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EXISTENCE OF SOLUTION TO NON-LINEAR BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATION OF THE SECOND ORDER IN HILBERT SPACE

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Summary. In this paper we deal with the boundary value problem in the Hilbert space. Existence of a solution is proved by using the method of lower and upper solutions. It is not necessary to suppose that the homogeneous problem has only the trivial solution. We use some results from functional analysis, especially the fixed-point theorem in the Banach space with a cone (Theorem 4.1, [5]).

Keywords: boundary value problem, existence of solutions, ordinary differential equations in Hilbert space

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In this paper we consider

— an infinite-dimensional Hilbert space H with a countable orthonormal base $\{e_i\}_{i=1}^{\infty}$, $(.,.)$ is a scalar product, $\|\cdot\|$ is a norm;

— the space $X = L_2(\langle a,b\rangle, H)$ of abstract functions $y\colon \langle a,b\rangle \to H$ such that $\|y\|_2 = \left(\int_a^b \|y(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} < \infty$;

— the cone K in X defined by $K = \{y \in X : y_i(t) = (y(t), e_i) \geqslant 0,\ i = 1,2,\dots t \in \langle a,b\rangle\}$.

It is proved in [6] that K is a normal, regular, strongly minihedral cone in X.

We deal with the boundary value problem

$$Ly = (p(t)\cdot y')' + q(t)\cdot y = f(t,y) \tag{1}$$

$$\begin{aligned} Uy\colon U_1y &= \alpha_0\cdot y(a) + \alpha_1\cdot y'(a) = 0 \\ U_2y &= \beta_0\cdot y(b) + \beta_1\cdot y'(b) = 0 \end{aligned} \tag{2}$$

where

1. $y\colon \langle a,b\rangle \to H$,

2. the functions p, $q\colon \langle a,b\rangle \to R$ are continuous and $p(t) > 0$ on $\langle a,b\rangle$,

3. $D \subseteq H$, $f \in L_2(\langle a,b\rangle \times D, H)$ and there exists $M \in R$ such that the function $f(t,y) + My$ is nonincreasing in y for every fixed $t \in \langle a,b\rangle$,

4. α_0, α_1, β_0, β_1 are real numbers such that $|\alpha_0| + |\alpha_1| > 0$, $|\beta_0| + |\beta_1| > 0$.

Remark 1. If there exists $M \in R$ such that the function $f(t,y) + M \cdot y$ is nonincreasing in $y \in D$, then for every $M_1 \in (-\infty, M)$ the function $f(t,y) + M_1 \cdot y$ is nonincreasing in $y \in D$. In the case $H = R$ we obtain the scalar problem (1), (2). Let us suppose that the scalar homogeneous problem has only the trivial solution. Then there exists the Green function $G_1(t,s)$ and the scalar problem is equivalent to the integral equation

$$y(t) = \int_a^b G_1(t,s) \cdot f(s, y(s))\, \mathrm{d}s. \tag{3}$$

We will use the following spaces:
$C(\langle a,b\rangle, H)$ with the norm $\|y\|_0 = \sup\limits_{\langle a,b\rangle} \|y(t)\|$
$C^1(\langle a,b\rangle, H)$ with the norm $\|y\|_1 = \|y\|_0 + \sup\limits_{\langle a,b\rangle} \|y'(t)\|$.
We are looking for a solution y of BVP (1), (2) in the space $C^1(\langle a,b\rangle, H)$.

Lemma 1. *(Lemma 1, [2]). If the scalar problem (1), (2) is equivalent to the equation (3), then also the problem (1), (2) in the Hilbert space H is equivalent to the equation (3) and the Green function $G_1(t,s)\colon \langle a,b\rangle \times \langle a,b\rangle \to R$ is given by the homogeneous scalar problem (1), (2).*

Lemma 2. *Let $f \in K$. Then also $\int_a^b f(t)\,\mathrm{d}t \in K$.*

Proof. Since $f \in K$ we have $f(t) = \sum\limits_{i=1}^{\infty} f_i(t) \cdot e_i$ $t \in \langle a,b\rangle$ where $f_i(t) = (f(t), e_i) \geqslant 0$ for $i = 1, 2, \ldots$ Further we have

$$\sum_{i=1}^{n} f_i(t) \cdot e_i \to f(t) \quad \text{for } n \to \infty,$$

$\Big|\big\|\sum\limits_{i=1}^{n} f_i(t)e_i\big\| - \|f(t)\|\Big| \leqslant \big\|\sum\limits_{i=1}^{n} f_i(t)e_i - f(t)\big\| \to 0$. Then $\big\|\sum\limits_{i=1}^{n} f_i(t)e_i\big\| < \varepsilon + \|f(t)\|$. Since $\Big(\int\limits_a^b \|f(t)\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} < \infty$ the integral $\int\limits_a^b \|f(t)\|\,\mathrm{d}t$ also exists. Using the Lebesgue dominated convergence theorem we get

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \left(\sum_{i=1}^{\infty} f_i(t)\cdot e_i\right)\mathrm{d}t = \sum_{i=1}^{\infty}\left(\int_a^b f_i(t)\,\mathrm{d}t\right)\cdot e_i = \sum_{i=1}^{\infty} F_i\cdot e_i$$

where real functions F_i satisfy $F_i = \int_a^b f_i(t)\,\mathrm{d}t = \text{const} \geqslant 0$. Hence the proof is complete. □

Lemma 3. *Let λ_0 be first characteristic number of the scalar BVP*

$$(4)\qquad (p(t)\cdot y')' + (q(t)+\lambda)\cdot y = 0$$

$$(5)\qquad Uy = 0.$$

Then for every $M \in (-\infty, \lambda_0)$ the scalar BVP

$$(6)\qquad (p(t)\cdot y')' + (q(t)+M)\cdot y = 0$$

$$(5)\qquad Uy = 0$$

has only the trivial solution and the Green function $G(t,s)$ satisfies $G(t,s) \leqslant 0$ on $\langle a,b\rangle \times \langle a,b\rangle$.

Proof. Let us denote the characteristic numbers of the scalar BVP (4), (5) by $\lambda_0, \lambda_1, \ldots$ supposing

$$\lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots$$

Then the characteristic numbers of the scalar BVP

$$(7)\qquad (p(t)\cdot y') + (q(t)+M+\lambda)\cdot y = 0$$

$$(5)\qquad Uy = 0$$

are $\lambda_0 - M, , \lambda_1 - M, , \ldots, , \lambda_n - M, , \ldots$. If we take $M < \lambda_0$ then all characteristic numbers of (7), (5) are positive and such that the Green function $G(t,s)$ satisfies $G(t,s) < 0$ on $(a,b)\times(a,b)$. Hence BVP (6), (5) has only the trivial solution. □

Corollary 1. *Let the Green function $G_1(t,s)$ of the scalar BVP $Ly = 0$, $Uy = 0$ exist and satisfy $G_1(t,s) \leqslant 0$ on $\langle a,b\rangle \times \langle a,b\rangle$. Then for every $M \leqslant 0$ the Green function $G(t,s)$ of the scalar BVP (6), (5) exists and $G(t,s) \leqslant 0$ on $\langle a,b\rangle \times \langle a,b\rangle$.*

Definition 1. An abstract function α or β from the space $C^1(\langle a,b\rangle, H)$ is called a *lower* or an *upper function* of equation (1), respectively, iff $L\alpha \geqslant f(t,\alpha)$ and $L\alpha \in X$ or $L\beta \leqslant f(t,\beta)$ and $L\beta \in X$, respectively, for $t \in \langle a,b\rangle$.

Theorem 1. *Let the following assumptions hold:*

(i) *Let there exist $M \in R$ such that the function $f(t,y) + M \cdot y$ is nonicreasing in $y \in D$ for every fixed $t \in \langle a,b\rangle$ and let the Green function $G(t,s)$ of BVP (6), (5) be such that $G(t,s) < 0$ on $(a,b) \times (a,b)$*

(ii) *Let α and β be respectively a lower and an upper function of equation (1) and let $\alpha \leqslant \beta$ and $\langle \alpha, \beta\rangle \subseteq D$ hold,*

(iii) *Let BVP*

$$\begin{gathered}\big(p(t) \cdot y'\big)' + \big(q(t) + M\big) \cdot y = 0\\ Uy = U\alpha\end{gathered}$$

have a solution $v \leqslant 0$. Similarly, let BVP

$$\begin{gathered}\big(p(t) \cdot y'\big)' + \big(q(t) + M\big) \cdot y = 0\\ Uy = U\beta\end{gathered}$$

have a solution $w \geqslant 0$.
Then there exists a solution y_0 of BVP (1), (2) and it satisfies

$$\alpha \leqslant y_0 \leqslant \beta$$

P r o o f. The equation (1) is equivalent to the equation

$$\big(p(t) \cdot y'\big)' + \big(q(t) + M\big) \cdot y = f(t,y) + M \cdot y.$$

Hence the existence of a solution of BVP (1), (2) is equivalent to the existence of a solution of the integral equation

$$y(t) = \int_a^b G(t,s) \cdot \Big[f\big(s, y(s)\big) + M \cdot y(s)\Big] \, \mathrm{d}s$$

Since $C\big(\langle a,b\rangle, H\big) \subseteq L_2\big(\langle a,b\rangle, H\big)$ we can define a set $D_1 = \{y \in C\big(\langle a,b\rangle, D\big) \colon y \in \langle \alpha, \beta\rangle\}$. Then $f\big(s, y(s)\big) \in X$ for $y(s) \in D_1$, $s \in \langle a,b\rangle$ so that the operator $T\colon D_1 \to X$,

$$Ty(t) = \int_a^b G(t,s) \cdot \Big[f\big(s, y(s)\big) + M \cdot y(s)\Big] \, \mathrm{d}s \quad t \in \langle a,b\rangle$$

is defined correctly.
We will show that

1. $\alpha \leqslant T\alpha, \quad \beta \geqslant T\beta$;

2. T is a monotone operator.

Thus it will be proved that $T\colon D_1 \to D_1$. Let us denote $h = L\alpha - f(t,\alpha)$. Then $h \geqslant 0$, $L\alpha = f(t,\alpha) + h$. The function α can be written as $\alpha = v + v_1$, where v is the function from the assumption (iii) and the function v_1 is a solution of BVP

$$(8) \qquad \left(p(t)\cdot y'\right)' + \left(q(t) + M\right)\cdot y = f(t,\alpha) + M\cdot\alpha + h$$

$$(5) \qquad Uy = 0,$$

The solution v_1 of BVP (8), (5) exists, because (i) implies that the Green function $G(t,s)$ of BVP (8), (5) exists. Hence we get

$$\begin{aligned}\alpha(t) &= v(t) + \int_a^b G(t,s)\cdot\left[f\left(s,\alpha(s)\right) + M\cdot\alpha(s) + h(s)\right]\mathrm{d}s\\ &= v(t) + \int_a^b G(t,s)\cdot h(s)\,\mathrm{d}s + \int_a^b G(t,s)\cdot\left[f\left(s,\alpha(s)\right) + M\cdot\alpha(s)\right]\mathrm{d}s\\ &= v(t) + \int_a^b G(t,s)\cdot h(s)\,\mathrm{d}s + T\alpha(t).\end{aligned}$$

Then $T\alpha(t) - \alpha(t) = -v(t) + \int\limits_a^b -G(t,s)\cdot h(s)\,\mathrm{d}s$. From Lemma 2 we obtain that $T\alpha - \alpha \geqslant 0$ and so $\alpha \leqslant T\alpha$. The inequality $\beta \geqslant T\beta$ can be verified similarly. Now we prove the monotonicity of the operator T. Let $y_1 \leqslant y_2$. Then

$$Ty_2 - Ty_1 = \int_a^b G(t,s)\cdot\left[f\left(s,y_2(s)\right) + M\cdot y_2(s) - f\left(s,y_1(s)\right) - M\cdot y_1(s)\right]\mathrm{d}s.$$

Again from Lemma 2 we get that $Ty_1 \leqslant Ty_2$. K is a strongly mnihedral cone in X, T is a monotone operator in D_1 so that there exists $y_0 \in D_1$ such that $y_0 = Ty_0$. Since $y_0 \in D_1$ we have $Ty_0 \in C\left(\langle a,b\rangle, H\right)$. Let Diag $= \{(t,s) \in \langle a,b\rangle \times \langle a,b\rangle : t = s\}$. The function $\frac{\partial G}{\partial t}$ is continuous on $\langle a,b\rangle \times \langle a,b\rangle$ except the set Diag. Then the theorem about parametric integrals yields that $y_0 \in C^1\left(\langle a,b\rangle, H\right)$. Hence y_0 is a solution of BVP (1), (2) and it satisfies $\alpha \leqslant y_0 \leqslant \beta$. □

R e m a r k 2. The verification of the assumptions of Theorem 1 is quite difficult. To simplify the assumption (i) we introduce a lemma:

Lemma 4. ([3], page 178). *Let BVP (1), (2) be given. Let $q(t) \leqslant 0$ on $\langle a,b\rangle$ and let $\alpha_0\cdot\alpha_1 < 0$, $\beta_0\cdot\beta_1 > 0$. Then the Green function $G_1(t,s)$ of BVP $Ly = 0$, $Uy = 0$ is such that $G_1(t,s) < 0$ on $\langle a,b\rangle \times \langle a,b\rangle$.*

Example. Let us prove the existence of a solution of BVP

$$y'' - y = -e^{\sqrt{y}-1}$$
$$\alpha_0 \cdot y(a) + \alpha_1 \cdot y'(a) = 0$$
$$\beta_0 \cdot y(b) + \beta_1 \cdot y'(b) = 0$$

wheie $\alpha_1 < 0 < \alpha_0$ and $\beta_0 > 0$, $\beta_1 > 0$.

Solution. It is sufficient to verify the assumptions of Theorem 1. Let $D = \{y \in C(\langle a,b\rangle, R) : y \geqslant 0\}$. Since the function $f(t,y) = -e^{\sqrt{y}-1}$ is decreasing in y, it is sufficient to put $M = 0$. The property (i) follows from Lemma 4. Let us verify (ii). $\alpha = 0$ is an element of D, $L\alpha = 0 \geqslant -e^{-1}$ and so α is a lower function of the given equation.
$\beta = 1$ is an element of D, $L\beta = -1 \leqslant -e^{1-1} = -1$ and so β is an upper function of the given equation. At the same time the inequality $\alpha \leqslant \beta$ holds. Now we verify (iii).

Let us consider BVP

$$v'' - v = 0$$

$$Uv = U\alpha \quad \text{i.e.} \quad \begin{aligned} \alpha_0 \cdot v(a) + \alpha_1 \cdot v'(a) &= 0 \\ \beta_0 \cdot v(b) + \beta_1 \cdot v'(b) &= 0. \end{aligned}$$

It follows from Lemma 4 that this BVP has only the trivial solution, i.e. $v = 0$. It remains to show that BVP

$$w'' - w = 0$$

$$Uw = U\beta \quad \text{i.e.} \quad \begin{aligned} \alpha_0 \cdot w(a) + \alpha_1 \cdot w'(a) &= \alpha_0 \\ \beta_0 \cdot w(b) + \beta_1 \cdot w'(b) &= \beta_0 \end{aligned}$$

has a solution $w \geqslant 0$. Solving this equation we get

$$w(t) = c_1 \cdot e^t + c_2 \cdot e^{-t},$$
$$w'(t) = c_1 \cdot e^t - c_2 \cdot e^{-t}$$

Substituting into the boundary conditions we get

$$\alpha_0 \cdot c_1 \cdot e^a + \alpha_0 \cdot c_2 \cdot e^{-a} + \alpha_1 \cdot c_1 \cdot e^a - \alpha_1 \cdot c_2 \cdot e^{-a} = \alpha_0,$$
$$\beta_0 \cdot c_1 \cdot e^b + \beta_0 \cdot c_2 \cdot e^{-b} + \beta_1 \cdot c_1 \cdot e^b - \beta_1 \cdot c_2 \cdot e^{-b} = \beta_0.$$

The determinant of the system is

$$\det = \begin{vmatrix} (\alpha_0+\alpha_1)\cdot e^{a} & (\alpha_0-\alpha_1)\cdot e^{-a} \\ (\beta_0+\beta_1)\cdot e^{b} & (\beta_0-\beta_1)\cdot e^{-b} \end{vmatrix}$$
$$= e^{a-b}\cdot(\alpha_0+\alpha_1)\cdot(\beta_0-\beta_1) - e^{b-a}\cdot(\alpha_0-\alpha_1)\cdot(\beta_0+\beta_1).$$

Since $0 < e^{a-b} < e^{b-a}$, we have $-e^{a-b}\cdot(\alpha_0-\alpha_1)\cdot(\beta_0+\beta_1) > -e^{b-a}\cdot(\alpha_0-\alpha_1)\cdot(\beta_0+\beta_1)$, hence

$$\det < e^{a-b}\cdot\left[(\alpha_0+\alpha_1)\cdot(\beta_0-\beta_1)-(\alpha_0-\alpha_1)\cdot(\beta_0+\beta_1)\right]$$
$$= e^{a-b}\cdot\left[2\cdot\alpha_1\cdot\beta_0 - 2\cdot\alpha_0\cdot\beta_1\right] < 0.$$

Similarly we get

$$\det_1 = \begin{vmatrix} \alpha_0 & (\alpha_0-\alpha_1)\cdot e^{-a} \\ \beta_0 & (\beta_0-\beta_1)\cdot e^{-b} \end{vmatrix}$$
$$= e^{-b}\cdot\alpha_0\cdot(\beta_0-\beta_1) - e^{-a}\cdot\beta_0\cdot(\alpha_0-\alpha_1) < 0$$

and also

$$\det_2 = \begin{vmatrix} (\alpha_0+\alpha_1)\cdot e^{a} & \alpha_0 \\ (\beta_0+\beta_1)\cdot e^{b} & \beta_0 \end{vmatrix}$$
$$= e^{a}\cdot\beta_0\cdot(\alpha_0+\alpha_1) - e^{b}\cdot\alpha_0\cdot(\beta_0+\beta_1) < 0.$$

Then $w(t) = \frac{\det_1}{\det}\cdot e^{t} + \frac{\det_2}{\det}\cdot e^{-t} > 0$, $t \in \langle a, b\rangle$, i.e. $w > 0$. Now assumptions of Theorem 1 hold so that there exists a solution y_0 of the given BVP and

$$0 \leqslant y_0 \leqslant 1.$$

Lemma 5. *Let $\alpha_1 < 0 < \alpha_0$, $\beta_0 > 0$, $\beta_1 > 0$ and let $M \in R$ be such that $q(t) + M \leqslant 0$ on $\langle a, b\rangle$. Then each of the scalar BVP's*

$$(p(t)\cdot y')' + (q(t)+M)\cdot y = 0 \tag{6}$$
$$Uy\colon U_1y = \alpha_0\cdot y(a) + \alpha_1\cdot y'(a) = 1 \tag{9}$$
$$U_2y = \beta_0\cdot y(b) + \beta_1\cdot y'(b) = 0$$

and

$$(p(t)\cdot y')' + (q(t)+M)\cdot y = 0 \tag{6}$$
$$Uy\colon U_1y = \alpha_0\cdot y(a) + \alpha_1\cdot y'(a) = 0 \tag{10}$$
$$U_2y = \beta_0\cdot y(b) + \beta_1\cdot y'(b) = 1$$

has one and only one solution. These solutions are linearly independent and positive on $\langle a, b\rangle$.

Proof. It follows from Lemma 4 that the solutions of BVP (6), (9) and (6), (10) exist and are uniquely determined. Let y_1 be the solution of (6), (9) and y_2 of (6), (10). Evidently y_1, y_2 are linearly independent. We shall show that $y_1(t) > 0$, $y_2(t) > 0$ on $\langle a, b\rangle$. We present the proof for the solution y_2, the proof for y_1 is similar.

First we prove by contradiction that $y_2 \not\equiv 0$.
Suppose that t_0 is the first number in $\langle a, b\rangle$ such that $y_2(t_0) = 0$. Since y_2 is the solution of the equation (6) we have

$$y_2'(t) = \frac{p(t_0) \cdot y_2'(t_0)}{p(t)} - \frac{1}{p(t)} \int_{t_0}^{t} [q(s) + M] \cdot y_2(s)\, ds \tag{11}$$

where $t \in \langle a, b\rangle$. Evidently $y_2'(t_0) \not\equiv 0$ because y_2 is a non-vanishing solution of the equation (6) with the condition $U_2 y_2 = 1$. The condition $U_1 y_2 = 0$ implies that $t_0 \not\equiv a$. Hence $t_0 > a$. Now (11) yields:
if $y_2'(t_0) < 0$ then $y_2(a) > 0$, $y_2'(a) < 0$,
if $y_2'(t_0) > 0$ then $y_2(a) < 0$, $y_2'(a) > 0$,
which contradicts the condition $U_1 y_2(a) = 0$. So we have proved that $y_2(t) \not\equiv 0$ on $\langle a, b\rangle$, i.e. $y_2(t) > 0$ or $y_2(t) < 0$ on $\langle a, b\rangle$.

Let us suppose that $y_2 < 0$. Form the condition $U_1 y_2 = 0$ we get

$$y_2'(a) = -\frac{\alpha_0}{\alpha_1} \cdot y_2(a) < 0.$$

From (11) for $t_0 = a$ it follows that $y_2'(t) < 0$ on $\langle a, b\rangle$. Then $y_2(b) < 0$, $y_2'(b) < 0$, which contradicts the boundary condition $U_2 y_2 = 1$. Hence that $y_2(t) > 0$ on $\langle a, b\rangle$, which completes the proof. □

Definition 2. Abstract functions $y_1, y_2, \ldots, y_n$ are called *linearly independent* iff every identity

$$\begin{gathered} d_1 \cdot y_1 + d_2 \cdot y_2 + \ldots + d_n \cdot y_n = 0, \quad d_i \in R \quad \text{for } i = 1, 2, \ldots, n, \\ \text{i.e.} \quad d_1 \cdot y_1(t) + d_2 \cdot y_2(t) + \ldots + d_n \cdot y_n(t) = 0 \quad t \in \langle a, b\rangle, \\ d_i \in R \quad \text{for } i = 1, 2, \ldots, n, \end{gathered}$$

implies that $d_i = 0$ for $i = 1, 2, \ldots, n$.

Theorem 2. *Let $\alpha_1 < 0 < \alpha_0$, $\beta_0 > 0$, $\beta_1 > 0$ and let $M \in R$ be such that $q(t) + M \leqslant 0$ on $\langle a, b\rangle$. Then each of BVP's*

$$(6) \qquad \left(p(t)\cdot y'\right)' + \left(q(t) + M\right)\cdot y = 0$$

$$(12) \qquad Uy\colon U_1 y = \alpha_0 \cdot y(a) + \beta_1 \cdot y'(a) = e_i$$
$$U_2 y = \beta_0 \cdot y(b) + \beta_1 \cdot y'(b) = 0$$

and

$$(6) \qquad \left(p(t)\cdot y'\right)' + \left(q(t) + M\right)\cdot y = 0$$

$$(13) \qquad Uy\colon U_1 y = \alpha_0 \cdot y(a) + \beta_1 \cdot y'(a) = 0$$
$$U_2 y = \beta_0 \cdot y(b) + \beta_1 \cdot y'(b) = e_i$$

where $i = 1, 2, \ldots$ has one and only one solution. These solutions are linearly independent and positive (by the cone K).

Proof. Lemma 5 implies that the scalar BVP (6), (9) has one and only one positive solution y_1. Let us define an abstract function $y_{1i}\ \langle a, b\rangle \to H$ by

$$y_{1i}(t) = y_1(t)\cdot e_i \quad t \in \langle a, b\rangle\,.$$

It is evident that y_{1i} is a solution of BVP (6), (12) in the Hilbert space H. Similarly, if y_2 is the solution of BVP (6), (10) then the abstract function $y_{2i}\colon \langle a, b\rangle \to H$ defined by

$$y_{2i}(t) = y_2(t)\cdot e_i \quad t \in \langle a, b\rangle$$

is a solution of BVP (6), (13). The uniqueness of y_{1i}, y_{2i} follows from the uniqueness of y_1, y_2. Now we prove that they are linearly independent. Let

$$d_1 \cdot y_{1i} + d_2 \cdot y_{2i} = 0,$$
$$\text{i.e.} \quad d_1 \cdot y_1 \cdot e_i + d_2 \cdot y_2 \cdot e_i = 0,$$
$$\text{i.e.} \quad d_1 \cdot y_1 + d_2 \cdot y_2 = 0.$$

Since y_1, y_2 are linearly independent we have $d_1 = d_2 = 0$ and so y_{1i}, y_{2i} are linearly independent. The continuity of y_{1i}, y_{2i} and y'_{1i}, y'_{2i} follows from the continuity of y_1, y_2 and y'_1, y'_2, respectively. □

Theorem 3. *Let $\alpha_1 < 0 < \alpha_0$, $\beta_0 > 0$, $\beta_1 > 0$ and let $M \in R$ be such that $q(t) + M \leqslant 0$ on $\langle a, b\rangle$. Then there exists a solution of BVP*

$$(6) \qquad \left(p(t)\cdot y'\right)' + \left(q(t) + M\right)\cdot y = 0$$

$$(14) \qquad Uy\colon U_1 y = a_1 \geqslant 0 \quad (\leqslant 0)$$
$$U_2 y = a_2 \geqslant 0 \quad (\leqslant 0)$$

where a_1, a_2 are given elements from H, and this solution is non-negative (non-positive).

Proof. Let y_1, y_2 be the solutions of (6), (9); (6), (10). Then $y = y_1 \cdot a_1 + y_2 \cdot a_2$ is a solution of (6), (14). Since y_1, y_2 are positive real functions, y is also a non-negative abstract function. □

Definition 3. The abstract function α or β from the space $C^1(\langle a,b\rangle, H)$ is called respectively a *lower* or an *upper solution* of BVP (1), (2) iff

$$L\alpha \geqslant f(t,\alpha), \quad U_1\alpha \leqslant 0, \quad U_2\alpha \leqslant 0$$
$$L\beta \leqslant f(t,\beta), \quad U_1\beta \geqslant 0, \quad U_2 \geqslant 0, \text{ respectively.}$$

Theorem 4. *Let the following assumptions hold:*

(i) $\alpha_1 < 0 < \alpha_0$, $\beta_0 > 0$, $\beta_1 > 0$, *let $M \in R$ be such that $q(t) + M \leqslant 0$ on $\langle a,b\rangle$ and let the function $f(t,y) + M \cdot y$ be nonincreasing in $y \in D$;*

(ii) *let functions α and β be a lower and an upper solution, respectively, of BVP* (6), (2) *such that $\alpha \leqslant \beta$ and $\langle \alpha, \beta\rangle \subseteq D$. Then BVP* (1), (2) *has at least one solution y_0 and*

$$\alpha \leqslant y_0 \leqslant \beta$$

Proof. The proof follows from Theorem 1 and 3. □

References

[1] *V. Šeda*: On some non-linear boundary value problems for ordinary differential equations, Archivum Mathematicum (Brno) *25* (1989), 207–222.

[2] *B. Rudolf*: Periodic boundary value problem in Hilbert space for differential equation of second order with reflection to the argument, Mathematica Slovaca *42* no. 1 (1992), 65–84.

[3] *M. Greguš, M. Švec, V. Šeda*: Ordinary differential equations, Alfa, Bratislava, 1985. (In Slovak.)

[4] *G. J. Šilov*: Mathematical analysis (Slovak translation), Alfa, Bratislava, 1985.

[5] *M. A. Krasnosel'skij*: Positive solutions of operators equations, Gosud. izd., Moskva, 1962. (In Russian.)

[6] *E. Rovderová*: A note on a Cone in the space $L_2(\langle a,b\rangle, H)$, Diploma Thesis, 1990.

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