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Mathematica Bohemica, Vol. 118 (1993), No. 3, 261–276

Persistent URL: http://dml.cz/dmlcz/125932

ON THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS FOR ORDINARY DIFFERENTIAL EQUATIONS

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(Received April 22, 1992)

Summary. Lepagean 2-form as a globally defined, closed counterpart of higher-order variational equations on fibered manifolds over one-dimensional bases is introduced, and elementary proofs of the basic theorems concerning the inverse problem of the calculus of variations, based on the notion of Lepagean 2-form and its properties, are given.

Keywords: Lepagean form, variational equations, local inverse problem to the calculus of variations, global inverse problem to the calculus of variations, Helmholtz conditions, minimal-order lagrangian.

AMS classification: 58E30, 70H35

1. INTRODUCTION

The inverse problem of the calculus of variations has been stated first by Helmholtz [4] as a problem of the existence of a lagrangian to a given system of second order ordinary differential equations. In his paper [4] Helmholtz has found necessary conditions for variationality of such equations, now called the *Helmholtz conditions*; it has been proved later by Mayer [14] that these conditions are also sufficient. Since that time the problem has been substantially generalized, enriched, and intensively studied by many authors using different methods.

Within the range of higher-order calculus of variations on fibered manifolds there are three basic questions concerned with the inverse problem:

(1) The *local inverse problem of the calculus of variations* means finding necessary and sufficient conditions for a system of higher-order partial differential equations to be identical with a system of the Euler-Lagrange equations of a lagrangian. For the case of higher-order mechanics it has been solved first by Vanderbauwhede [19] using Veinberg's potential operator method [18], and by Macjuk [13] who developed

Tulczyjew's approach based on studies of the "Lagrange derivative" [17]. For the general case (higher-order field theory) necessary and sufficient variationality conditions in an explicit form have been first proved independently by Anderson and Duchamp [2], using the variational bicomplex (Dedecker and Tulczyjew [3], Takens [15], Vinogradov [20]), and by Krupka in [7] (see also [8]), using properties of Lepagean equivalents of lagrangians.

(2) A closely related question to the local inverse problem concerns explicit construction of a lagrangian to a given system of variational equations. It is well-known how to construct to a system of order r a (local) lagrangian of the same order, e.g. the so-called *Veinberg-Tonti lagrangian* [16]. However, there arises a question of the possibility of *lowering the order* of this lagrangian. This problem has been solved completely for the case of higher-order mechanics by Vanderbauwhede [19] who has shown, within the range of the potential operators theory, that every system of higher-order ordinary differential equations possesses a lagrangian of the lowest possible order, and has found an explicit formula for the construction of such a lagrangian. In the case of higher-order field theory the problem of possibility of lowering the order of a lagrangian has been touched in [11].

(3) A system of Euler-Lagrange equations on a fibered manifold is globally represented by the so-called Euler-Lagrange form (Krupka [6]), which in general arises from *locally* defined lagrangians. The question of the existence of a globally defined lagrangian—the *global inverse problem of the calculus of variations*—has been studied by many authors (Takens [15], Vinogradov [20, 21], Anderson [1], Krupka [11], and references therein). The paper [11] also solves the problem of the *order* of global lagrangians (cf. Anderson and Duchamp [2]).

In the present paper we develop the theory of Lepagean 2-forms, initiated in [12], and apply it to the inverse problem in higher-order mechanics. We show that the above mentioned results, which have been obtained by different methods and rather complicated tools, can be explained and proved within the range of the calculus of variations on fibered manifolds by straightforward and elementary techniques, based in fact only on the definition of the Lepagean 2-form and on the Poincaré Lemma for contact forms.

We work in the category of smooth, finite-dimensional manifolds. We suppose the reader to be familiar with basic structures used in higher-order calculus of variations—the theory of higher-order jet prolongations of fibered manifolds, and the related calculus—the theory of horizontal and contact forms and projectable vector fields. These prerequisites can be found e.g. in [9]. We use the following (more or less standard) notation: d the exterior derivative, i the inner product, $*$ the pull-back, $\pi\colon Y \to X$ a fibered manifold ($\dim X = 1$), J^s the s-jet prolongation functor, $\pi_s\colon J^sY \to X$ or simply J^sY the s-jet prolongation of π, $\pi_{s,k}\colon J^sY \to J^kY$,

$0 \leqslant k < s$, the canonical jet projections, (V, ψ), $\psi = (t, q^\sigma)$ a fiber chart on Y, (V_s, ψ_s), $\psi_s = (t, q^\sigma, \ldots, q^\sigma_s)$ the associated fiber char on J^sY, $V_s = \pi^{-1}_{s,0}(V)$, $\Omega^p(J^sY)$ the module of (exterior) p-forms on J^sY, $\Omega^p_X(J^sY)$ the module of π_s-horizontal p-forms on J^sY, $\Omega^p_{J^kY}(J^sY)$, $0 \leqslant k < s$ the module of $\pi_{s,k}$-horizontal p-forms on J^sY, and $\Omega^{p-k,k}(J^sY)$, $1 \leqslant k < p$ the module of k-contact p-forms on J^sY. In particular, $\Omega^{1,1}_Y(J^sY)$ denotes the module of 1-contact 2-forms on J^sY, horizontal with respect to the projection onto Y. Note that in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y a form $E \in \Omega^{1,1}_Y(J^sY)$ is represented by the chart expression $E = E_\sigma \mathrm{d}q^\sigma \wedge \mathrm{d}t$, where E_σ are functions on V_s. By h we denote the horizontalization with respect to π, by p the contactization with respect to π, by p_1 the 1-contactization, and by p_2 the 2-contactization. Recall that for the 1-forms $\mathrm{d}t$, $\mathrm{d}q^\sigma_i$, $1 \leqslant i \leqslant s$ on V_s we have

$$h\mathrm{d}t = \mathrm{d}t, \quad h\mathrm{d}q^\sigma_j = q^\sigma_{j+1}\mathrm{d}t, \quad p\mathrm{d}t = 0, \quad p\mathrm{d}q^\sigma_j = \omega^\sigma_j, \quad 0 \leqslant j \leqslant s, \tag{1.1}$$

where the standard notation

$$\omega^\sigma_j = \mathrm{d}q^\sigma_j - q^\sigma_{j+1}\mathrm{d}t, \quad 0 \leqslant j \leqslant s \tag{1.2}$$

is used. Finally, $E|_W$ denotes the restriction of a form E to a set W (i.e. the pull-back by the canonical inclusion).

2. Lepagean 2-forms and the inverse problem

First we recall the notion and basic properties of a Lepagean form, adapted to the case of a fibered manifold with one-dimensional base. For details and proofs we refer to Krupka [6, 10].

Proposition 1. *Let $\varrho \in \Omega^1(J^sY)$. The following five conditions are equivalent:*

(1) *It holds*

$$\pi^*_{s+1,s}\mathrm{d}\varrho = E + F, \tag{2.1}$$

where $E \in \Omega^{1,1}_Y(J^{s+1}Y)$ and $F \in \Omega^{0,2}(J^{s+1}Y)$.

(2) *$p_1(\pi^*_{s+1,s}\mathrm{d}\varrho)$ is a $\pi_{s+1,0}$-horizontal form.*

(3) *For each $\pi_{s,0}$-projectable vector field ξ on J^sY the form $h(i_\xi \mathrm{d}\varrho)$ depends on the $\pi_{s,0}$-projection of ξ only.*

(4) *For each $\pi_{s,0}$-vertical vector field ξ on J^sY, $h(i_\xi \mathrm{d}\varrho) = 0$.*

(5) *In each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y*

$$\pi^*_{s+1,s}\varrho = L\mathrm{d}t + \sum_{i=0}^{s} f^{i+1}_\sigma \omega^\sigma_i, \tag{2.2}$$

where

$$(2.3)\qquad f_\sigma^{i+1} = \sum_{k=0}^{s-i-1} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q^\sigma_{i+1+k}}, \quad 0 \leqslant i \leqslant s.$$

Any 1-form ϱ satisfying one of the equivalent conditions of Proposition 1 is called a *Lepagean 1-form.* If ϱ is a Lepagean 1-form then the form E defined by (2.1), or equivalently by

$$(2.4)\qquad i_{J^{s+1}\xi} E = h(i_{J^s\xi}\mathrm{d}\varrho)$$

for each π-vertical vector field ξ on Y, is called the *Euler-Lagrange form.* In each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y where ϱ is expressed by (2.2), (2.3) we get

$$(2.5)\qquad E = E_\sigma(L)\omega^\sigma \wedge \mathrm{d}t,$$

$$(2.6)\qquad E_\sigma(L) = \sum_{k=0}^{s} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q^\sigma_k}$$

and the functions (2.6) are called the *Euler-Lagrange expressions* associated with L. Note that both ϱ and E depend on $h(\varrho)$ only.

Let $r \geqslant 1$. Any π_r-horizontal 1-form λ on J^rY is called a *lagrangian of order* r *for* π. A Lepagean 1-form ϱ is called a *Lepagean equivalent of* λ if $h(\varrho) = \lambda$. To every lagrangian there exists a unique Lepagean equivalent, it is denoted by θ_λ. If $\lambda \in \Omega^1_X(J^rY)$ then, in general, $\theta_\lambda \in \Omega^1(J^{2r-1}Y)$. The Euler-Lagrange form of λ is denoted by E_λ. The mapping $\mathscr{E}\colon \lambda \to E_\lambda$ assigning to every lagrangian its Euler-Lagrange form is called the *Euler-Lagrange mapping.*

Proposition 2. *Let $\lambda \in \Omega^1_X(J^rY)$ be a lagrangian, E_λ the Euler-Lagrange form of λ, θ_λ the Lepagean equivalent of λ. Let γ be a section of π. The following four conditions are equivalent:*

(1) *For each π-projectable vector field ξ on Y*

$$(2.7)\qquad J^{2r-1}\gamma^* i_{j^{2r-1}\xi}\mathrm{d}\theta_\lambda = 0.$$

(2) *For each π-projectable vector field ξ on Y*

$$(2.8)\qquad J^{2r}\gamma^* i_{j^{2r}\xi} E_\lambda = 0.$$

(3) *The Euler-Lagrange form E_λ vanishes along $J^{2r}\gamma$, i.e.*

$$(2.9)\qquad E_\lambda \circ J^{2r}\gamma = 0.$$

(4) *For any fiber chart (V,ψ) the restriction of γ to the set $\pi(V)$ satisfies the system of ordinary differential equations of order $2r$*

$$E_\sigma(L)\circ J^{2r}\gamma = 0, \quad 1\leqslant\sigma\leqslant m \tag{2.10}$$

where $E_\sigma(L)$ are defined (2.5), (2.6).

Any of the equivalent equations (2.7)–(2.10) are called the *Euler-Lagrange equations* of the lagrangian λ, and any (local) section γ of π satisfying these conditions is called an *extremal* of λ.

Let $s\geqslant 1$, let $E\in\Omega^{1,1}_Y(J^sY)$ be a form. E is called *variational*, or *globally variational*, if there exists an integer $r\geqslant 1$ and a lagrangian $\lambda\in\Omega^1_X(J^rY)$ such that $E=E_\lambda$ (up to the projection $\pi_{2r,s}$ or $\pi_{s,2r}$.) E is called *locally variational* if there exists an open covering $\{W_\iota\}$ of J^sY such that for every ι, $E\big|_{W_\iota}$ is variational.

Consider the fibered manifold $pr_1\colon R\times R^m\to R$ and a system of ordinary differential equations of order $s\geqslant 1$

$$E_\sigma\left(t,\gamma^\nu,\frac{d\gamma^\nu}{dt},\dots,\frac{d^s\gamma^\nu}{dt^s}\right)=0, \quad 1\leqslant\sigma\leqslant m \tag{2.11}$$

for sections $\gamma\colon R\to R\times R^m$, $\gamma(\mathrm{id}_{\mathrm{dom}\,\gamma}(\gamma^\nu))$, defined on open subsets of R. Denoting by (q^σ) the canonical coordinates on R^m we can write (2.11) in an equivalent form

$$E_\sigma(t,q^\nu,\dots,q^\nu_s)\circ J^s\gamma=0, \quad 1\leqslant\sigma\leqslant m. \tag{2.12}$$

Put $E=E_\sigma dq^\sigma\wedge dt$. The system (2.12) is called *variational* if the form E is variational.

Let r, k be integers, $r\leqslant k$, let $W_1\subset J^rY$, $W_2\subset J^kY$ be open sets such that $W_1\cap\pi_{k,r}(W_2)\neq\emptyset$. Consider two lagrangians $\lambda_1\in\Omega^1_X(W_1)$, $\lambda_2\in\Omega^1_X(W_2)$. We say that λ_1 and λ_2 are *equivalent* if (up to the projection) $E_{\lambda_1}=E_{\lambda_2}$ on the common domain of definition.

We shall start with a fundamental lemma, called the *Poincaré Lemma for contact forms* (Krupka [8]). Although the proof is standard we recall it briefly to fix the notations.

Lemma. *Let $\tau\colon W\to U$ be fibered manifold such that $W=U\times V$, where $U\subset R$ and open interval and $V\subset R^m$ is an open ball with the centre at the origin. Let $k\geqslant 1$ be an integer, and $\varrho\in\Omega^p_{J^{s-1}W}(J^sW)$ a closed p-form of the order of contactness k. Then there exists a $(p-1)$-form η on J^sW of the order of contactness $k-1$ such that $d\eta=\varrho$.*

Proof. Let (t, q^σ) be the canonical coordinates on W. Define a mapping χ_s: $[0,1] \times J^s W \to J^s W$ by

$$\chi_s\big(u(t, q^\sigma, \dots, q^\sigma_s)\big) = (t, uq^\sigma, \dots, uq^\sigma_s). \tag{2.13}$$

Let $\varrho \in \Omega^p_{J^{s-1}W}(J^s W)$ be a form. Denote $\varrho = \Sigma c_K \Omega^{(p)}_K$, where K is a multi-index labelling exterior products of p factors of $\mathrm{d}t$, ω^σ, ..., ω^σ_{s-1}, denoted by $\Omega^{(p)}_K$. Now since

$$\chi^*_s \omega^\sigma_j = q^\sigma_j \,\mathrm{d}u + u\omega^\sigma_j, \quad 0 \leqslant j \leqslant s-1, \tag{2.14}$$

we obtain

$$\chi^*_s \varrho = \mathrm{d}u \wedge \varrho_0 + \varrho', \tag{2.15}$$

where ϱ_0 and ϱ' do not contain $\mathrm{d}u$. More precisely,

$$\varrho_0 = \Sigma a_j \Omega^{(p-1)}_j, \quad \varrho' = \Sigma b_K \Omega^{(p)}_K, \tag{2.16}$$

where b_K are given by $b_K = u^K(c_K \circ \chi_s)$, J is a multi-index labelling exterior products of $p-1$ factors of $\mathrm{d}t$, ω^σ, ..., ω^σ_{s-1}, which are denoted by $\Omega^{(p-1)}_J$, and a_J are functions on $[0,1] \times J^s W$. Put

$$A\varrho = \Sigma\Big(\int_0^1 a_J \,\mathrm{d}u\Big)\Omega^{(p-1)}_J. \tag{2.17}$$

$A\varrho$ is $(p-1)$-form on $J^s W$, and its order of contactness is $k-1$. By a straightforward computation we get

$$\mathrm{d}A\varrho + A\mathrm{d}\varrho = \varrho. \tag{2.18}$$

By assumption, $\mathrm{d}\varrho = 0$. Hence, putting $\eta = A\varrho$, the proof is completed. □

Note that the mapping A defined by (2.17) maps, for every k, the module $\Omega^{p-k,k}_{J^{s-1}W}(J^s W)$ into $\Omega^{p-k,k-1}_{J^{s-1}W}(J^s W)$.

Now, we shall return to the inverse problem.

Let $s \geqslant 1$ be an integer. A *closed* form $\alpha \in \Omega^2_{J^{s-1}Y}(J^s Y)$ is called a *Lepagean 2-form* if $\alpha = E + F$, where $E \in \Omega^{1,1}_Y(J^s Y)$ and $F \in \Omega^{0,2}(J^s Y)$ [12]. Taking into account the definition of a Lepagean 1-form we see immediately that locally $\alpha = \mathrm{d}\varrho$, where ϱ is a Lepagean 1-form.

Proposition 3. *Let $\alpha \in \Omega^2_{J^{s-1}Y}(J^sY)$ be a Lepagean 2-form. Then the form $E = p_1\alpha$ is locally variational, and J^sY can be covered by open sets in such a way that on each of these sets $\lambda = h(A\alpha) = AE$ is a lagrangian for E.*

P r o o f. By the Poincaré Lemma for contact forms J^sY can be covered by open sets in such a way that, on each of these sets, $\alpha = \mathrm{d}\eta$ where $\eta = A\alpha = AE + AF$. Evidently, each of the forms η is a (locally defined) Lepagean 1-form. This means that $E = E_\lambda$ where $\lambda = h(\eta) = AE$. □

The (local) lagrangian

$$\lambda = AE = \left(q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\,\mathrm{d}u\right)\mathrm{d}t \tag{2.19}$$

is called the *Veinberg-Tonti lagrangian*; note that its order is equal to the order of the corresponding form E.

Proposition 4. *Let $s \geqslant 1$, let $\alpha \in \Omega^2_{J^{s-1}Y}(J^sY)$ be a form. The following conditions are equivalent:*

(1) *α is a Lepagean 2-form.*

(2) *In each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y, α is expressed in the form*

$$\alpha = E_\sigma \omega^\sigma \wedge \mathrm{d}t + \sum_{j,k=1}^{s-1} F^{jk}_{\sigma\nu}\omega^\sigma_j \wedge \omega^\nu_k, \quad F^{jk}_{\sigma\nu} = -F^{kj}_{\nu\sigma}, \tag{2.20}$$

where the functions E_σ satisfy

$$\frac{\partial E_\sigma}{\partial q^\nu_\ell} - (-1)^\ell \frac{\partial E_\nu}{\partial q^\sigma_\ell} - \sum_{k=\ell+1}^{s} (-1)^k \binom{k}{\ell} \frac{\mathrm{d}^{k-\ell}}{\mathrm{d}t^{k-\ell}} \frac{\partial E_\nu}{\partial q^\sigma_k} = 0 \tag{2.21}$$

for $0 \leqslant \ell \leqslant s$, and $F^{jk}_{\sigma\nu}$ are expressed by means of E_ϱ, $1 \leqslant \varrho \leqslant m$ in the form

$$F^{jk}_{\sigma\nu} = \frac{1}{2} \sum_{\ell=0}^{s-j-k-1} (-1)^{j+\ell} \binom{j+\ell}{\ell} \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell} \frac{\partial E_\sigma}{\partial q^\nu_{j+k+\ell+1}}, \quad 0 \leqslant j+k \leqslant s-1, \tag{2.22}$$

$$F^{jk}_{\sigma\nu} = 0, \quad s \leqslant j+k \leqslant 2s-2. \tag{2.23}$$

P r o o f. We recall the proof from [12] (cf. also [5]).

Suppose (1). Expressing the relation $d\alpha = 0$ in fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y we get the following identities:

$$\frac{\partial E_\sigma}{\partial q^\nu} - \frac{\partial E_\nu}{\partial q^\sigma} + 2\frac{d}{dt}F_{\nu\sigma} = 0, \tag{2.24}$$

$$\frac{\partial E_\sigma}{\partial q^\nu_k} - 2\frac{d}{dt}F^{0k}_{\sigma\nu} - F^{0,k-1}_{\sigma\nu} = 0, \quad 1 \leqslant k \leqslant s-1, \tag{2.25}$$

$$\frac{\partial E_\sigma}{\partial q^\nu_s} - 2F^{0,s-1}_{\sigma\nu} = 0, \tag{2.26}$$

$$\frac{d}{dt}F^{jk}_{\sigma\nu} + F^{j-1,k}_{\sigma\nu} + F^{j,k-1}_{\sigma\nu} = 0, \quad 1 \leqslant j,k \leqslant s-1, \tag{2.27}$$

$$F^{s-1,k}_{\sigma\nu} = 0, \quad 1 \leqslant k \leqslant s-1, \tag{2.28}$$

$$\frac{\partial F^{jk}_{\sigma\nu}}{\partial q^\varrho_\ell} + \frac{\partial F^{\ell j}_{\varrho\sigma}}{\partial q^\nu_k} + \frac{\partial F^{k\ell}_{\nu\varrho}}{\partial q^\sigma_j} = 0, \quad 0 \leqslant j,k,\ell \leqslant s-1. \tag{2.29}$$

The relations (2.25) and (2.26) enable us to express the functions $F^{0k}_{\sigma\nu}$, $0 \leqslant k \leqslant s-1$ by means of E_σ in the form

$$F^{0k}_{\sigma\nu} = \frac{1}{2}\sum_{\ell=0}^{s-k-1}(-1)^\ell \frac{d^\ell}{dt^\ell}\frac{\partial E_\sigma}{\partial q^\nu_{k+\ell+1}}, \quad 0 \leqslant k \leqslant s-1. \tag{2.30}$$

From (2.27) and (2.28) we get (2.23) and the relations

$$F^{jk}_{\sigma\nu} = \sum_{\ell=0}^{s-j-k-1}(-1)^{k+\ell}\binom{k+\ell-1}{\ell}\frac{d^\ell}{dt^\ell}F^{j+k+\ell,0}_{\sigma\nu}, \quad 2 \leqslant j+k \leqslant s-1, \tag{2.31}$$

and

$$F^{0k}_{\sigma\nu} = \sum_{\ell=0}^{s-k-1}(-1)^{k+\ell}\binom{k+\ell-1}{\ell}\frac{d^\ell}{dt^\ell}F^{k+\ell,0}_{\sigma\nu}, \quad 1 \leqslant k \leqslant s-1. \tag{2.32}$$

Now, the relations (2.21) and (2.22) are obtained after obvious and straightforward calculations. Finally, we shall show that (2.29) are fulfilled identically. Put

$$G^{jk\ell}_{\sigma\nu\varrho} = \frac{\partial F^{jk}_{\sigma\nu}}{\partial q^\varrho_\ell} + \frac{\partial F^{\ell j}_{\varrho\sigma}}{\partial q^\nu_k} + \frac{\partial F^{k\ell}_{\nu\varrho}}{\partial q^\sigma_j}, \quad 0 \leqslant j,k,\ell \leqslant s, \quad 1 \leqslant \sigma,\nu,\varrho \leqslant m. \tag{2.33}$$

Differentiating the relations (2.24)–(2.27) with respect to q^ϱ_{s+1} we obtain

$$G^{jks}_{\sigma\nu\varrho} = \frac{\partial F^{jk}_{\sigma\nu}}{\partial q^\varrho_s} = 0, \quad 0 \leqslant j,k \leqslant s, \quad 1 \leqslant \sigma,\nu,\varrho \leqslant m. \tag{2.34}$$

Using (2.27) we get for $1 \leqslant j,k,\ell \leqslant s$, $1 \leqslant \sigma,\nu,\varrho \leqslant m$ the relation

$$\frac{d}{dt}G^{jk\ell}_{\sigma\nu\varrho} + G^{j-1,k,\ell}_{\sigma\nu\varrho} + G^{j,k-1,\ell}_{\sigma\nu\varrho} + G^{j,k,\ell-1}_{\sigma\nu\varrho} = 0. \tag{2.35}$$

Proceeding by induction starting from (2.34) the desired relations $G^{jk\ell}_{\sigma\nu\varrho} = 0$ for $0 \leqslant j,k,\ell \leqslant s$, $1 \leqslant \sigma,\nu,\varrho \leqslant m$ are obtained.

The implication (2) $\Rightarrow$ (1) is proved in an obvious way. □

Corollary. *Let $\alpha \in \Omega^2_{J^{s-1}Y}(J^sY)$ be a form satisfying the following two conditions:*

(1) $\alpha = E + F$, *where* $E \in \Omega^{1,1}_Y(J^sY)$ *and* $F \in \Omega^{0,2}(J^sY)$,

(2) $p_2\,d\alpha = 0$.

Then α is closed.

Let E be a locally variational for on J^sY. A Lepagean 2-form is called a *Lepagean equivalent* of E if $p_2\alpha = E$.

Theorem 1. *Every locally variational form on J^sY has a unique Lepagean equivalent, and this is projectable onto $J^{s-1}Y$.*

Proof. Let $E \in \Omega^{1,1}_Y(J^sY)$ be a locally variational form. Consider an open covering $\{W_\iota\}$ of J^sY such that (i) $W_\iota \subset V_s$, for every ι, where (V,ψ) is a fiber chart on Y, (ii) $E|_{W_\iota}$ is variational. Let ι, κ be arbitrary such that $W_\iota \cap W_\kappa \neq \emptyset$. Denote by α_ι (resp. α_κ)) a Lepagean equivalent of $E|_{W_\iota}$ (resp. $E|_{W_\kappa}$) on W_ι (resp. W_κ) (we can take e.g. $\alpha_\iota = d\theta_{\lambda_\iota}$ where λ_ι is a lagrangian for $E|_{W_\iota}$, and similarly for α_κ). Hence, on $W_\iota \cap W_\kappa$ we have $\alpha_\iota = E + F_\iota$ and $\alpha_\kappa = E + F_\kappa$ where F_ι and F_κ are 2-contact 2-forms. Now, $\alpha_\iota - \alpha_\kappa = F_\iota - F_\kappa$, i.e., since the form $\alpha_\iota - \alpha_\kappa$ is closed by assumption, $F_\iota - F_\kappa = d\eta$ for a 1-comtact 1-form η. This means that $\alpha_\iota = \alpha_\kappa + d\eta$, i.e. $p_1 d\eta = 0$. This implies, however, $p_2 d\eta = 0$. Hence $d\eta = 0$, i.e. $\alpha_\iota = \alpha_\kappa$ on $W_\iota \cap W_\kappa$, proving the global existence of a Lepagean equivalent of E. Uniqueness is a direct consequence of Proposition 4.

It remains to show that the Lepagean equivalent of E is projectable onto $J^{s-1}Y$. Let (V,ψ), $\psi = (t, q^\sigma)$ be a fiber chart on Y, let $E = E_\sigma \omega^\sigma \wedge dt$ be the chart expression of E. According to Proposition 4, the form α defined by (2.20)–(2.23) is the Lepagean equivalent of E. Since $F^{ik}_{\sigma\nu}$, $1 \leqslant \sigma,\nu \leqslant m$, $0 \leqslant i,k \leqslant s-1$ are dined on $V_{2s-1-i-k}$, α is defined on $J^{2s-1}Y$. Writing α in the form (of a non-invariant decomposition)

$$(2.36)\qquad \alpha = \Big(E_\sigma - \sum_{k=0}^{s-1} 2F^{0k}_{\sigma\nu} q^\nu_{k+1}\Big) dq^\sigma \wedge dt - \sum_{i=1}^{s-1}\sum_{k=0}^{s-1-i} 2F^{ik}_{\sigma\nu} q^\nu_{k+1} dq^\sigma_i \wedge dt + \sum_{i=0}^{s-1}\sum_{k=0}^{s-1-i} F^{jk}_{\sigma\nu} dq^\sigma_j \wedge dq^\nu_k$$

it suffices to show that the functions $F^{ik}_{\sigma\nu}$, $E_\sigma - 2F^{0,s-1}_{\sigma\nu} q^\nu_s$, $1 \leqslant \sigma, \nu \leqslant m$, $0 \leqslant i, k \leqslant s-1$ are defined on V_{s-1}. Differentiating the relations (2.24) and (2.25) consecutively with respect to $q^\varrho_{2s}, q^\varrho_{2s-1}, \ldots, q^\varrho_{s+1}$ and taking into account that E_σ are defined on V_s, we obtain that $F^{0k}_{\sigma\nu}$ are defined on V_{s-1}. Similar conclusions are made for $F^{ik}_{\sigma\nu}$ with the help of (2.27). Finally, since the conditions (2.21) imply that

$$(2.37)\qquad E_\sigma = A_\sigma + B_{\sigma\nu} q^\nu_s,$$

where A_σ and $B_{\sigma\nu}$, $1 \leqslant \sigma, \nu \leqslant m$ are functions on V_{s-1}, and obviously $B_{\sigma\nu} = 2F^{0,s-1}_{\sigma\nu}$, we get the desired result. □

If E its a locally variational form the Lepagean equivalent of E ill be denoted by α_E.

Theorem 1 has the following important consequences:

Corollary 1. *Let $E \in \Omega^{1,1}_Y(J^sY)$ be a locally variational form. There exists a unique 2-contact 2-form $F \in \Omega^{0,2}(J^sY)$ such that $d(E+F) = 0$. In each fiber chart (V,ψ), $\psi = (t, q^\sigma)$ on Y*

$$(2.38)\qquad F = \sum_{j,k=1}^{s-1} F^{jk}_{\sigma\nu} \omega^\sigma_j \wedge \omega^\nu_k,$$

where the functions $F^{jk}_{\sigma\nu} = -F^{kj}_{\nu\sigma}$ are defined by (2.22) and (2.23).

Corollary 2. *Let r, s be integers. Let $E \in \Omega^{1,1}_Y(J^sY)$ be a locally variational form, α_E its Lepagean equivalent. Let λ be a lagrangian for E, defined on an open subset of J^rY. Then the form $d\theta_\lambda$ is projectable onto an open subset $W \subset J^{s-1}Y$ and $\pi^*_{2r-1,s-1} d\theta_\lambda = \alpha_E|_W$.*

Corollary 3. *Consider the Euler-Lagrange mapping $\mathscr{E}$, denote by $\ker \mathscr{E}$ its kernel. A lagrangian $\lambda \in \Omega^1_X(W)$ defined on an open subset W of J^rY belongs to $\ker \mathscr{E}$ if and only if there exists a closed 1-form ϱ on $\pi_{r,r-1}W$ such that $\lambda = h\varrho$.*

Proof. Let $\lambda \in \ker \mathscr{E}$, i.e. $E_\lambda = 0$. Then, by Theorem 1, $\alpha_{E_\lambda} = d\theta_\lambda = 0$. Since $\lambda = h\theta_\lambda$, the form $h\theta_\lambda$ is projectable onto W, and locally $h\theta_\lambda = h df$ where f is a function. Since hdf is defined on an open subset of $J^r Y$ we conclude that f does not depend on q^σ_r, $1 \leqslant \sigma \leqslant m$. Now $\theta_\lambda = df$, i.e. θ_λ is projectable onto $\pi_{r,r-1}W$.

Conversely, if $\varrho \in \Omega^1(\pi_{r,r-1}W)$ is closed and such that $\lambda = h\varrho$ we get $\theta_\lambda = \theta_{h\varrho} = \varrho$, i.e. $d\theta_\lambda = d\varrho = 0$. Hence, $E_\lambda = 0$. □

Notice that by Corollary 3, two lagrangians $\lambda_1 \in \Omega^1_X(W_1)$, $\lambda_2 \in \Omega^1_X(W_2)$ defined on open sets $W_1 \subset J^r Y$, $W_2 \subset J^k Y$ where $r \leqslant k$, such that $W_1 \cap \pi_{k,r}(W_2) \neq \emptyset$, are equivalent if and only if they differ locally by a "total derivative" hdf where f is a function depending on t, q^σ, ..., q^σ_{k-1} only.

Theorem 1 provides, for the case of higher-order mechanics, an easy solution of both the *local* and the *global inverse problems of the calculus of variations.*

Theorem 2. *A form $E \in \Omega^{1,1}_Y(J^s Y)$ is locally variational if and only if in each fiber chart (V, ψ), $\psi = (t, q^\sigma)$ on Y the functions E_σ, $1 \leqslant \sigma \leqslant m$, defined by $E = E_\sigma dq^\sigma \wedge ddt$ satisfy the identities*

$$\frac{\partial E_\sigma}{\partial q^\nu_\ell} - (-1)^\ell \frac{\partial E_\nu}{\partial q^\sigma_\ell} - \sum_{k=\ell+1}^{s} (-1)^k \binom{k}{\ell} \frac{d^{k-\ell}}{dt^{k-\ell}} \frac{\partial E_\nu}{\partial q^\sigma_k} = 0, \quad 0 \leqslant \ell \leqslant s. \tag{2.39}$$

Proof. If E is locally variational then the assertion follows from Theorem 1 and Proposition 4.

Conversely, suppose that the identities (2.39) are satisfied. Then we can construct a 2-contact 2-form F and a 2-form α using (2.20) and (2.22). Since, by Proposition 4, α is a Lepagean 2-form we get from Proposition 3 that E is locally variational. □

The identities (2.39), for the general case (i.e. a system of higher order partial differential equations) are called *Anderson-Duchamp-Krupka conditions.* For $\dim X = 1$ and $s = 2$ they are usually called the *Helmholtz conditions.*

Theorem 3. *Let $E \in \Omega^{1,1}_Y(J^s Y)$ be a locally variational form, let α_E be its Lepagean equivalent. Then following four assertions are equivalent:*

(1) *E is globally variational,*
(2) *α_E is exact,*
(3) *there exists a global Lepagean 1-form ϱ on $J^{s-1}Y$ such that $p_1 d\varrho = E$,*
(4) *there exists a global lagrangian λ on $J^s Y$ such that $E = E_\lambda$.*

Proof. (a) If E is globally variational then there exists a lagrangian λ such that (up to a projection) $E = E_\lambda$. Taking the Lepagean equivalent θ_λ of λ we get $d\theta_\lambda = \alpha_E$, proving the exactness of α_E.

(b) If α_E is exact then there is a 1-form ϱ on $J^{s-1}Y$ such that $\mathrm{d}\varrho = \alpha_E$. By Proposition 2.4 $p_1 \mathrm{d}\varrho = E$, hence ϱ is a Lepagean 1-form.

(c) If (3) holds then $h\varrho$ is a global lagrangian of order s for E.

(d) The last implications is trivial.

Note that a sufficient condition for any of the assertions of Theorem 3 to be satisfied is that the de Rham cohomology group $H^2(Y)$ be trivial.

In the sequel we shall be interested in the problem of existence of lagrangians of the lowest possible order, the so-called *minimal-order lagrangians* for a given locally variational form. We shall show that this property is closely related to the existence of a certain canonical form of the Lepagean 2-form.

In what follows, s will be an integer, denoting the minimal order of a locally variational form E, i.e. $E \in \Omega_Y^{1,1}(J^sY)$ is supposed not to be $\pi_{s,k}$-projectable for any $k < s$ (equivalently, there is a point $x \in J^sY$ such that

$$\frac{\partial E_\sigma}{\partial q_s^\nu}(x) \neq 0 \tag{2.40}$$

for at least one σ and ν). Further, we shall denote by c the integer equal to $\frac{1}{2}s$ if s is odd, or to $\frac{1}{2}(s-1)$ if s is even (i.e. $s = 2c$, or $s = 2c+1$). □

Proposition 5. *Let $E \in \Omega_Y^{1,1}(J^sY)$ be a locally variational form, let α_E be its Lepagean equivalent. $J^{s-1}Y$ can be covered by open sets W such that*

(1) for each W there exists a fiber chart (V,ψ), $\psi = (t, q^\sigma)$ on Y such that $W \subset V_{s-1}$;

(2) there exist functions H, p_ν^k, $1 \leqslant \nu \leqslant m$, $0 \leqslant k \leqslant s-c-1$ on W such that the restriction of α_E to W can be expressed in the form

$$\alpha_E = -\mathrm{d}H \wedge \mathrm{d}t + \sum_{k=1}^{s-c-1} \mathrm{d}p_\nu^k \wedge \mathrm{d}q_k^\nu. \tag{2.41}$$

Proof. (cf. [12]). The form α_E is closed, i.e. there exists a covering of $J^{s-1}Y$ by open sets W such that (1) every W is a subset of V_{s-1} where (V,ψ), $\psi = (t, q^\sigma)$ is a fiber chart on Y, (2) on each W, $\alpha_E = \mathrm{d}\varrho$ for a 1-form ϱ on W. Using the Poincaré Lemma for contact forms we obtain (up to a projection)

$$\varrho = A\alpha_E = \left[q^\sigma \int_0^1 (E_\sigma \circ \chi_s)\,\mathrm{d}u\right]\mathrm{d}t + \sum_{k=0}^{s-1}\left[\sum_{j=0}^{s-1} 2q_j^\sigma \int_0^1 (F_{\sigma\nu}^{jk} \circ \chi_{s-1})u\,\mathrm{d}u\right]\omega_k^\nu. \tag{2.42}$$

We shall show that there exist functions f, H, p^k_ν, $1 \leqslant \nu \leqslant m$, $0 \leqslant k \leqslant s-c-1$ on W such that (2.42) can be equivalently expressed in the form

$$\varrho = -H\mathrm{d}t + \sum_{k=0}^{s-c-1} p^k_\nu \mathrm{d}q^\nu_k + \mathrm{d}f. \tag{2.43}$$

We define a mapping $\chi_{s-1,s-c}\colon [0,1] \times W \to W$ by

$$\begin{aligned}\chi_{s-1,s-c}(\nu, (t, q^\sigma, \ldots, q^\sigma_{s-c-1}, q^\sigma_{s-c}, \ldots, q^\sigma_{s-1}))\\ = (t, q^\sigma, \ldots, q^\sigma_{s-c-1}, \nu q^\sigma_{s-c}, \ldots, \nu q^\sigma_{s-1}).\end{aligned} \tag{2.44}$$

Put

$$\begin{aligned} f = {} & \sum_{k=s-c}^{s-1} \sum_{j=0}^{s-1-k} 2q^\nu_k q^\sigma_j \int_0^1 \left(\int_0^1 (F^{jk}_{\sigma\nu} \circ \chi_{s-1}) u \,\mathrm{d}u \right) \circ \chi_{s-1,s-c} \mathrm{d}\nu \\ & + \varphi(t, q^\varrho, \ldots, q^\varrho_{s-c-1}), \end{aligned} \tag{2.45}$$

where φ is an arbitrary function, and define

$$p^k_\nu = \sum_{j=0}^{s-k-1} 2q^\sigma_j \int_0^1 (F^{jk}_{\sigma\nu} \circ \chi_{s-1}) u \,\mathrm{d}u - \frac{\partial f}{\partial q^\nu_k}, \quad 1 \leqslant \nu \leqslant m, \quad 0 \leqslant k \leqslant s-c-1, \tag{2.46}$$

$$\begin{aligned} -H = {} & q^\sigma \int_0^1 (E_\sigma \circ \chi_s) \,\mathrm{d}u \\ & - \sum_{k=0}^{s-1} \sum_{j=0}^{s-k-1} 2q^\sigma_j q^\nu_{k+1} \int_0^1 (F^{jk}_{\sigma\nu} \circ \chi_{s-1}) u \,\mathrm{d}u - \frac{\partial f}{\partial t}. \end{aligned} \tag{2.47}$$

Since obviously

$$\frac{\partial f}{\partial q^\nu_k} = \sum_{j=0}^{s-k-1} 2q^\sigma_j \int_0^1 (F^{jk}_{\sigma\nu} \circ \chi_{s-1}) u \,\mathrm{d}u, \quad s-c \leqslant k \leqslant s-1, \tag{2.48}$$

we obtain, substituting into (2.43), the formula (2.42). This completes the proof. □

The expression of α_E in the form (2.43) will be called the *canonical form* of the Lepagean equivalent of a locally variational for E. For any fixed function $\varphi(t, q^\varrho, \ldots, q^\varrho_{s-c-1})$ the functions H and p^k_ν, $1 \leqslant \nu \leqslant m$, $0 \leqslant k \leqslant s-c-1$, defined by (2.47) and (2.46) will be called a *Hamilton function*, or a *Hamiltonian* of E, and a *family of momenta* of a locally variational form E (relative to φ).

Theorem 4. *Let E be a locally variational form of order s, let c be as above. There exists an open covering of $J^{s-c}Y$ such that on each of its elements there exists a lagrangian $\lambda_{\min}$ satisfying $E_{\lambda_{\min}} = E$.*

Proof. Consider an open covering of J^sY such that on each of its elements there exists a Veinberg-Tonti lagrangian for E. Denote by λ this (local) lagrangian and put $\lambda_{\min} = \lambda - hdf$, where f is defined by (2.45). Obviously, $\lambda_{\min}$ is a local lagrangian for E. Using (2.48) and the formulas (2.25), (2.27) we obtain

$$\begin{aligned}(2.49)\qquad \frac{\partial L_{\min}}{\partial q^\nu_k} = q^\sigma \int_0^1 \left(\frac{\partial E_\sigma}{\partial q^\nu_k} - -2\frac{\mathrm{d}}{\mathrm{d}t}F^{0k}_{\sigma\nu} - F^{0,k-1}_{\sigma\nu}\right) \circ \chi_s u\, \mathrm{d}u \\ - \sum_{j=1}^{s-k-1} 2q^\sigma_j \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}F^{jk}_{\sigma\nu} + F^{j-1,k}_{\sigma\nu} + F^{j,k-1}_{\sigma\nu}\right) \circ \chi_s u\, \mathrm{d}u \\ - 2q^\sigma_{s-k} \int_0^1 (F^{s-k-1,k}_{\sigma\nu} + F^{s-k,k-1}_{\sigma\nu}) \circ \chi_{s-1} u\, \mathrm{d}u = 0\end{aligned}$$

for $s - c + 1 \leqslant k \leqslant s - 1$, and

$$(2.50)\qquad \frac{\partial L_{\min}}{\partial q^\nu_s} = q^\sigma \int_0^1 \left(\frac{\partial E_\sigma}{\partial q^\nu_s} - F^{0,s-1}_{\sigma\nu}\right) \circ \chi_s u\, \mathrm{d}u = 0.$$

Hence for every function φ, $\lambda_{\min} = L_{\min}\mathrm{d}t$ is a lagrangian for E, defined on an open subset of $J^{s-c}Y$. □

From the construction of minimal-order lagrangians and from Proposition 5 we get the relation between minimal-order lagrangians on the one side and Hamiltonians and momenta on the other.

Proposition 6. (1) *Let H, p^k_ν, $1 \leqslant \nu \leqslant m$, $0 \leqslant k \leqslant s - c - 1$, be a Hamiltonian and a family of momenta of a locally variational form $E \in \Omega^{1,1}_Y(J^sY)$, relative to an arbitrary but fixed function $\varphi(t, q^\varrho, \ldots, q^\varrho_{s-c-1})$, and defined on an open set $U \subset J^{s-1}Y$. Then there exists a minimal-order lagrangian $\lambda_{\min}$ of E on $\pi_{s-1,s-c}U$ such that*

$$(2.51)\qquad p^k_\nu = (f_{\min})^k_\nu, \quad 1 \leqslant \nu \leqslant m, \quad 0 \leqslant k \leqslant s - c - 1,$$

$$(2.52)\qquad H = -L_{\min} + \sum_{i=0}^{s-c-1} (f_{\min})^{i+1}_\sigma q^\sigma_{i+1},$$

where $L_{\min}$ is defined by $\lambda_{\min} = L_{\min}\mathrm{d}t$ and

$$(2.53)\qquad (f_{\min})^i_\sigma = \sum_{k=0}^{s-c-i} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L_{\min}}{\partial q^\sigma_{i+k}}, \quad 1 \leqslant i \leqslant s - c, \quad 1 \leqslant \sigma \leqslant m.$$

(2) *Let* λ_{min} *be a minimal-order lagrangian of a locally variational form* $E \in \Omega_Y^{1,1}(J^sY)$ *defined on an open set* $U \subset J^{s-c}Y$. *Let* (V, ψ), $\psi = (t, q^\sigma)$ *be a fiber chart on* Y *such that* $V_{s-c} \subset U$. *Then the functions* H, p_ν^k, $1 \leqslant \nu \leqslant m$, $0 \leqslant k \leqslant s-c-1$, *defined by*

$$p_\nu^k = (f_{\text{min}})_\nu^{k+1}, \quad 1 \leqslant \nu \leqslant m, \quad 0 \leqslant k \leqslant s-c-1, \tag{2.54}$$

$$H = -L_{\text{min}} + \sum_{i=0}^{s-c-1} (f_{\text{min}})_\sigma^{i+1} q_{i+1}^\sigma, \tag{2.55}$$

where L_{min} *is defined by* $\lambda_{\text{min}} = L_{\text{min}}\,\mathrm{d}t$ *and*

$$(f_{\text{min}})_\sigma^i = \sum_{k=0}^{s-c-i} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L_{\text{min}}}{\partial q_{i+k}^\sigma}, \quad 1 \leqslant i \leqslant s-c, \quad 1 \leqslant \sigma \leqslant m, \tag{2.56}$$

are a Hamiltonian and a family of momenta of E.

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