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NOTE ON GRAPHS COLOURING

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Summary. In this paper, we show that the maximal number of minimal colourings of a graph with n vertices and the chromatic number k is equal to k^{n-k}, and the single graph for which this bound is attained consists of a k-clique and $n-k$ isolated vertices.

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The graphs considered here are finite, undirected and simple (without loops or multiple edges), [1] being followed for terminology and notation. Let $G = (V, E)$ be a graph with V the set of vertices and E the set of edges. A colouring of G is a partition of V such that every class of the partition induces a subgraph consisting only of isolated vertices. We denote by $C(G)$ the number of minimal colourings of G. Obviously, if $\gamma(G)$ is the chromatic number of G, then $\gamma(G)$ represents the minimal number of classes of a colouring of G.

In the sequel, we suppose that G has n vertices and $\gamma(G) = k$, and we prove the following

Theorem. *$C(G) \leqslant k^{n-k}$, and the single graph G for which this upper bound is attained consists of a k-clique and $n-k$ isolated vertices.*

Proof. We shall prove this result by induction on n. Obviously, for $n = 1, 2$ the result is true. So, suppose that the result is true for all graphs with at most $n-1$ vertices, and let G be a graph with n vertices and the chromatic number $\gamma(G) = k$, $1 \leqslant k \leqslant n$. Let v be an arbitrary vertex of G and $G - v$ the subgraph obtained from G by deleting v. If $\gamma(G - v) = k$, then, by induction hypothesis, we have

$$C(G) \leqslant kC(G - v) \leqslant k \cdot k^{n-k-1} = k^{n-k},$$

the equality holding only if v is an isolated vertex and $G - v$ has the maximal number of minimal colourings, since v may be added to a minimal partition of $G - v$ in at most k different ways.

If $\gamma(G - v) = k - 1$, then a minimal colouring of G is $\{v\}$, $I_1, I_2, \ldots, I_{k-1}$, where $I_1, I_2, \ldots, I_{k-1}$ are independent sets and there exist vertices $v_1 \in I_1$, $v_2 \in I_2, \ldots$, $v_{k-1} \in I_{k-1}$ which are joined by an edge with v since, otherwise, $\gamma(G) = k - 1$.

Obviously, every minimal colouring of G has a class which contains v and a subset of $V - \{v_1, v_2, \ldots, v_{k-1}\}$. However, $|V - \{v, v_1, v_2, \ldots, v_{k-1}\}| = n - k$ and, therefore, the number of minimal partitions of V which contain in the same class the vertex v together with r vertices from $V - \{v, v_1, v_2, \ldots, v_{k-1}\}$, $0 \leqslant r \leqslant n - k$, is less than or equal to $\binom{n-k}{r}(k-1)^{n-k-r}$. Indeed, r vertices may be chosen from a set of $n - k$ vertices in $\binom{n-k}{r}$ different ways, and the maximal number of minimal partitions of a graph H with $n - k - r$ vertices and $\gamma(H) = k - 1$ is equal to $(k-1)^{n-k-r}$, by induction hypothesis. Hence

$$C(G) \leqslant \sum_{r=0}^{n-k} \binom{n-k}{r}(k-1)^{n-k-r} = k^{n-k},$$

the equality holding only if $\{v, v_1, v_2, \ldots, v_{k-1}\}$ induces a k-clique and the remaining vertices are isolated. Thus, the theorem is proved. □

Corollary. *The maximal number of minimal colourings of a graph with n vertices is equal to*

$$\max_{r=\lfloor x\rfloor, \lceil x\rceil} (r^{n-r}),$$

where x is the real number which verifies the equation $x(1 + \ln x) = n$.

Proof. By the above theorem, the maximal number of minimal colourings of a graph with n vertices is equal to

$$\max_{1\leqslant k\leqslant n} (k^{n-k}),$$

and the equation $x(1 + \ln x) = n$ is obtained by equalizing to zero the derivative of the function x^{n-x}. □

References

[1] *C. L. Liu*: Introduction to Combinatorial Mathematics, Mc Graw-Hill Book Co., New York, 1968.

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