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ON ESSENTIAL NORM OF THE NEUMANN OPERATOR

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Summary. One of the classical methods of solving the Dirichlet problem and the Neumann problem in $\mathbf{R}^m$ is the method of integral equations. If we wish to use the Fredholm-Radon theory to solve the problem, it is useful to estimate the essential norm of the Neumann operator with respect to a norm on the space of continuous functions on the boundary of the domain investigated, where this norm is equivalent to the maximum norm. It is shown in the paper that under a deformation of the domain investigated by a diffeomorphism, which is conformal (i.e. preserves angles) on a precisely specified part of boundary, for the given norm there exists a norm on the space of continuous functions on the boundary of the deformated domain such that this norm is equivalent to the maximum norm and the essential norms of the corresponding Neumann operators with respect to these norms are the same.

Keywords: Neumann operator, compact operator, reduced boundary, interior normal in Federer's sense, Hausdorff measure

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The present paper follows the papers [D], [ME1], in which it was proved that the Fredholm radius of the Neumann operator does not change under a deformation of the domain investigated by a diffeomorphism which is conformal (i.e. preserves angles) on a precisely specified part of the boundary (roughly speaking, at the angular points of the boundary). We will prove that the result holds if we substitute the maximum norm on the space of continuous functions on the boundary of the domain investigated by an equivalent norm. By this substitution we may achieve the essential norm of the Neumann operator that with respect to the new norm decreases (see [AKK], [KW]), and we may use well-known results about the convergence of the Neumann series (see [S], [ME2]).

If $H \subset \mathbf{R}^m$ $(m \geqslant 2)$ is an open set with a compact boundary ∂H, we denote by $\mathscr{C}(\partial H)$ the space of all bounded continuous functions on ∂H and by $\mathscr{C}'(\partial H)$ the space of all finite signed measures on ∂H. For a given function h harmonic on H we define the weak normal derivative $N^H h$ as a distribution

$$\langle \varphi, N^H h \rangle = \int_H \operatorname{grad} \varphi \cdot \operatorname{grad} h \, \mathrm{d}\mathscr{H}_m$$

for $\varphi \in \mathscr{D}$, the space of all infinitely differentiable functions in $\mathbf{R}^m$ with a compact support spt φ. Here $\mathscr{H}_k$ is the k-dimensional Hausdorff measure. We formulate the Neumann problem for the Laplace equation with a boundary condition $\mu \in \mathscr{C}'(\partial H)$ as follows: determine a harmonic function h on H for which $N^H h = \mu$. We wish to find the function h in the form of the single layer potential

$$U\nu(x) = \int_{\mathbf{R}^m} h_x(y) d\nu(y)$$

where $\nu \in \mathscr{C}'(\partial H)$,

$$h_x(y) = (m-2)^{-1} A^{-1} |x-y|^{2-m} \quad \text{for } m \geqslant 2,$$
$$A^{-1} \log |x-y|^{-1} \quad \text{for } m = 2,$$

A is the area of the unit sphere in $\mathbf{R}^m$.

For a Borel set B denote

$$v_r^B(x) = \sup \Big\{ \int_B \operatorname{grad} h_x(y) \cdot \operatorname{grad} \varphi(y) \, \mathrm{d}\mathscr{H}_m(y);$$
$$\varphi \in \mathscr{D}, \operatorname{spt} \varphi \subset U(x;r) - \{x\}, |\varphi| \leqslant 1 \Big\}.$$

Here $U(x;r) = \{y \in \mathbf{R}^m; |x-y| < r\}$. More generally, we denote $U(M;r) = \{y \in \mathbf{R}^m;\ \operatorname{dist}(y, M) < r\}$ where M is a set, r is a positive constant and $\operatorname{dist}(y, M)$ is the distance of the point y from the set M. Further, denote

$$V^B = \sup_{x \in \partial H} v_\infty^B(x),$$

$$\tau_B = \{z \in \partial B; \exists \varrho > 0 \lim_{r \to 0} \sup_{y \in U(z;\varrho)} V_r^B(y) = 0\}.$$

The operator $N^H U$ is a bounded linear operator on $\mathscr{C}'(\partial H)$ if and only if $V^H < \infty$. Under the assumption $V^H < \infty$ we look for a solution of the Dirichlet problem for the Laplace equation on the set $\mathbf{R}^m - \operatorname{cl} H$ with the boundary condition g (where $\operatorname{cl} H$ is the closure of the set H) in the form $u(x) = \langle f, N^H h_x \rangle$, where $f \in \mathscr{C}(\partial H)$. A solution f of the problem satisfies

$$W^H f(x) \equiv \langle f, N^H h_x \rangle = g(x).$$

The operator $W^B: \mathscr{C}(\partial B) \to \mathscr{C}(\partial B)$ may be introduced analogously even for the Borel set B with a compact boundary under the assumption $V^B < \infty$ (see [K1]).

If $\| \ \|$ is a norm on $\mathscr{C}(\partial H)$ equivalent to the maximum norm and U is a continuous linear operator on $\mathscr{C}(\partial H)$ then we denote

$$\omega(U, \| \ \|) = \inf\{\|U - K\|; K \in \mathscr{K}\},$$

where $\mathscr{K}$ is the space of linear compact operators on $\mathscr{C}(\partial H)$. If $\omega(\beta W^H - I, \| \ \|)$ is less than 1 then the Riesz-Schauder theory permits to apply the Fredholm theorems to the dual equations

$$[I + (\beta W^H - I)]f = \beta g,$$
$$[I + (\beta N^H U - I)]\nu = \beta\mu.$$

Definition. Let $D \subset \mathbf{R}^m$, $x \in D$. A mapping $\psi: D \to \mathbf{R}^m$ is said to be *conformal at the point* x if there is $\delta > 0$ such that $U(x;\delta) \subset D$, $\psi \in \mathscr{C}^1(U(x;\delta))$ and the angle of the curves $\{\psi(x + t\theta_j); 0 \leqslant t < \delta\}$ $(j = 1, 2)$ at the point $\psi(x)$ is the same as the angle of the curves $\{x + t\theta_j; 0 \leqslant t < \delta\}$ $(j = 1, 2)$ at the point x for all pairs of unit vectors θ_1, θ_2.

The aim of this paper is to deduce the following result:

Theorem 1. *Let $D \subset \mathbf{R}^m$ be an open set, $\psi: D \to \mathbf{R}^m$ a homeomorphism, H a bounded Borel set, $V^H < \infty$, $\mathrm{cl}\, H \subset D$, let ψ be a diffeomorphism of the class $C^{1+\alpha}$ on a neighborhood of ∂H where $\alpha > 0$, let ψ be conformal on $\partial H - \tau_H$. If $\| \ \|$ is a norm on $\mathscr{C}(\partial H)$ equivalent to the maximum norm we define on $(\partial\psi(H))$ the norm $\|\varphi\|_\psi \equiv \|\varphi \circ \psi\|$. Then*

$$\omega(\beta I + W^H, \| \ \|) = \omega(\beta I + W^{\psi(H)}, \| \ \|_\psi)$$

for every real number β. (Here I is the identical operator.)

Convention. In the sequel we will consider an open set $D \subset \mathbf{R}^m$, a homeomorphism $\psi: D \to \mathbf{R}^m$ and a bounded Borel set H such that $\mathrm{cl}\, H \subset D$ and the mapping ψ is a diffeomorphism of class $C^{1+\alpha}$ in a neighborhood of ∂H, where $0 < \alpha < 1$.

Lemma 1. $V^{\psi(H)} < \infty \Leftrightarrow V^H < \infty$.

P r o o f. See [ME1], Theorem 2. □

Definition. Let $B \subset \mathbf{R}^m$ be a Borel set, $y \in \mathbf{R}^m$. A unit vector θ is termed the *interior normal of B at y in Federer's sense*, if the symmetric difference of B and the half-space $\{x \in \mathbf{R}^m; (x-y)\cdot\theta > 0\}$ has m-dimensional density zero at y. If there is such a vector θ, then it is unique and we will denote it by $n^B(y)$; if there is no interior normal of B at y in this sense, we denote by $n^B(y)$ the zero vector in $\mathbf{R}^m$. The set $\{y \in \mathbf{R}^m; |n^B(y)| > 0\}$ is called the reduced boundary of B and will be denoted by $\partial_r B$.

Lemma 2. *$\partial_r\psi(H) = \psi(\partial_r H)$, and $n^{\psi(H)}(\psi(x))$ is a normal vector to the surface $\psi(\{z \in D; (z-x)\cdot n^H(x) = 0\})$ at $\psi(x)$ for each $x \in \partial_r H$.*

Proof. See [ME1], Lemma 7. □

Note 1. Let $B \subset \mathbf{R}^m$ be an open set, $\varphi\colon B \to \mathbf{R}^m$ a diffeomorphism of class C^1, K a constant such that $\|D\varphi^{-1}(x)\| \leqslant K$ for each $x \in \varphi(B)$, where $D\varphi^{-1}(x)$ is the differential of the mapping φ^{-1} at the point x. Then $|D\varphi(x)u| \geqslant |u|/K$ for every $x \in B$, $u \in \mathbf{R}^m$.

Proof. See [ME1], Note 1. □

Lemma 3. *Let $x \in \partial H$, let ψ be conformal at the point x. Then for every $\varepsilon > 0$ there is $r > 0$ such that for every $z \in \partial_r H$, $|z-x| < r$*

$$\left|n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z) - \|D\psi(x)\|\right| < \varepsilon.$$

Proof. Suppose that $\varepsilon < 1$. By the assumption there are positive constants r_0, L such that ψ is a diffeomorphism of class C^1 on $\{z;\ \mathrm{dist}(z,\partial H) < r_0\}$ and for each z, $\mathrm{dist}(z,\partial H) < r_0$ we have $\|D\psi(z)\| \leqslant L$, $\|D\psi^{-1}(\psi(z))\| \leqslant L$. There is $r \in (0, r_0)$ such that for every $z, |z-x| < r$ we have $\|D\psi(x) - D\psi(z)\| < \varepsilon/(2L^4 m)$. Now let $z \in \partial_r H$, $|z-x| < r$. Since $D\psi(z)$ maps the tangent plane of H at the point z to the tangent plane of $\psi(H)$ at the point $\psi(z)$ according to Lemma 2, there are tangential vectors $v^1, \ldots, v^{m-1}$ of H at the point z such that $n^{\psi(H)}(\psi(z))$, $D\psi(z)v^1/\|D\psi(z)v^1\|, \ldots, D\psi(z)v^{m-1}/\|D\psi(z)v^{m-1}\|$ form an orthonormal system. Thus

$$\|D\psi(z)n^H(z)\|^2 = \sum_{i=1}^{m-1}\left(\frac{D\psi(z)v^i}{\|D\psi(z)v^i\|}\cdot D\psi(z)n^H(z)\right)^2 + (n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z))^2.$$

Since ψ is conformal at the point x and thus

$$\frac{D\psi(x)n^H(z)}{\|D\psi(x)n^H(z)\|}\cdot\frac{D\psi(x)v^i}{\|D\psi(x)v^i\|} = n^H(z)\cdot v^i = 0,$$

we have

$$
\begin{aligned}
\|D\psi(z)n^H(z)\|^2 &= (n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z))^2\\
&\geqslant \|D\psi(z)n^H(z)\|^2 - \sum_{i=1}^{m-1}\Big[\Big|\frac{D\psi(z)v^i}{\|D\psi(z)v^i\|}\cdot(D\psi(z)-D\psi(x))n^H(z)\Big|\\
&\quad + \Big|D\psi(x)n^H(z)\cdot\Big(\frac{D\psi(z)v^i}{\|D\psi(z)v^i\|}-\frac{D\psi(x)v^i}{D\psi(z)v^i\|}\Big)\Big|\\
&\quad + \Big|D\psi(x)n^H(z)\cdot\frac{D\psi(x)v^i}{\|D\psi(z)v^i\|}\Big|\Big]^2\\
&\geqslant \|D\psi(z)n^H(z)\|^2 - \sum_{i=1}^{m-1}\Big(\frac{\varepsilon}{2L^4m}+\frac{\varepsilon}{2L^2m}\Big)^2 \geqslant\\
&\geqslant \|D\psi(z)n^H(z)\|^2 - \frac{\varepsilon^2}{L^4m} \geqslant \|D\psi(z)n^H(z)\|^2\Big(1-\frac{\varepsilon^2}{mL^2}\Big),
\end{aligned}
$$

because $\|D\psi(z)v^i\| \geqslant 1/L$, $\|D\psi(z)n^H(z)\| \geqslant 1/L$ according to Note 1. Therefore

$$
\begin{aligned}
\|D\psi(z)n^H(z)\| &\geqslant |n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z)| \geqslant \|D\psi(z)n^H(z)\|\sqrt{1-\frac{\varepsilon^2}{2L^2}}\\
&\geqslant \|D\psi(z)n^H(z)\|(1-\frac{\varepsilon^2}{2L^2}) \geqslant \|D\psi(z)n^H(z)\| - \frac{\varepsilon}{2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\big|\|D\psi(x)n^H(z)\| - |n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z)|\big|\\
&\quad\leqslant \big|\|D\psi(z)n^H(z)\| - |n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z)|\big| + \|D\psi(x)-D\psi(z)\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}.
\end{aligned}
$$

It suffices to prove that $n^{\psi(H)}(\psi(z))\cdot D\psi(z)n^H(z) \geqslant 0$. The vector $n^{\psi(H)}(\psi(z))$ points toward the set $\psi(\{z+v;\ v\cdot n^H(z)>0\})$. Similarly, the vector $n^H(z)$ points into the set $\psi^{-1}(\{\psi(z)+v;\ v\cdot n^{\psi(H)}(\psi(z))>0\})$. Therefore, there is $\tau_0>0$ such that for $\tau\in(0,\tau_0)$ we have $z+\tau n^H(z)\in\psi^{-1}(\{\psi(z)+v;\ v\cdot n^{\psi(H)}(\psi(z))>0\})$ and thus $\psi(z+\tau n^H(z))\in\{\psi(z)+v;\ v\cdot n^{\psi(H)}(\psi(z))>0\}$, which yields $(\psi(z+\tau n^H(z))-\psi(z))\cdot n^{\psi(H)}(\psi(z))>0$. Therefore

$$
D\psi(z)n^H(z)\cdot n^{\psi(H)}(\psi(z)) = \lim_{\tau\to 0_+}\frac{\psi(z+\tau n^H(z))-\psi(z)}{\tau}\cdot n^{\psi(H)}(\psi(z)) \geqslant 0.
$$

□

Lemma 4. *Let $x\in\partial H$, let ψ be conformal at the point x. Then there are $R>0$ and a constant K such that for $r\in(0,R)$, $y\in\partial H$, $z\in\partial_r H$, $|y-x|<r$, $|x-z|<r$, $y\neq z$ we have*

$$
\Big|\frac{|z-y|^m}{|\psi(z)-\psi(y)|^m}-\frac{1}{\|D\psi(x)\|^m}\Big| \leqslant Kr^\alpha.
$$

Proof. By the assumption there are positive constants L, R such that ψ is a diffeomorphism of class $C^{1+\alpha}$ on $U(\partial H, R)$, for $y, z \in U(\partial H, R)$ we have $\|D\psi(z)\| \leqslant L$, $\|D\psi(z) - D\psi(y)\| \leqslant L|z-y|^\alpha$; for $y, z \in \psi(U(\partial H, R))$ we have $\|D\psi^{-1}(z)\| \leqslant L$, $|\psi^{-1}(z) - \psi^{-1}(y) \leqslant L|z-y|$.

Now let $y \in \partial H$, $z \in \partial_r H$, $y \neq z$, $|y-x| < r < R$, $|z-x| < r$. Then

$$\left|\frac{|y-z|^m}{|\psi(z)-\psi(y)|^m} - \frac{1}{\|D\psi(x)\|^m}\right| = \frac{\left|\|D\psi(x)\|^m - \left|\int_0^1 D\psi(z+t(y-z))\frac{z-y}{|z-y|}\,\mathrm{d}t\right|^m\right|}{\frac{|\psi(z)-\psi(y)|^m}{|z-y|^m}\|D\psi(x)\|^m}.$$

Since $|z-y|/|\psi(z)-\psi(y)| \leqslant L$ by the assumption and $\|D\psi(x)\| \geqslant 1/L$ according to Note 1, we have

$$\begin{aligned}\left|\frac{|z-y|^m}{|\psi(z)-\psi(y)|^m} - \frac{1}{\|D\psi(x)\|^m}\right| &\leqslant L^{2m}\left|\|D\psi(x)\|^m - \left|\int_0^1 D\psi(z+t(y-z))\frac{y-z}{|y-z|}\,\mathrm{d}t\right|^m\right| \\ &= L^{2m}\left|\|D\psi(x)\| - \left|\int_0^1 D\psi(z+t(y-z))\frac{y-z}{|y-z|}\,\mathrm{d}t\right|\right| \times \\ &\quad \times \sum_{i=0}^{m-1}\|D\psi(x)\|^i\left|\int_0^1 D\psi(z+t(y-z))\frac{y-z}{|y-z|}\,\mathrm{d}t\right|^{m-1-i} \\ &\leqslant mL^{3m-1}\left|\|D\psi(x)\| - \left|\int_0^1 D\psi(z+t(y-z))\frac{y-z}{|y-z|}\,\mathrm{d}t\right|\right|.\end{aligned}$$

Since ψ is conformal at x we have $\|D\psi(x)\| = \|D\psi(x)\frac{y-z}{|y-z|}\|$ and thus

$$\begin{aligned}&\left|\frac{|z-y|^m}{|\psi(z)-\psi(y)|^m} - \frac{1}{\|D\psi(x)\|^m}\right| \\ &\quad\leqslant mL^{3m-1}\left|D\psi(x)\frac{y-z}{|y-z|} - \int_0^1 D\psi(z+t(y-z))\frac{y-z}{|y-z|}\,\mathrm{d}t\right| \\ &\quad= mL^{3m-1}\left|\int_0^1 (D\psi(x) - D\psi(z+t(y-z)))\frac{y-z}{|y-z|}\,\mathrm{d}t\right| \leqslant mL^{3m}r^\alpha.\end{aligned}$$

□

Lemma 5. *If* $y \in \partial_r H$, $u \in \mathbf{R}^m$ *then*

$$n^{\psi(H)}(\psi(y)) \cdot D\psi(y)u = (u \cdot n^H(y))(n^{\psi(H)}(\psi(y)) \cdot D\psi(y)n^H(y)).$$

Proof. See [ME1], Lemma 8. □

Lemma 6. *There exist $R > 0$ and a constant K such that for every $y \in \partial H$, $z \in \partial_r H$, $0 < |y - z| < R$ we have*

$$\Big|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(H)}(\psi(z)) - \frac{1}{A|\psi(z) - \psi(y)|^m}[(z - y) \cdot n^H(z)]\times$$
$$\times[n^{\psi(H)}(\psi(z)) \cdot D\psi(z)n^H(z)]\Big| \leqslant K|y - z|^{1+\alpha-m}.$$

Proof. By assumption there are positive constants L, R such that ψ is a diffeomorphism of class $C^{1+\alpha}$ on $U(\partial H, R)$, for $y, z \in U(\partial H, R)$ we have $\|D\psi(z) - D\psi(y)\| \leqslant L|z - y|^\alpha$; for $y, z \in \psi(U(\partial H, R))$ we have $|\psi^{-1}(z) - \psi^{-1}(y)| \leqslant L|y - z|$. Now let $y \in \partial H$, $z \in \partial_r H$, $0 < |y - z| < R$. Then

$$\begin{aligned}\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(H)}(\psi(z)) &= \frac{1}{A}\frac{\psi(z) - \psi(y)}{|\psi(z) - \psi(y)|^m} \cdot n^{\psi(H)}(\psi(z))\\
&= \frac{1}{A|\psi(z) - \psi(y)|^m}\int_0^1 D\psi(z + t(y - z))(z - y) \cdot n^{\psi(H)}(\psi(z))\,\mathrm{d}t\\
&= \frac{1}{A|\psi(z) - \psi(y)|^m}\Big\{D\psi(z)(z - y) \cdot n^{\psi(H)}(\psi(z))\\
&\quad + \int_0^1 [D\psi(z + t(y - z)) - D\psi(z)](y - z) \cdot n^{\psi(H)}(\psi(z))\,\mathrm{d}t\Big\}\\
&= \frac{1}{A|\psi(z) - \psi(y)|^m}\Big\{[(z - y) \cdot n^H(z)] \cdot [n^{\psi(H)}(\psi(z)) \cdot D\psi(z)n^H(z)]\\
&\quad + \int_0^1 [D\psi(z + t(y - z)) - D\psi(z)](y - z) \cdot n^{\psi(H)}(\psi(z))\,\mathrm{d}t\Big\}.\end{aligned}$$

according to Lemma 5. Therefore

$$\Big|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(H)}(\psi(z)) - \frac{1}{A|\psi(z) - \psi(y)|^m}[(z - y) \cdot n^H(z)]\times$$
$$\times[n^{\psi(H)}(\psi(z)) \cdot D\psi(z)n^H(z)]\Big| \leqslant \frac{L|y - z|^{1+\alpha}}{A|\psi(z) - \psi(y)|^m} \leqslant \frac{L^{m+1}}{A}|y - z|^{1+\alpha-m}.$$

□

Lemma 7. $\tau_{\psi(H)} = \psi(\tau_H)$.

Proof. See [ME1], Lemma 14. □

Lemma 8. *If $V^H < \infty$ then there is a positive constant K such that for every $x \in \mathbf{R}^m$, $r \in (0, 1)$ we have*

$$\int_{\partial_r H \cap U(x;r)} |x - y|^{\alpha+1-m}\,\mathrm{d}\mathcal{H}_{m-1}(y) \leqslant Kr^\alpha.$$

Proof. See [K 1], Corollary 2.17, [ME1], Lemma 9. □

Lemma 9. *Let $V^H < \infty$, let ψ be conformal on $\partial H - \tau_H$. Then for $x \in \partial H$, $\varepsilon > 0$ there is $R > 0$ such that for each $y \in \partial H \cap U(x; R)$, $\varphi \in \mathscr{C}(\partial H)$, $\|\varphi\| \leqslant 1$, $r \in (0, R)$ we have*

$$\left| \int_{\psi(\partial H \cap U(x;r))} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} - \int_{\partial H \cap U(x;r)} \varphi n^H \cdot \operatorname{grad} h_y \right| < \varepsilon.$$

Proof. First, let $x \in \tau_H$. According to Lemma 7 we have $\psi(x) \in \tau_{\psi(H)}$. Therefore there is a positive number ϱ such that for every $y \in \partial H \cap U(x;\varrho)$, $z \in \partial\psi(H) \cap U(\psi(x);\varrho)$ we have $v_\varrho^H(y) < \frac{1}{2}\varepsilon$, $v_\varrho^{\psi(H)}(z) < \frac{1}{2}\varepsilon$. Since ψ is a homeomorphism there is $R \in (0, \frac{1}{2}\varrho)$ such that $\psi(U(x;R)) \subset U(\psi(x); \frac{1}{2}\varrho)$. Thus for $y \in \partial H \cap U(x;R)$, $r \in (0,R)$, $\varphi \in \mathscr{C}(\partial H)$, $\|\varphi\| \leqslant 1$ we have

$$\left| \int_{\psi(\partial H \cap U(x;r))} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} - \int_{\partial H \cap U(x;r)} \varphi n^H \cdot \operatorname{grad} h_y \right| \leqslant v_\varrho^{\psi(H)}(\psi(y)) + v_{2R}^H(y) < \varepsilon$$

according to [K1], Lemma 2.15 and Lemma 2.8.

Now let $x \in \partial H - \tau_H$. We may suppose that $\varepsilon < (V^H + 1)/(\|D\psi(x)\|^{m-1} + 1)$. By assumption there are constants $r_0 \in (0,1)$, $L \geqslant 1$ such that ψ is a diffeomorphism on $U(\partial H; r_0)$, for $u, v \in U(\partial H; r_0)$ we have $\|D\psi(u)\| \leqslant L$, $\|D\psi(u) - D\psi(v)\| \leqslant L|u-v|^\alpha$, for $u, v \in \psi(U(\partial H; r_0))$ we have $\|D\psi^{-1}(u)\| \leqslant L$, $|\psi^{-1}(u) - \psi^{-1}(v)| \leqslant L|u - v|$. By Lemma 6 and Lemma 4 there are positive constants K, r_1 such that $r_1 \in (0, r_0)$ and for $y \in \partial H$, $z \in \partial_r H$, $|x - y| < R$, $|x - z| < R$, $R \in (0, r_1)$ we have

$$\begin{aligned}\Big|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(H)}(\psi(z)) - \frac{1}{\|D\psi(x)\|^m} \operatorname{grad} h_y(z) \cdot n^H(z) \times \\ \times [n^{\psi(H)}(\psi(z)) \cdot D\psi(z) n^H(z)]\Big| \\ \leqslant K|y - z|^{1+\alpha-m} + KR^\alpha |\operatorname{grad} h_y(z) \cdot n^H(z)| \cdot \|D\psi(z)\| \\ \leqslant KL\big(|y - z|^{1+\alpha-m} + R^\alpha |\operatorname{grad} h_y(z) \cdot n^H(z)|\big).\end{aligned}$$

Thus

$$\begin{aligned}|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(H)}(\psi(z)) - \|D\psi(x)\|^{1-m} \operatorname{grad} h_y(z) \cdot n^H(z)| \\ \leqslant KL(|y - z|^{1+\alpha-m} + R^\alpha |\operatorname{grad} h_y(z) \cdot n^H(z)|) \\ + \|D\psi(x)\|^{1-m} |\operatorname{grad} h_y(z) \cdot n^H(z)| \times \\ \times \left|1 - \frac{1}{\|D\psi(x)\|} [D\psi(z) n^H(z) \cdot n^{\psi(H)}(\psi(z))]\right|.\end{aligned}$$

Put

$$(1) \qquad B = \frac{\|D\psi(x)\|^{m-1}\varepsilon}{2(V^H+1)},$$

$$(2) \qquad C = \min\left(r_1, \left[\frac{\varepsilon}{8KL^m(V^H+1)}\right]^{1/\alpha}, \left\{\frac{\|D\psi(x)\|}{L}[1-(1-B)^{\frac{1}{m-1}}]\right\}^{1/\alpha}, \left\{\frac{\|D\psi(x)\|}{L}[(1+B)^{\frac{1}{m-1}}-1]\right\}^{1/\alpha}\right).$$

By Lemma 3 there is $r_2 \in (0, C)$ such that for $R \in (0, r_2)$ we have

$$\begin{aligned}
&|\operatorname{grad} h_{\psi(y)}(\psi(z)) \cdot n^{\psi(H)}(\psi(z)) - \|D\psi(x)\|^{1-m} \operatorname{grad} h_y(z) \cdot n^H(z)| \\
&\qquad \leqslant KL\left[|y-z|^{1+\alpha-m} + \left|R^\alpha + \frac{\varepsilon}{4L^m(V^H+1)K}\right| |\operatorname{grad} h_y(z) \cdot n^H(z)|\right] \\
&\qquad \leqslant KL[|y-z|^{1+\alpha-m} + \frac{3\varepsilon}{8L^m(V^H+1)K} |\operatorname{grad} h_y(z) \cdot n^H(z)|].
\end{aligned}$$

If $\varphi \in \mathscr{C}(\partial H)$, $\|\varphi\| \leqslant 1$, $r \in (0, R)$ then the Lipschitz condition for ψ^{-1} yields

$$\begin{aligned}
&\left| \int_{\psi(\partial_r H \cap U(x;r))} \varphi_0 \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} - \int_{\partial_r H \cap U(x;r)} \varphi n^H \cdot \operatorname{grad} h_y \right| \\
&\quad \leqslant \Big| \int_{\psi(\partial_r H \cap U(x;r))} \varphi_0 \psi^{-1} \|D\psi(x)\|^{1-m} \operatorname{grad} h_y(\psi^{-1}(z)) \cdot n^H(\psi^{-1}(z)) \, \mathrm{d}\mathscr{H}_{m-1}(z) \\
&\qquad - \int_{\partial_r H \cap U(x;r)} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathscr{H}_{m-1} \Big| \\
&\qquad + KL \int_{\psi(\partial_r H \cap U(x;r))} |y - \psi^{-1}(z)|^{1+\alpha-m} \, \mathrm{d}\mathscr{H}_{m-1}(z) \\
&\qquad + \frac{3\varepsilon}{8L^{m-1}(V^H+1)} \cdot \int_{\psi(\partial_r H \cap U(x;r))} |\operatorname{grad} h_y(\psi^{-1}(z)) \cdot n^H(\psi^{-1}(z))| \, \mathrm{d}\mathscr{H}_{m-1}(z) \\
&\quad \leqslant \Big| \int_{\psi(\partial_r H \cap U(x;r))} \varphi_0 \psi^{-1} \|D\psi(x)\|^{1-m} \cdot \operatorname{grad} h_y(\psi^{-1}(z)) \cdot n^H(\psi^{-1}(z)) \, \mathrm{d}\mathscr{H}_{m-1}(z) \\
&\qquad - \int_{\partial_r H \cap U(x;r)} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathscr{H}_{m-1} \Big| + KL^m \int_{\partial_r H \cap U(x;r)} |y - z|^{1+\alpha-m} \, \mathrm{d}\mathscr{H}_{m-1}(z) \\
&\qquad + \frac{3}{8} \frac{\varepsilon}{V^H+1} \int_{\partial_r H \cap U(x;r)} |\operatorname{grad} h_y(z) \cdot n^H(z)| \, \mathrm{d}\mathscr{H}_{m-1}(z)
\end{aligned}$$

$$\leqslant \Big| \int_{\psi(\partial_r H\cap U(x;r))} \varphi_0\psi^{-1}\|D\psi(x)\|^{1-m}\,\mathrm{grad}\, h_y(\psi^{-1}(z))\cdot n^H(\psi^{-1}(z))\,\mathrm{d}\mathscr{H}_{m-1}(z)$$

$$- \int_{\partial_r H\cap U(x;r)} \varphi n^H\cdot \mathrm{grad}\, h_y\,\mathrm{d}\mathscr{H}_{m-1}\Big|$$

$$+ KL^m \int_{\partial_r H\cap U(x;r)} |y-z|^{1+\alpha-m}\,\mathrm{d}\mathscr{H}_{m-1}(z) + \frac{3}{8}\varepsilon.$$

By Lemma 8 there is $r_3 \in (0, r_2)$ such that for $R \in (0, r_3)$, $r \in (0, R)$ we have

$$(3)\qquad \Big| \int_{\psi(\partial_r H\cap U(x;r))} \varphi_0\psi^{-1} n^{\psi(H)}\cdot \mathrm{grad}\, h_{\psi(y)} - \int_{\partial_r H\cap U(x;r)} \varphi n^H\cdot \mathrm{grad}\, h_y\Big|$$

$$\leqslant \Big| \int_{\psi(\partial_r H\cap U(x;r))} \varphi_0\psi^{-1}\|D\psi(x)\|^{1-m}\,\mathrm{grad}\, h_y(\psi^{-1}(z))\cdot n^H(\psi^{-1}(z))\,\mathrm{d}\mathscr{H}_{m-1}(z)$$

$$- \int_{\partial_r H\cap U(x;r)} \varphi n^H\cdot \mathrm{grad}\, h_y\,\mathrm{d}\mathscr{H}_{m-1}\Big| + \frac{\varepsilon}{2}.$$

We know that for $u, v \in U(x;r)$

$$\begin{aligned}|\psi(u)-\psi(v)| &= |D\psi(u+\theta(v-u))(u-v)|\\ &= |D\psi(x)(u-v) + [D\psi(u+\theta(v-u)) - D\psi(x)](u-v)|\end{aligned}$$

holds, where $\theta \in (0,1)$. Since ψ is conformal at the point x, we have

$$|D\psi(x)(u-v)| = \|D\psi(x)\|\,|u-v|.$$

By the assumption

$$\big|[D\psi(u+\theta(v-u)) - D\psi(x)](u-v)\big| \leqslant Lr^\alpha |u-v|.$$

Hence

$$(\|D\psi(x)\| - Lr^\alpha)|u-v| \leqslant |\psi(u)-\psi(v)| \leqslant (\|D\psi(x)\| + Lr^\alpha)|u-v|.$$

Thus for every Borel nonnegative function f we have

$$\begin{aligned}\int_{\partial_r H\cap U(x;r)} f(\|D\psi(x)\| - Lr^\alpha)^{m-1} &\leqslant \int_{\psi(\partial_r H\cap U(x;r))} f\circ\psi^{-1}\\ &\leqslant (\|D\psi(x)\| + Lr^\alpha)^{m-1} \int_{\partial_r H\cap U(x;r)} f.\end{aligned}$$

Since r < *C* we háve (see (1), (2))

d_rH(\U{x\r) *ý(d_rHnU(x\r))*

d_rfřOC/(a?;r)

Hence

Jf\\rhi>(*)\r'djein.i - *j for'dje_m.:\ <*

d_r/fnl/(*;r) ^l(d_rffnl/(ai;n))

||m-l

* 2(v* + i) y^m-

d_rHC\U(x\r)

We get this relation for every integrable function / by decomposing it into the positive and negative parts. Thus we get from (3)

i <po xl)'^ln^{H)} • grád*h^_{y)}* - *[<pn^H* • grád *h_y*

if>(d_rHnU(x_tr)) *d_rHC\U(x_tr)*

l)M*_)li^m^f / IP^-g^M d J ř ^ + £

^ 2(V" + 1) /J HDV-WII"^1-1

d_rJ/nl/(r;r)

iiz?v>(x)ir-^i_g V" Z

* 2(V^ff +1) ||DV(*)||^m-^1 2

D

Lemma 10. *Let V^H* < co, *let* V> be conforma/ on *dH — TH- Then for every e* > 0, *o 6 ## tAere is í > 0 such that for each *x,y* € d#, |y - aio| < ^» |* — *o| < í, y> € *V(dH),* \\<*p*\\ < 1 we háve

\(*í* *<po1,-^ln*W-e»dhn*)dJGn-i-* *[* ^" g r a d M ^ m - i)

^1 v *Ja^H)* *JaH* '

-(f *(po^-tntW-gtsAh^dJíSn-i-* *I* ^n^w g r a d h_y d J ^ _n_l)|<c.

^K *Ja^,(H)* *JaH* *'i*

P r o o f. By Lemma 9 there is r > 0 such that for *y* € *dH,* *y* - z<j| < r we háve

| /'^o^-^1n*<^Jř)-grad^_(f)dJR_l_i

*<l>(U(*_0;r)naH)*

/vn* gradh,d^,_i < -.

*U(*_p;r)ndH*

If $x, y \in \partial H$, $|x - x_0| < \frac{1}{2}r$, $|y - x_0| < \frac{1}{2}r$ then

$$\Big|\Big(\int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(x)} \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_x \, \mathrm{d}\mathcal{H}_{m-1}\Big)$$
$$- \Big(\int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathcal{H}_{m-1}\Big)\Big| <$$
$$< \frac{\varepsilon}{2} + \int\limits_{\psi(\partial_r H - U(x_0, r))} |\operatorname{grad} h_{\psi(x)} - \operatorname{grad} h_{\psi(y)}| \, \mathrm{d}\mathcal{H}_{m-1}$$
$$+ \int\limits_{\partial_r H - U(x_0, r)} |\operatorname{grad} h_x - \operatorname{grad} h_y| \, \mathrm{d}\mathcal{H}_{m-1}.$$

Since $\operatorname{grad} h_u(v)$, $\operatorname{grad} h_{\psi(u)}\psi(v))$ are finite continuous functions on the compact $\{(u, v) \in \operatorname{cl} U(x_0; \frac{1}{2}r) \times (\partial_r H - U(x_0; r))\}$ and $\mathcal{H}_{m-1}(\partial_r H) < \infty$, $\mathcal{H}_{m-1}(\partial_r \psi(H)) < \infty$ according to [K1], Corollary 2.17 and Lemma 2, there is $\delta \in (0, \frac{1}{2}r)$ such that for $x, y \in \partial H$, $|x - x_0| < \delta$, $|y - x_0| < \delta$, $z \in (\partial_r H - U(x_0; r))$ we have

$$|\operatorname{grad} h_x(z) - \operatorname{grad} h_y(z)| < \frac{\varepsilon}{4(\mathcal{H}_{m-1}(\partial_r H) + 1)},$$
$$|\operatorname{grad} h_{\psi(x)}(\psi(z)) - \operatorname{grad} h_{\psi(y)}(\psi(z))| < \frac{\varepsilon}{4(\mathcal{H}_{m-1}(\partial_r \psi(H)) + 1)}.$$

Therefore

$$\Big|\Big(\int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(x)} \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_x \, \mathrm{d}\mathcal{H}_{m-1}\Big)$$
$$- \int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathcal{H}_{m-1}\Big| < \varepsilon.$$

□

N o t a t i o n. Let $x \in \mathbf{R}^m$, let B be a Borel set. Then we denote by

$$d_B(x) = \lim_{r \to 0_+} \frac{\mathcal{H}_m(B \cap U(x; r))}{\mathcal{H}_m(U(x; r))}$$

the m-dimensional density of the set B at the point x, if this density exists.

N o t a t i o n. For $B \subset \mathbf{R}^m$ we call the set of $y \in \mathbf{R}^m$ for which

$$\mathcal{H}_m(U(y; r) \cap B) > 0 \quad \text{and} \quad \mathcal{H}_m(U(y; r) - B) > 0$$

for each $r > 0$ the essential boundary of B and denote it by $\partial_e B$.

Lemma 11. *Let $V^H < \infty$, let ψ be conformal on $\partial H - \tau_H$. Then $d_{\psi(H)}(\psi(x)) = d_H(x)$ for each $x \in \partial H$.*

Proof. First, let $x \in \partial_e H$. Put $\varphi \equiv 1$. According to Lemma 1 and [K1], Theorem 2.19 and Theorem 2.15 we have $W^H \varphi \in \mathscr{C}(\partial H)$, $W^{\psi(H)}(\varphi \circ \psi^{-1}) \in \mathscr{C}(\psi(\partial H))$. By [K1], Lemma 2.15 and Lemma 2.8 we have for $y \in \partial H$

$$W^H \varphi(y) = d_H(y) + \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathscr{H}_{m-1},$$

$$W^{\psi(H)}(\varphi \circ \psi^{-1})(\psi(y)) = d_{\psi(H)}(\psi(y)) + \int_{\partial \psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathscr{H}_{m-1}.$$

Choose $\varepsilon > 0$. The continuity of $W^H \varphi$ and $W^{\psi(H)}$ and $W^{\psi(H)}(\varphi \circ \psi^{-1})$ yields that there is $r > 0$ such that for $y \in \partial H$, $|y - x| < r$ we have

$$|W^H \varphi(y) - W^H \varphi(x)| < \varepsilon,$$
$$|W^{\psi(H)}(\varphi \circ \psi^{-1})(\psi(y)) - W^{\psi(H)}(\varphi \circ \psi^{-1})(\psi(x))| < \varepsilon.$$

Hence

$$\begin{aligned}
&|d_H(x) - d_{\psi(H)}(\psi(x))| \\
&\quad \leqslant |d_H(y) - d_{\psi(H)}(\psi(y))| \\
&\qquad + |W^{\psi(H)}(\varphi \circ \psi^{-1})(\psi(y)) - W^{\psi(H)}(\varphi \circ \psi^{-1})(\psi(x))| \\
&\qquad + |W^H \varphi(y) - W^H \varphi(x)| \\
&\qquad + \Big| \Big(\int_{\partial \psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(x)} \, \mathrm{d}\mathscr{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_x \, \mathrm{d}\mathscr{H}_{m-1} \Big) \\
&\qquad - \Big(\int_{\partial \psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathscr{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathscr{H}_{m-1} \Big) \Big| \\
&\quad < |d_H(y) - d_{\psi(H)}(\psi(y))| + 2\varepsilon \\
&\qquad + \Big| \Big(\int_{\partial \psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(x)} \, \mathrm{d}\mathscr{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_x \, \mathrm{d}\mathscr{H}_{m-1} \Big) \\
&\qquad - \Big(\int_{\partial \psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathscr{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathscr{H}_{m-1} \Big) \Big|.
\end{aligned}$$

By Lemma 10 there is $r_1 \in (0, r)$ such that for $y \in \partial H$, $|y - x| < r_1$ we have

$$|d_H(x) - d_{\psi(H)}(\psi(x))| < |d_H(y) - d_{\psi(H)}(\psi(y))| + 3\varepsilon.$$

$\partial_r H$ is dense in $\partial_e H$ by the Isoperimetric Lemma (see [K1], p. 50). Thus we can choose $y \in \partial_r H$ such that $|y - x| < r_1$. Since $\psi(y) \in \partial_r \psi(H)$ by Lemma 2 we have

$$|d_H(x) - d_{\psi(H)}(\psi(x))| < 3\varepsilon.$$

Thus $d_{\psi(H)}(\psi(x)) = d_H(x)$ because ε was arbitrary.

Now let $x \notin \partial_e H$. By the assumption there are positive constants R, L such that for $y, z \in U(x; R)$ we have $|\psi(y) - \psi(z)| \leqslant L|y-z|$ and for $y, z \in U(\psi(x); R)$ we have $|\psi^{-1}(y) - \psi^{-1}(z)| \leqslant L|y-z|$. If there is $r \in (0, R)$ such that $\mathcal{H}_m(H \cap U(x;r)) = 0$ then $\mathcal{H}_m(\psi(H) \cap U(\psi(x); r/L)) \leqslant \mathcal{H}_m(\psi(H \cap U(x;r))) \leqslant L^m \mathcal{H}_m(H \cap U(x;r)) = 0$ and thus

$$d_{\psi(H)}(\psi(x)) = d_H(x) = 0.$$

Similarly, if there is $r \in (0, R)$ such that $\mathcal{H}_m(U(x;r) - H) = 0$ then

$$d_{\psi(H)}(\psi(x)) = d_H(x) = 1.$$

□

Lemma 12. *Let $V^H < \infty$, let ψ be conformal on $\partial H - \tau_H$. Then*

$$V : \varphi \mapsto [W^{\psi(H)}(\varphi \circ \psi^{-1})] \circ \psi - W^H \varphi$$

is a compact mapping on $\mathscr{C}(\partial H)$.

Proof. By the Arzelà-Ascoli theorem it suffices to prove that $\{V\varphi;\ \varphi \in \mathscr{C}(\partial H), \|\varphi\| \leqslant 1\}$ is a set of uniformly continuous and uniformly bounded functions. If $\|\varphi\| \leqslant 1$ then by [K1], Theorem 2.5 we have

$$\|V\varphi\| \leqslant V^{\psi(H)} + V^H + 2 < \infty.$$

Thus it remains to prove that $\{V\varphi;\ \varphi \in \mathscr{C}(\partial\psi(H)), \|\varphi\| \leqslant 1\}$ is a set of uniformly continuous functions. Let $\varepsilon > 0$. By Lemma 10 there is a positive number $\delta(x)$ for each $x \in \partial H$ such that for every $y, z \in \partial H$, $|y - x| < \delta(x)$, $|z - x| < \delta(x) < \varphi \in \mathscr{C}(\partial H)$, $\|\varphi\| \leqslant 1$ we have

$$(4)\quad \Big| \Big(\int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathcal{H}_{m-1} \Big) - \Big(\int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(z)} \, \mathrm{d}\mathcal{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_z \, \mathrm{d}\mathcal{H}_{m-1} \Big) \Big| < \varepsilon.$$

Since ∂H is a compact set there exist $x^1, \dots, x^n \in \partial H$ such that $\bigcup\{U(x^i; \delta(x^i)/2);\ i = 1, \dots, n\} \supset \partial H$.

If we put

$$\delta = \min_{i=1,\dots,n} \frac{\delta(x^i)}{2},$$

(4) holds for each $y, z \in \partial H$, $|y-z| < \delta$, $\varphi \in \mathscr{C}(\partial H)$, $\|\varphi\| \leqslant 1$. By (4) Lemma 11 and [K1], Lemma 2.15 and Lemma 2.8 we have for $y, z \in \partial H$, $|y-z| < \delta$, $\varphi \in \mathscr{C}(\partial H)$, $\|\varphi\| \leqslant 1$ the estimate

$$
\begin{aligned}
|V\varphi(y) - V\varphi(z)| = \Big| &\Big[\varphi(y) d_{\psi(H)}(\psi(y)) + \int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(y)} \, \mathrm{d}\mathscr{H}_{m-1} \\
&- \varphi(y) d_H(y) - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_y \, \mathrm{d}\mathscr{H}_{m-1}\Big] \\
&- \Big[\varphi(z) d_{\psi(H)}(\psi(z)) + \int_{\partial\psi(H)} \varphi \circ \psi^{-1} n^{\psi(H)} \cdot \operatorname{grad} h_{\psi(z)} \, \mathrm{d}\mathscr{H}_{m-1} \\
&- \varphi(z) d_H(z) - \int_{\partial H} \varphi n^H \cdot \operatorname{grad} h_z \, \mathrm{d}\mathscr{H}_{m-1}\Big]\Big| \\
< \varepsilon. &
\end{aligned}
$$

□

Lemma 13. *Let $B \subset \mathbf{R}^m$, $\psi \colon B \to \mathbf{R}^m$ be an injective mapping which is conformal at a point $z \in B$. Then ψ^{-1} is conformal at the point $\psi(z)$.*

Proof. See [ME1], Lemma 16. □

Theorem. *Let $V^H < \infty$, let ψ be conformal on $\partial H - \tau_H$. If $\| \; \|$ is a norm on $\mathscr{C}(\partial H)$ equivalent to the maximum norm we define on $\mathscr{C}(\partial\psi(H))$ the norm $\|\varphi\|_\psi \equiv \|\varphi \circ \psi\|$. Then*

$$\omega(\beta I + W^H, \| \; \|) = \omega(\beta I + W^{\psi(H)}, \| \; \|_\psi)$$

for each real number β. (Here I is the identical operator.)

Proof. Since $V^H < \infty$, we have $V^{\psi(H)} < \infty$ by Lemma 1. By Lemma 7 and Lemma 13 it suffices to prove

$$\omega(\beta I + W^{\psi(H)}, \| \; \|_\psi) \leqslant \omega(\beta I + W^H, \| \; \|).$$

Let $\varepsilon > 0$. Then there exists a compact linear operator K on $\mathscr{C}(\partial H)$ such that

$$\|\beta I + W^H - K\| < \omega(\beta I + W^H, \| \; \|) + \varepsilon.$$

Since the mapping K_1 on $\mathscr{C}(\partial\psi(H))$ defined by

$$K_1\varphi = [K(\varphi \circ \psi)] \circ \psi^{-1}$$

is compact because it maps a system of uniformly bounded continuous functions to a set of uniformly bounded and uniformly continuous functions, we have by Lemma 12

$$\begin{aligned}\omega(\beta I + W^{\psi(H)}, \|\ \|_\psi) &= \omega(\beta I + W_1^H - K_1, \|\ \|_\psi)\\ &\leqslant \|\beta I + W_1^H - K_1\|_\psi\\ &= \|\beta I + W^H - K\| < \omega(\beta I + W^H, \|\ \|) + \varepsilon,\end{aligned}$$

where $W_1^H : \varphi \mapsto [W^H(\varphi \circ \psi)] \circ \psi^{-1}$. □

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