with the centre S create a circle K of the diameter y_t. The distance r_p of every point of the circle K to the origin is equal to

$$r_p = \sqrt{[r_s^2 - (y/2)^2]}, \tag{2}$$

where r_s is the distance of the centre S to the origin. For the diameter y_t it holds

$$y_t = \frac{y}{r_s} \sqrt{[r_s^2 - (y/2)^2]}. \tag{3}$$

Due to the property (2) of the circle K, the disc of diameter y_t is recorded in the

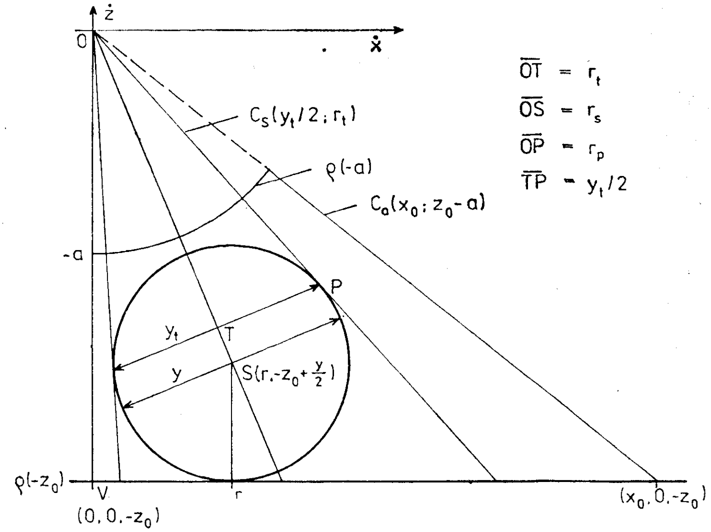

Fig. 3. The product of the $\dot{x}\dot{z}$-plane and of the incomplete cone $C_a(x_0, z_0 - a)$ and a sphere with the centre $S \in C_a(x_0, z_0 - a)$ having its coordinates $(r, 0, -(z_0 - (y/2)))$.

image plane $\varrho(b)$ as a disc of diameter y equal to

$$y = \frac{b}{r_p} y_t . \tag{4}$$

In comparison with the actual observable diameter y_t the image diameter y_* is enlarged for $r_p < b$ and reduced for $r_p > b$. For $r_p = b$ we reach the maximum possible focussing and we have $y = y_t$. The relationship between the observable size of the object in $\varrho(-b)$ and that of its image in $\varrho(b)$ remains the same as in the simplified scheme and is achieved by a suitable lens.

Inserting (2) and (3) into (4) we get

$$y_* = \frac{b}{r_s} y = \frac{b}{\sqrt{[r^2 + \{z_0 - (y/2)\}^2]}} y . \tag{5}$$

Therefore, the image diameter y_* depends on the actual diameter y of the particular sphere recorded, on the distance b of the surface of focussing $\varrho(-b)$ from the camera pinhole and on the distance z_0 of the flat plane from $\varrho(0)$ and finally on the distance r of the sphere centre S from the $\dot{z}$-axis.

4. THE MODEL

Let the sphere diameter y and the distance r of the centre sphere S from the $\dot{z}$-axis be random variable Y and R, respectively. Therefore, the image diameter Y_* of the corresponding disc on the photograph is a random variable, too. By (5) these three random variables are related in this way

$$Y_* = bY/\sqrt{[R^2 + \{z_0 - (Y/2)\}^2]} . \tag{6}$$

By the assumption, Y has an unknown pdf $g(y)$, $0 < y < \infty$.

Now, let N_A be the expected number of orthogonally projected sphere centres per unit area in the plane $\varrho(-z_0)$. Then the expected number of sphere centres orthogonally projected into an annulus bounded by circles of radius r and $r + \mathrm{d}r$, respectively, and having the common centre in the point $(0, 0, -z_0)$ is equal to $2\pi r N_A \, \mathrm{d}r$. Then the conditional pdf of R relative to the hypothesis $Y = y$ is

$$f(r \mid y) = 2rt^{-2}\{z_0 - (y/2)\}^{-2} , \tag{7}$$

where the corresponding definition region $Q = \langle 0; t\{z_0 - (y/2)\}\rangle$ of $(R \mid Y = y)$ follows from the model assumption e) and

$$t = x_0/z_0 . \tag{8}$$

In the next step we shall pay attention to the moment relationship between $\mathsf{E}(Y_*^i)$ and $\mathsf{E}(Y^i)$, $i = 1, 2, \ldots$. The drawback of its derivation inheres in the analytical form of the fraction on the right hand of Eq. (6), a nonseparable function of two random variables Y and R, enlightened above. We have namely

$$\begin{aligned} \mathsf{E}(Y_*^i) &= \mathsf{E}\{Y^i U^{-i}(R, Y \mid Y = y)\} \\ &= \int_0^\infty [\mathsf{E}\{U^{-i}(R, Y \mid Y = y)\}] \, y^i g(y) \, \mathrm{d}y , \end{aligned} \tag{9}$$

where

$$U(R, Y) = b^{-1} \sqrt{[R^2 + \{z_0 - (Y/2)\}^2]}$$

and further with respect to (7)

$$\mathsf{E}\{U^{-i}(R, Y \mid Y = y)\} = \int_Q b^i [r^2 + \{z_0 - (y/2)\}^2]^{-i/2} f(r \mid y) \, \mathrm{d}r . \tag{10}$$

The substitution

$$v = r^2 + \{z_0 - (y/2)\}^2$$

applied in the foregoing integral leads to

$$\begin{aligned} \mathsf{E}\{U^{-i}(R, Y \mid Y = y)\} &= \frac{2b^i}{(2 - i)\, t^2} \{(1 + t^2)^{(2-i)/2} - 1\} \{z_0 - (y/2)\}^{-i} \quad \text{for} \quad i \neq 2 , \\ &= \frac{b^2 \ln (1 + t^2)}{t^2} \{z_0 - (y/2)\}^{-2} \quad \text{for} \quad i = 2 , \end{aligned} \tag{11}$$

where t is defined in (8) and ln denotes the natural logarithm. Inserting (11) into the integrand of (9) we see the weak point of solution of the moment relationship pointed out above.

One way how to overcome this obstacle might be to consider the denominator of (6) in the form $\sqrt{(R^2 + z_0^2)}$, taking into account the inequality $z_0 \gg (y/2)$, valid in practice. But we need not employ such a rough approach. As a tractable approximation appears even

$$Y_* \approx bY[R^2 + \{z_0 - (\mu/2)\}^2]^{-1/2} \quad \text{where} \quad \mu = \mathsf{E}(Y), \tag{12}$$

giving the relevant error much smaller than in neglecting Y in the square-root in (6).

For solving the moment relationship under the validity of (12) we can use the general relationships derived above. Putting $y = \mu$ in (11) and inserting it into (8) we obtain

$$\begin{aligned} \mathsf{E}(Y_*^i) &\approx \frac{2b^i}{(2-i)\,t^2} \{(1+t^2)^{(2-i)/2} - 1\}\{z_0 - (\mu/2)\}^{-i}\,\mathsf{E}(Y^i) \quad \text{for} \quad i \neq 2, \\ &\approx \frac{b^2 \ln(1+t^2)}{t^2} \{z_0 - (\mu/2)\}^{-2}\,\mathsf{E}(Y^2) \quad \text{for} \quad i = 2, \end{aligned} \tag{13}$$

when $\mathsf{E}(Y^i)$ exists. Hence, for $i = 1$ we attain

$$\mu \approx \frac{2t^2 z_0\,\mathsf{E}(Y_*)}{4b\{\sqrt{(1+t^2)} - 1\} + t^2\,\mathsf{E}(Y_*)}. \tag{14}$$

Applying this result in (13) we get the ith moment $\mathsf{E}(Y^i)$ of the particle size distribution

$$\begin{aligned} \mathsf{E}(Y^i) &\approx \frac{(2-i)\,2^{2i-1} t^2 z_0^i \{\sqrt{(1+t^2)} - 1\}^i}{\{(1+t^2)^{(2-i)/2} - 1\}\,[4b\{\sqrt{(1+t^2)} - 1\} + t^2\,\mathsf{E}(Y_*)]^i}\,\mathsf{E}(Y_*^i) \text{ for } i \neq 2, \\ &\approx \frac{2^4 t^2 z_0^2 \{\sqrt{(1+t^2)} - 1\}^2}{\{\ln(1+t^2)\}\,[4b\{\sqrt{(1+t^2)} - 1\} + t^2\,\mathsf{E}(Y_*)]^2}\,\mathsf{E}(Y_*^2) \quad \text{for} \quad i = 2. \end{aligned} \tag{15}$$

The constants b, x_0 and z_0 are taken as known, t can be calculated from (8) and the moments $\mathsf{E}(Y_*^i)$ estimated from the results gained by using the automatic image analyzer (see Sections 5 and 6).

5. THE DISCUSSION OF RESULTS

The following conclusions can be drawn from the derived relationships:

1) From (2) it follows that in contrast with the orthogonal projection in the non-orthogonal projection the observable diameter y_t of any sphere lying on the plate is always smaller than its actual diameter y. That holds true even for the maximum possible focussing of the particular sphere, when the surface of focussing $\varrho(-b) = \varrho(-r_p)$ – see Figure 3. The size of this difference decreases with increasing b.

2) The spheres to be photographically recorded lie on a plane, but contrary to it $\varrho(-b)$ is a spherical surface. Further by model assumptions the sphere diameter is a random variable independent of the sphere position on the plate. Owing to these facts it is impossible to focus all spheres on the plate simultaneously. Therefore one part of spheres is recorded on the photograph with reduced diameters ($r_p > b$), the other part with enlarged diameters ($r_p < b$) and only a few spheres are recorded in actual size ($r_p = b$). From this aspect there is only one way how to gain the in-

formation on the actual sphere size distribution from a collective record processed simultaneously: to use a stereological model respecting the geometrical relationships following from the optics used and the statistical behaviour of particles from the point of view of their size and position on the plate.

3) The model presented suppresses the bias in estimating the moments of the sphere size distribution arising in evaluating the photograph of spheres (lying on a plate) by using an orthogonal projection model, hitherto generally used. In the orthogonal projection the equality

$$\mathsf{E}(Y^i) = \mathsf{E}(Y_*^i) \tag{16}$$

holds for all $i = 1, 2, \ldots$, for which $\mathsf{E}(Y^i)$ exists. In the nonorthogonal projection the relationship between $\mathsf{E}(Y^i)$ and $\mathsf{E}(Y_*^i)$ depends on a series of technical parameters characterizing the optics used. The model presented is based on the principle of the pinhole camera optics and brings the technical parameters (a, b, c, t, x_0, z_0, etc.) into calculation. In contrast with the quoted non-orthogonal projection model by Horálek and Coleman [3] in the model presented we are not able to express the moment relationship converting $\mathsf{E}(Y_*^i)$ into $\mathsf{E}(Y^i)$ by means of a correction factor independent on Y. The complexity of the optics'geometry permits only the construction of an approximate model with a correction factor depending on $\mathsf{E}(Y_*)$.

4) The role of the distance b of the surface of focussing to the camera pinhole can be viewed in various contexts. Three of them will be investigated:

a) It is required to fulfill the relationship

$$\mathsf{E}(Y) \approx \mathsf{E}(Y_*) \tag{17}$$

even in the non-orthogonal case. Then from (13), putting there $i = 1$, it follows for b

$$b = \{z_0 - (\mu/2)\}\, b_1(t) = \{z_0 - (\mu/2)\} \frac{t^2}{2\{\sqrt{(1 + t^2)} - 1\}}\,. \tag{18}$$

b) From considerations in point 2) of this section it can be deduced another logical requirement: to maintain an approximate equilibrium (in statistical sense) in the numbers of photographically recorded spheres with diameters reduced and enlarged, respectively, in comparison with their actual observable sizes. In this case the product of the plane $\varrho\{-z_0 + (\mu/2)\}$ and the surface of focussing $\varrho(-b)$ is formed by a circle with the radius $t\{z_0 - (\mu/2)\}/\sqrt{2}$, t defined in (8), and the centre in the optical axis. Such a circle halves the area $\pi t^2\{z_0 - (\mu/2)\}^2$. Hence the corresponding distance b is

$$b = \{z_0 - (\mu/2)\}\, b_2(t) = \{z_0 - (\mu/2)\} \sqrt{[1 + (t^2/2)]}\,. \tag{19}$$

c) Finally, the requirement can be concentrated to gaining a maximum possible number of spheres recorded in their observable size or in size very near to it. It necessitates to evaluate only a periphery region represented by an annulus bounded by the circles of radii $t\{z_0 - (\mu/2)\}$ and $x_0 - \mu\{2 + (t/2)\}$, respectively, and the common centre in the optical axis. Then b corresponding to the mean value of this annulus of the width 2μ has the form

$$b = \sqrt{[\{z_0 - (\mu/2)\}^2 + \{tz_0 - 0{\cdot}5\mu(t + 2)\}^2]} . \tag{20}$$

Comparing $b_1(t)$ and $b_2(t)$, introduced in (18) and (19), respectively, we ascertain that the function $b_1(t)$ very closely fits the function $b_2(t)$ for $t \in (0, 1)$. On the other hand, the method c) appears as a very suitable one for gaining a preliminary estimate $\hat{\mu}$ of μ. This method gives not only the maximum possible number of spheres of a quoted quality of their records but it leads to the largest possible value of b and therefore it reduces the influence of μ in the highest possible degree.

5) By the model assumption e) on the photograph submitted for automatic processing only diameters of those image discs are measured the centres of which are located inside the circular mask with the radius $t\{z_0 - (\mu/2)\}$. In applying the method c) the image disc centres must be situated inside the annulus specified in point 4 of this section.

6. APPLICATION

In spray drying research so called impact methods have been widely used [4]. The principle of these methods consists in falling spray droplets on a glass slide covered by a thin layer of an adhesive material fit for fixing the position of these droplets. The slide with the droplets caught has been illuminated and photographed. The method presented was used for the analysis of the size distribution of spray particles caught in the adhesive film on the slide.

The model derived can be applied in the following steps:

1) Before the preparation of the photograph to verify the fulfilling of the model assumptions a), b) and d).

2) To apply the method c) – see point 4 in the Section 5:

a) to prepare the first photograph of the scene:

- to choose the maximum possible radius x_0,
- on the periphery part of the field of view to demarcate an annulus A of a width approximately 1·5 times of the maximum sphere diameter observable on the field of view,
- to locate the surface of focussing in the middle of the width of the annulus A,

b) by using the automatic image analyzer to measure the diameters only of spheres with centres inside the annulus A (see point 5 of the preceding section),

c) by using the data gained from the image analyzer to calculate a rough estimate $\hat{\mu}$ of $\mathsf{E}(Y)$.

3) To prepare the second photograph of the same scene but for the surface of focussing $\varrho(-b)$ where b has been calculated by (19), putting there $\mu = \hat{\mu}$. Further

a) to make note of the corresponding values of b, x_0, z_0 and to mark the position V of the optical axis (the z-axis) on the photograph;

b) on the second photograph to draw a circular mask with centre in V and the radius $t\{z_0 - (\mu/2)\}$;

c) by using the automatic image analyzer to process this circular part of the photograph, keeping the rule specified in point 5 of the preceding section and by means of the data gained from the image analyzer to calculate the estimations of the required moments $\mathsf{E}(Y_*^i)$, $i = 1, 2, \ldots$, of the disc diameter distribution.

4) To insert the corresponding numerical values of b, t, z_0 and $\mathsf{E}(Y_*^i)$ into (15) and to calculate the searched estimates of the moments $\mathsf{E}(Y^i)$, $i = 1, 2, \ldots$, of the sphere diameter distribution.

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