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Mathematica Bohemica, Vol. 119 (1994), No. 2, 143–149

119 (1994) MATHEMATICA BOHEMICA No. 2, 143–149

ABOUT A GENERALIZATION OF TRANSVERSALS

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(Received May 14, 1992)

Summary. The aim of this paper is to generalize several basic results from transversal theory, primarily the theorem of Edmonds and Fulkerson.

Keywords: Matroid, transversal, system of representatives

AMS classification: 05D15, 05B35

1. INTRODUCTION

There are two fundamental results concerning both transversals and matroids. The first was proved by Rado [14], who established a necessary and sufficient condition for a finite family of sets to possess a transversal which is independent in a given matroid. Perfect [13] extended this theorem to partial transversals. The second result, proved by Edmonds and Fulkerson [2] (and independently also by Mirsky and Perfect [12]), says that the partial transversals of a finite family of sets form a matroid.

There are plenty of generalizations of these two results. A comprehensive survey of this field is in [11], [12] and [16], for later results see e. g. [6], [7], [17]. In this paper we introduce $\mathscr{M}$-polytransversals, which are in fact characteristic vectors of some special (matroid relative) systems of representatives. We show that $\mathscr{M}$-polytransversals of a finite family of sets form an integral polymatroid. Using this fact we can extend the Rado–Perfect theorem and also the result of Ford and Fulkerson [3] about common transversals of two families of sets. Our results generalize the classical theorems known for transversals and also some recent results of [7] and [6].

[1] This research was partially supported by Grant of Slovak Academy of Sciences No. 88 and by EC Cooperation action IC 1000 "Algorithms for Future Technologies"

2. Preliminaries

We expect the reader to be familiar with the theory of matroids. All terminology related to matroids and polymatroids is essentially the same as that of Welsh [16].

By $\mathbb{Z}_+$ ($\mathbb{R}_+$) we denote the set of nonnegative integral (real) numbers and the symbol $\mathbb{Z}_+^S$ ($\mathbb{R}_+^S$) denotes the space of integer (real) valued nonnegative vectors with coordinates indexed by a finite set S. For each $\mathbf{u} \in \mathbb{R}_+^S$ and $s \in S$ denote the sth coordinate of $\mathbf{u}$ by $\mathbf{u}(s)$. For $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}_+^S$ we write $\mathbf{u} \leqslant \mathbf{v}$ iff $\mathbf{u}(s) \leqslant \mathbf{v}(s)$ for any $s \in S$. For $\mathbf{u} \in \mathbb{R}_+^S$ and $X \subseteq S$ define $\mathbf{u}(X) = \sum_{s \in X} \mathbf{u}(s)$, and call the quantity $|\mathbf{u}| = \mathbf{u}(S) = \sum_{s \in S} \mathbf{u}(s)$ the *modulus* $|\mathbf{u}|$ of $\mathbf{u}$.

A *polymatroid* $\mathbf{P}$ *on* S is a pair (S, ϱ) where S, are the *ground set*, is a nonempty finite set an ϱ, the *ground set rank function*, is a function $\varrho\colon 2^S \to \mathbb{R}_+$, such that ϱ is *nondecreasing* (i.e., $\varrho(X) \leqslant \varrho(Y)$ for any $X \subseteq Y \subseteq S$), *submodular* (i.e., $\varrho(X) + \varrho(Y) \geqslant \varrho(X \cup Y) + \varrho(X \cap Y)$ for any X, $Y \subseteq S$) and $\varrho(\emptyset) = 0$. The vectors $\mathbf{u} \in \mathbb{R}_+^S$ such that $\mathbf{u}(X) \leqslant \varrho(X)$ for all $X \subseteq S$ are the *independent vectors of* $\mathbf{P}$. For each vector $\mathbf{a} \in \mathbb{R}_+^S$, the *vector rank* $r(\mathbf{a})$ of $\mathbf{a}$ is given by

$$r(\mathbf{a}) = \min_{X \subseteq S} \big(\mathbf{a}(X) + \varrho(S \setminus X)\big) \tag{1}$$

or equivalently, $r(\mathbf{a}) = \max(|\mathbf{u}|;\ \mathbf{u} \leqslant \mathbf{a},\ \mathbf{u}$ is independent in $\mathbf{P})$.

A polymatroid $\mathbf{P} = (S, \varrho)$ is *integral* if ϱ is integral. Moreover, if $\varrho(\{s\}) = 0, 1$ for any $s \in S$ then $\mathbf{P}$ is a polymatroid of a matroid on S with rank function ϱ. The following theorem is one of the basic results of matroid theory (see [1], [10]).

Theorem 1. *Let* $\mathbf{P}_1 = (S, \varrho_1)$ *and* $\mathbf{P}_2 = (S, \varrho_2)$ *be two polymatroids on* S *and let* $k \in \mathbb{R}_+$. *Then there exists a vector* $\mathbf{u}$ *of* $\mathbb{R}_+^S$ *independent in both* $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *and with modulus at least* k *iff for any* $X \subseteq S$,

$$\varrho_1(X) + \varrho_2(S \setminus X) \geqslant k.$$

Furthermore, if $\mathbf{P}_1$ *and* $\mathbf{P}_2$ *are both integral we may insist that the independent vector* $\mathbf{u}$ *be integer valued.*

Throughout this paper S and T denote finite sets, $\mathscr{A}$ denotes the family $(A_t\colon t \in T)$ of subsets of S and $\mathscr{M}$ denotes the family $(M_s\colon s \in S)$ of matroids on T. For any $J \subseteq T$ and $s \in S$, denote

$$A(s, J) = \{t \in J; s \in A_t\} \quad (\subseteq T).$$

A family $(x_t\colon t \in J)$ $(J \subseteq T)$ of elements of S is called a *partial system of representatives* (in abbreviation *partial SR*) of $\mathscr{A}$ if $x_t \in A_t$ for any $t \in J$. $|J|$

$(|T \setminus J|)$ is called the *length (defect)* of the partial SR $(x_t : t \in J)$ of $\mathscr{A}$. A partial SR $(x_t : t \in J)$ of $\mathscr{A}$ will be called a *partial $\mathscr{M}$-system of representatives (partial $\mathscr{M}$-SR)* of $\mathscr{A}$ if the set $\{t \in J : x_t = s\}$ is independent in M_s for any $s \in S$.

If $(x_t : t \in J)$ is a partial $\mathscr{M}$-SR of $\mathscr{A}$, then the vector $\mathbf{u} \in \mathbb{Z}_+^S$ satisfying $\mathbf{u}(s) = |\{t \in J : x_t = s\}|$ for any $s \in S$ is called the *partial $\mathscr{M}$-polytransversal* of $\mathscr{A}$. We will call $|J|$ $(|T \setminus J|)$ the *length (defect)* of the partial $\mathscr{M}$-polytransversal $\mathbf{u}$. Clearly $\sum_{s \in S} \mathbf{u}(s) = |J|$.

As usual, the partial SR, partial $\mathscr{M}$-SR and partial $\mathscr{M}$-polytransversal of $\mathscr{A}$ with defect 0 are called the *system of representatives, $\mathscr{M}$-system of representatives* and *$\mathscr{M}$-polytransversal of $\mathscr{A}$*, respectively.

If $\mathscr{M}$ is a family of uniform matroids of rank 1 then the partial $\mathscr{M}$-polytransversals of $\mathscr{A}$ are the characteristic vectors of the classical partial transversals of $\mathscr{A}$. We dealt with $\mathscr{M}$-SR also in [7] and proved the following variant of Hall's theorem [4] for $\mathscr{M}$-SR.

Lemma 1. *Let $\mathscr{A} = (A_t : t \in T)$ be a finite family of subsets of a finite set S and let $\mathscr{M}$ be a family $(M_s : s \in S)$ of matroids on T with rank functions ϱ_s, respectively. Then the maximal length of a partial $\mathscr{M}$-system of representatives of $\mathscr{A}$ (thus also the maximal length of a partial $\mathscr{M}$-polytransversal of $\mathscr{A}$) is equal to*

$$\min_{J \subseteq T} \Big(\sum_{s \in S} \varrho_s(A(s, J)) + |T \setminus J| \Big).$$

It is straightforward to check the following lemma (see [9]).

Lemma 2. *Let M' be a matroid on a finite set T with rank function ϱ' and let $\mathscr{B} = (B_s : s \in S)$ be a finite family of subsets of T. Then the function $\varrho : 2^S \to \mathbf{R}_+$ satisfying*

$$\varrho(X) = \varrho'(\cup\{B_s ; s \in X\}) \tag{2}$$

for any $X \subseteq S$ is the ground set rank function of an integral polymatroid $\mathbf{P}$ on S.

3. Properties of $\mathscr{M}$-polytransversals

Primarily we can extend the theorem of Edmonds and Fulkerson [2] to $\mathscr{M}$-polytransversals.

Theorem 2. *Let $\mathscr{A} = (A_t\colon t \in T)$ be a finite family of subsets of a finite set S and let $\mathscr{M}$ be a family $(M_s\colon s \in S)$ of matroids on T with rank functions ϱ_s, respectively. Then the partial $\mathscr{M}$-polytransversals of $\mathscr{A}$ are the integer valued independent vectors of the integral polymatroid $\mathsf{P} = (S, \varrho)$ such that for any $X \subseteq S$,*

$$\varrho(X) = \min_{J\subseteq T} \Big(\sum_{s\in X} \varrho_s\big(A(s,J)\big) + |T \setminus J| \Big). \tag{3}$$

P r o o f. Let ϱ be the function defined by (3). Then, by Lemma 1, $\varrho(X)$ denotes the maximal length of a partial $\mathscr{M}$-polytransversal of the family $\mathscr{A}_X = (A_t \cap X\colon t \in T)$ of subsets of X.

Take the family $\mathscr{B} = (B_s\colon s \in S)$ of subsets of T such that $B_s = A(s,T)$ for any $s \in S$. Let M_s' be the restriction of M_s to $A(s,T)$ $(s \in S)$ and M' the union of all M_s', $s \in S$. Let ϱ' be the rank of M'.

It is easy to check that there exists a one-to-one correspondence between the $\mathscr{M}$-SR of $\mathscr{A}_X$ and the subsets of $\cup\{B_s;\ s \in X\}$ which are independent in M'. Then, by Lemmas 1 and 2, (2) and (3) determine the same function, i.e. $\mathsf{P} = (S, \varrho)$ is an integral polymatroid and any $\mathscr{M}$-polytransversal of $\mathscr{A}$ is independent in P.

On the other hand, let $\mathbf{u} \in \mathbb{Z}_+^S$ be independent in P, i.e. $\mathbf{u}(X) \leqslant \varrho(X)$ for any $X \subseteq S$. Denote by $M_s^{\mathbf{u}}$ the truncation of M_s at $\mathbf{u}(s)$, i.e. the rank $\varrho_s^{\mathbf{u}}$ of $M_s^{\mathbf{u}}$ satisfies

$$\varrho_s^{\mathbf{u}}(J) = \min\{\varrho_s(J), \mathbf{u}(s)\} \quad (s \in S, J \subseteq T).$$

Denote by $\mathscr{M}^{\mathbf{u}}$ the family of matroids $(M_s^{\mathbf{u}}\colon s \in S)$ on T. We assert that

$$\mathbf{u}(S) \leqslant \min_{J\subseteq T} \Big(\sum_{s\in S} \varrho_s^{\mathbf{u}}\big(A(s,J)\big) + |T \setminus J| \Big). \tag{4}$$

Indeed, if this is not the case, take $K \subseteq T$ such that

$$\mathbf{u}(S) > \sum_{s\in S} \varrho_s^{\mathbf{u}}\big(A(s,K)\big) + |T \setminus K| = \sum_{s\in S} \Big(\min\big\{\varrho_s\big(A(s,K)\big), \mathbf{u}(s)\big\}\Big) + |T \setminus K|,$$

and let $Y = \big\{s \in S; \varrho_s\big(A(s,K)\big) \leqslant \mathbf{u}(s)\big\}$. Then

$$\mathbf{u}(S) > \sum_{s\in Y} \varrho_s\big(A(s,K)\big) + \mathbf{u}(S \setminus Y) + |T \setminus K| \geqslant \mathbf{u}(S \setminus Y) + \varrho(Y).$$

Therefore $\mathbf{u}(Y) > \varrho(Y)$ – a contradiction. Thus (4) holds.

Let $\mathbf{v} \in \mathbb{Z}_+^S$ be a partial $\mathcal{M}^{\mathbf{u}}$-polytransversal of $\mathcal{A}$ with the maximal length. Then, by Lemma 1 and (4), $\mathbf{u}(S) \leqslant \mathbf{v}(S)$. But, by definition of $M_s^{\mathbf{u}}$, $\mathbf{u}(s) \geqslant \mathbf{v}(s)$ for any $s \in S$. Thus $\mathbf{u} = \mathbf{v}$ and $\mathbf{u}$ is a partial $\mathcal{M}^{\mathbf{u}}$-polytransversal (and also a partial $\mathcal{M}$-polytransversal) of $\mathcal{A}$. Thus the partial $\mathcal{M}$-polytransversals of $\mathcal{A}$ are the integer valued independent vectors of the integral polymatroid $\mathsf{P} = (S, \varrho)$, which concludes the proof. □

The polymatroid $\mathsf{P} = (S, \varrho)$ from Theorem 2 will be called the *polymatroid of partial $\mathcal{M}$-polytransversals of $\mathcal{A}$*.

Theorem 2 has interesting consequences. Primarily, we can extend the theorems of Rado and Perfect.

Corollary 1. *Let $\mathcal{A} = (A_t : t \in T)$ be a finite family of subsets of a finite set S and let $\mathcal{M}$ be a family $(M_s : s \in S)$ of matroids on T with rank functions ϱ_s, respectively. Let $\mathsf{P}_1 = (S, \varrho_1)$ be an integral polymatroid on S with vector rank r_1 and $d \in \mathbb{Z}_+$, $d \leqslant |T|$. Then $\mathcal{A}$ has a partial $\mathcal{M}$-polytransversal of $\mathcal{A}$ with defect d which is independent in P_1 if and only if for all $J \subseteq T$,*

$$r_1\big(\varrho_s(A(s,J)) : s \in S\big) \geqslant |J| - d$$

(note that $\big(\varrho_s(A(s,J)) : s \in S\big)$ denotes a vector in $\mathbb{Z}_+^S$).

Proof. Let $\mathsf{P} = (S, \varrho)$ be the (integral) polymatroid of partial $\mathcal{M}$-polytransversals of $\mathcal{A}$. Then Theorems 1 and 2 imply that $\mathcal{A}$ has the required property if and only if

$$|T| - d \leqslant \min_{X \subseteq S} \big(\varrho(X) + \varrho_1(S \setminus X)\big)$$
$$= \min_{X \subseteq S} \min_{J \subseteq T} \Big(\sum_{s \in X} \varrho_s(A(s,J)) + |T \setminus J| + \varrho_1(S \setminus X) \Big).$$

Thus, by (1),

$$|T| - d \leqslant \min_{J \subseteq T} \big(r_1\big(\varrho_s(A(s,J)) : s \in S\big) + |T \setminus J| \big),$$

concluding the proof. □

Ford and Fulkerson's theorem [3] gives a condition for two families of sets to have a common transversal. We extend this result.

Corollary 2. *For $j = 1, 2$, let $\mathcal{A}^{(j)} = (A_t^{(j)} : t \in T^{(j)})$ be a finite family of subsets of a finite set S and let $\mathcal{M}^{(j)}$ be a family $(M_s^{(j)} : s \in S)$ of matroids on $T^{(j)}$ with* *rank functions $\varrho_s^{(j)}$, respectively. Then there exists* $\mathbf{u} \in \mathbb{Z}_+^S$, $|\mathbf{u}| \geqslant k$ $(k \in \mathbb{Z}_+)$, *such that* $\mathbf{u}$ *is a partial* $\mathscr{M}^{(j)}$*-polytransversal of* $\mathscr{A}^{(j)}$ *for both* $j = 1, 2$, *if and only if for any* $J \subseteq T^{(1)}$, $K \subseteq T^{(2)}$,

$$\sum_{s \in S} \left(\min \left\{ \varrho_s^{(1)}\left(A^{(1)}(s, J)\right), \varrho_s^{(2)}\left(A^{(2)}(s, K)\right) \right\} \right) \geqslant |J| + |K| - |T^{(1)}| - |T^{(2)}| + k.$$

Proof. follows immediately from Theorems 1 and 2. □

$\mathscr{M}$-polytransversals and $\mathscr{M}$-SR generalize several known notions from transversal theory.. For instance, if $\mathscr{M}$ is a system of uniform matroids of rank k then we get in fact the k-transversals from [15] and [16]. A little more complicated "k-transversals" were introduced in [6], but they can be also described by a special class of $\mathscr{M}$-polytransversals. In [7] we dealt with another generalization of transversals, the so called "$\mathscr{M}$-transversals". Note that from Theorem 2 some of the results from [7] can be obtained, too.

As pointed out in [9] (see also [5], [8], [10]), any integral polymatroid on S can be represented by the construction of Lemma 2. Then it follows from the proof of Theorem 2 that any integral polymatroid on S can be represented as a polymatroid of $\mathscr{M}$-polytransversals of a family of sets $\mathscr{A}$. This contrasts with the known fact that transversal matroids from a proper subclass of matroids.

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