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REPRESENTATION OF UNDIRECTED GRAPHS BY ANTICOMMUTATIVE CONSERVATIVE GROUPOIDS

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Summary. The paper studies tolerances and congruences on anticommutative conservative groupoids. These groupoids can be assigned in a one-to-one way to undirected graphs.

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Various authors have studied graphs by algebraic methods. Among these methods there was also assigning certain algebraic structures to graphs in a one-to-one way. But usually only special classes of graphs were considered, e.g. directed graphs assigned to unary algebras. Representation of trees by certain ternary algebras was done by L. Nebeský [2], G. F. McNulty and C. R. Shallon [1] and R. Pöschel [3] have represented directed graphs by groupoids. In this case the support of the groupoid was equal to the union of the vertex set of the graph with some one-element set and thus not to the vertex set itself. Here we shall study another way of expressing graphs algebraically, namely by anticommutative conservative groupoids.

The multiplication in a groupoid will be denoted by simple juxtaposition and a groupoid will be identified with its support. Graphs will be always undirected, without loops and multiple edges.

A groupoid Γ is called anticommutative, if

$$xy = yx \Rightarrow x = y$$

for any x, y of Γ.

A groupoid Γ is called conservative, if

$$xy = x \vee xy = y$$

for any x, y of Γ.

Obviously every conservative groupoid is idempotent.

Let Γ be an anticommutative conservative groupoid, let x, y be two elements of Γ. Then either $xy = x$ and $yx = y$, or $xy = y$ and $yx = x$. Therefore we may introduce a one-to-one correspondence between undirected graphs and anticommutative conservative groupoids.

Let G be an undirected graph. Define the groupoid $\Gamma(G)$ on the vertex set $V(G)$ of G in such a way that $xx = x$ for each $x \in V(G)$, $xy = x$ for any two adjacent vertices x, y of G and $xy = y$ for any two distinct non-adjacent vertices x, y of G. On the other hand, to every anticommutative conservative groupoid we may assign an undirected graph in such a way that the vertices of the graph are the elements of the groupoid and two vertices x, y are adjacent if and only if $x \neq y$ and $xy = x$.

Theorem 1. *Let G be an undirected graph. The groupoid $\Gamma(G)$ is a semigroup if and only if G is either a complete graph, or a totally disconnected graph.*

Remark. A graph is called totally disconnected, if it has no edges.

Proof. If G is a complete graph, then for any three elements x, y, z of $\Gamma(G)$ we have

$$(xy)z = xz = x = xy = x(yz)$$

and the multiplication is associative. If G is a totally disconnected graph, then

$$(xy)z = yz = z = xz = x(yz)$$

and the multiplication is again associative.

Now suppose that G is neither complete, nor totally disconnected. Then there exist three distinct vertices x, y, z of G such that x, y are adjacent, while x, z are not. If y, z are adjacent, then

$$(xy)z = xz = z \neq x = xy = x(yz).$$

If y, z are not adjacent, then

$$(xz)y = zy = y \neq x = xy = x(zy).$$

□

We shall study tolerances and congruences on anticommutative conservative groupoids. A tolerance on a groupoid Γ is a reflexive and symmetric binary relation T on Γ with the property that $(x_1, y_1) \in T$, $(x_2, y_2) \in T$ imply $(x_1x_2, y_1y_2) \in T$

for any four elements x_1, x_2, y_1, y_2 of Γ. If moreover T is transitive, it is called a congruence on Γ.

Let a groupoid Γ and a tolerance T on it be given. A subset B of Γ is called a block of T, if $(x,y) \in T$ for any two elements of B and B is a maximal set with this property (it is not a proper subset of another set with this property). If T is a congruence, then its blocks are called congruence classes.

We shall prove a lemma.

Lemma. *Let G be a graph, let T be a tolerance on $\Gamma(G)$. Let M be a subset of a block of T. Let $u \in \Gamma(G) - M$, let u be adjacent to at least one vertex of M and non-adjacent to at least one vertex of M in G. Then $(u,x) \in T$ for each $x \in M$.*

P r o o f. Let X (or Y) be the set of all vertices of M which are adjacent (or non-adjacent respectively) to u. According to the assumption $X \neq \emptyset$, $Y \neq \emptyset$. Let $x \in X$, $y \in Y$. As both x, y are in M, we have $(x,y) \in T$. By reflexivity $(u,u) \in T$. Then $(ux,uy) = (u,y) \in T$, $(xu,yu) = (x,u) \in T$ and by symmetry $(u,x) \in T$. The vertex x was chosen arbitrarily in X, the vertex y was chosen arbitrarily in Y and $X \cup Y = M$, which proves the assertion. □

Now we prove a theorem.

Theorem 2. *Let G be a graph, let B be a non-empty subset of $\Gamma(G)$. Then the following two assertions are equivalent:*

(i) *Each vertex $x \in \Gamma(G) - B$ is either adjacent to all vertices of B, or non-adjacent to all vertices of B.*

(ii) *There exists a tolerance T on $\Gamma(B)$ such that B is a block of T.*

P r o o f. (i)⇒(ii). Let (i) be satisfied. Let us define a tolerance T such that $(x,y) \in T$ if and only if either $x = y$, or $x \in B$ and $y \in B$. Evidently T is reflexive and symmetric (and moreover transitive). Let x_1, y_1, x_2, y_2 be four elements of $\Gamma(B)$ such that $(x_1,y_1) \in T$, $(x_2,y_2) \in T$. If $x_1 = y_1$, $x_2 = y_2$, then $(x_1x_2,y_1y_2) = (x_1x_2,x_1x_2) \in T$. Suppose $x_1 \in B$, $y_1 \in B$, $x_2 = y_2 \notin B$. Then by (i) either $x_2 = y_2$ is adjacent to all vertices of B, or non-adjacent to all of them. In the first case $(x_1x_2,y_1y_2) = (x_1,y_1) \in T$, in the second case $(x_1x_2,y_1y_2) = (x_2,x_2) \in T$. Analogously in the case where $x_1 = y_1 \notin B$, $x_2 \in B$, $y_2 \in B$. If all the elements x_1, x_2, y_1, y_2 are in B, then so are the products x_1x_2, y_1y_2, because $\Gamma(G)$ is conservative; again $(x_1x_2,y_1y_2) \in T$ and T is a tolerance on $\Gamma(G)$.

(ii)⇒(i). Suppose that there exists $x \in \Gamma(G) - B$ adjacent to at least one vertex of B and non-adjacent to at least one vertex of B. Then, by Lemma, the set $B \cup \{x\}$ has the property that any two of its elements are in T and thus B is not maximal with this property, i.e. it is not a block of T. □

The family of all non-empty subsets of $\Gamma(G)$ satisfying the condition (i) will be denoted by $\mathcal{B}(G)$.

We shall prove a theorem concerning $\mathcal{B}(G)$.

Theorem 3. *Let G be an undirected graph. Then $\mathcal{B}(G) \cup \{\emptyset\}$ is a complete lattice with respect to set inclusion.*

Proof. Let $\mathcal{C}$ be a non-empty subset of $\mathcal{B}(G)$ and consider the intersection $D = \bigcap_{C \in \mathcal{C}} C$. If $D = \emptyset$, then $D \in \mathcal{B}(G) \cup \{\emptyset\}$. If $D \neq \emptyset$, then let $x \in \Gamma(G) - D$. Then there exists $C_0 \in \mathcal{C}$ such that $x \in \Gamma(G) - C_0$. As $C_0 \in \mathcal{B}(G)$, the vertex x is either adjacent to all vertices of C_0 and thus also to all vertices of $D \subseteq C_0$, or non-adjacent to all of them; we have proved that $D \in \mathcal{B}(G)$. Therefore there exists the meet $\bigwedge_{C \in \mathcal{C}} C = \bigcap_{C \in \mathcal{C}} C$. Now consider the set $\mathcal{D}$ of all elements of $\mathcal{B}(G)$ which contain $\bigcup_{C \in \mathcal{C}} C$ as a subset; this set is non-empty, because $\Gamma(G) \in \mathcal{D}$. There exists the meet $\bigwedge_{D \in \mathcal{D}} D = \bigcap_{D \in \mathcal{D}} D$ and this is $\bigvee_{C \in \mathcal{C}} C$. □

Theorem 4. *Let G be an undirected graph, let $B \in \mathcal{B}(G)$, $C \in \mathcal{B}(G)$, $B \cap C \neq \emptyset$. Then $B \vee C = B \cup C$.*

Proof. Let $x \in \Gamma(G) - (B \cup C)$. Then $x \in \Gamma(G) - B$ and $x \in \Gamma(G) - C$. As $x \in \Gamma(G) - B$, it is either adjacent to all vertices of B, or non-adjacent to all vertices of B. In the first case it is adjacent to all vertices of $B \cap C \subseteq B$. As $B \cap C \neq \emptyset$, it is adjacent to at least one vertex of C and, as $C \in \mathcal{B}(G)$, to all vertices of C and hence also to all vertices of $B \cup C$. In the second case it is non-adjacent to all vertices of $B \cup C$. Therefore $B \cup C \in \mathcal{B}(G)$ and $B \vee C = B \cup C$. □

Proposition 1. *The lattice $\mathcal{B}(G) \cup \{\emptyset\}$ is not distributive in general, but each of its complete sublattices not containing $\emptyset$ as an element is distributive.*

Proof. Let the vertex set of G be $V(G) = \{v, x, y, z\}$, let G have exactly one edge vx. Evidently each one-element subset of $V(G)$ is in $\mathcal{B}(G)$ and thus the sets $\{x\}$, $\{y\}$, $\{z\}$ are in $\mathcal{B}(G)$. Evidently

$$\{x\} \vee (\{y\} \wedge \{z\}) = \{x\} \vee \emptyset = \{x\}.$$

The set $\{x\} \vee \{y\}$ is the least set which contains x and y and is in $\mathcal{B}(G)$. The vertex v is adjacent to x and not to y, therefore $v \in \{x\} \vee \{y\}$. The set $\{v, x, y\} \in \mathcal{B}(G)$ and therefore $\{x\} \vee \{y\} = \{v, x, y\}$. Analogously $\{x\} \vee \{z\} = \{v, x, z\}$. We have

$$(\{x\} \vee \{y\}) \wedge (\{x\} \vee \{z\}) = \{v, x, y\} \cap \{v, x, z\} = \{v, x\} \neq \{x\}$$

and the lattice $\mathcal{B}(G) \cup \{\emptyset\}$ is not distributive.

Now let G be an arbitrary undirected graph. Let $\mathcal{B}_0$ be a sublattice of $\mathcal{B}(G) \cup \{\emptyset\}$ which does not contain $\emptyset$. Let B_0 be the meet of all elements of $\mathcal{B}_0$; as $\mathcal{B}_0$ is complete, B_0 is the least element of $\mathcal{B}_0$ and $B_0 \neq \emptyset$. Any two elements of $\mathcal{B}_0$ have a non-empty intersection, because they both contain $\mathcal{B}_0$. Therefore the join in $\mathcal{B}_0$ is equal to the set union and $\mathcal{B}_0$ is a sublattice of the lattice of all subsets of $\Gamma(G)$, hence it is distributive. □

Note that $\mathcal{B}(G)$ contains always the set $\Gamma(G)$ and all of its one-element subsets.

Proposition 2. *Let G be an undirected graph with at least two vertices. Then the lattice $\mathcal{B}(G) \cup \{\emptyset\}$ is generated by its atoms.*

P r o o f. As it was mentioned above, every one-element subset of $\Gamma(G)$ is in $\mathcal{B}(G)$ and therefore the set of all atoms of $\mathcal{B}(G) \cup \{\emptyset\}$ is equal to the set of all one-element subsets of $\Gamma(G)$. If $B \in \mathcal{B}(G)$, then evidently $B = \bigvee_{x \in B} x$. If x, y are two different elements of $\Gamma(G)$, then $\{x\} \wedge \{y\} = \emptyset$. This implies the assertion. □

Now we shall study the lattice $\mathrm{Tol}\,(\Gamma(G))$ of all tolerances on $\Gamma(G)$.

Theorem 5. *Let G be an undirected graph. The lattice* $\mathrm{Tol}\,(\Gamma(G))$ *is a sublattice of the lattice of all reflexive and symmetric binary relations on $\Gamma(G)$.*

P r o o f. Let T_1, T_2 be two tolerances on $\Gamma(G)$. It is well-known that the meet of two tolerances on an algebra is equal to their intersection, $T_1 \wedge T_2 = T_1 \cap T_2$.

Consider the relation $T_1 \cup T_2$. Let $(x_1, y_1) \in T_1 \cup T_2$ and $(x_2, y_2) \in T_1 \cup T_2$. If they both belong to T_1 or they both belong to T_2, it is evident that $(x_1x_2, y_1y_2) \in T_1 \cup T_2$. Thus suppose $(x_1, y_1) \in T_1$, $(x_2, y_2) \in T_2$. If x_1 is adjacent to x_2 or $x_1 = x_2$ and y_1 is adjacent to y_2 or $y_1 = y_2$, then $(x_1x_2, y_1y_2) = (x_1, y_1) \in T_1 \subseteq T_1 \cup T_2$. If x_1 is non-adjacent to x_2 or $x_1 = x_2$ and y_1 is non-adjacent to y_2 or $y_1 = y_2$, then $(x_1x_2, y_1y_2) = (x_2, y_2) \in T_2 \subseteq T_1 \cup T_2$. Now suppose that x_1 is adjacent to x_2 and y_1 is non-adjacent to y_2. Then $(x_1x_2, y_1y_2) = (x_1, y_2)$. If x_1 is adjacent to y_2, then $(x_1, y_2) = (x_1y_2, y_1y_2) \in T_1 \subseteq T_1 \cup T_2$. If x_1 is non-adjacent to y_2, then $(x_1, y_2) = (x_1y_1, x_1y_2) \in T_2 \subseteq T_1 \cup T_2$. If $x_1 = y_2$, then by reflexivity $(x_1, y_2) \in T_1 \cup T_2$. Hence $T_1 \cup T_2 \in \mathrm{Tol}\,(\Gamma(G))$ and $T_1 \vee T_2 = T_1 \cup T_2$. We have proved that $\mathrm{Tol}\,(\Gamma(G))$ is a sublattice of the lattice of all reflexive and symmetric relations on $\Gamma(G)$. □

Let x, y be two distinct elements of $\Gamma(G)$. By $T(x, y)$ we shall denote the least tolerance on $\Gamma(G)$ containing the pair (x, y), i.e. the intersection of all tolerances on $\Gamma(G)$ containing that pair.

Theorem 6. *Let G be an undirected graph, let x, y be two distinct vertices of G. Then $T(x,y)$ is a congruence on $\Gamma(G)$ which has exactly one class with more than one element.*

Proof. Let $\mathcal{B}(x,y)$ be the set of all elements of $\mathcal{B}(G)$ which contain the vertices x, y. This set is non-empty, because $\Gamma(G) \in \mathcal{B}(x,y)$. Let $B_0(x,y)$ be the intersection of all elements of $\mathcal{B}(x,y)$, i.e. their meet in the lattice $\mathcal{B}(G) \cup \{\emptyset\}$. Obviously $\{x,y\} \subseteq B_0(x,y)$. In any tolerance T every pair of elements being in T belongs to at least one block of T. Therefore there exists a block B of $T(x,y)$ such that $\{x,y\} \subseteq B$. By Theorem 2 we have $B \in \mathcal{B}(G)$ and hence $B_0(x,y) \subseteq B$. We have then $(u,v) \in T(x,y)$ whenever $u \in B_0(x,y)$ and $v \in B_0(x,y)$. On the other hand, let the relation T_0 be defined so that $(u,v) \in T_0$ if and only if either $u \in B_0(x,y)$ and $v \in B_0(x,y)$, or $u = v$. Then by Theorem 2 the relation T_0 is a tolerance on $\Gamma(G)$; hence $T_0 \subseteq T(x,y)$ and by the minimality of $T(x,y)$ we have $T_0 = T(x,y)$. From the definition of T_0 it is clear that it has the required properties. □

At the end we shall prove a theorem concerning the relationship between different blocks of a tolerance.

Theorem 7. *Let G be an undirected graph, let $T \in \mathrm{Tol}\,(\Gamma(G))$, let B_1, B_2 be two distinct blocks of T. Then $B_1 - B_2 \neq \emptyset$, $B_2 - B_1 \neq \emptyset$ and either all vertices of $B_1 - B_2$ are adjacent to all vertices of B_2 and all vertices of $B_2 - B_1$ are adjacent to all vertices of B_1, or all vertices of $B_1 - B_2$ are non-adjacent to all vertices of B_2 and all vertices of $B_2 - B_1$ are non-adjacent to all vertices of B_1.*

Proof. We have $B_1 - B_2 \neq \emptyset$ and $B_2 - B_1 \neq \emptyset$, because no block of a tolerance is a proper subset of another block. Let $x_1 \in B_1 - B_2$, $x_2 \in B_2 - B_1$. If x_1 is adjacent to x_2, then it is adjacent to all vertices of B_2, because $B_2 \in \mathcal{B}(G)$. But then x_2 is adjacent to x_1 and thus x_2 is adjacent to all vertices of B_1, because $B_1 \in \mathcal{B}(G)$. As x_1, x_2 were chosen arbitrarily, the assertion holds. If x_1 is non-adjacent to x_2, the proof is analogous. □

We shall add some final remarks.

We may introduce a factor-graph G/T of the graph G by the tolerance $T \in \mathrm{Tol}\,(\Gamma(G))$ in such a way that the vertex set of G/T is the set of all blocks of T and two such blocks B_1, B_2 are adjacent in G/T if and only if all vertices of $B_1 - B_2$ are adjacent to all vertices of B_2. The corresponding groupoid $\Gamma(G/T)$ is called the factor-groupoid of $\Gamma(G)$ by T and may be denoted by $\Gamma(G)/T$. If T is a congruence, this is the factor-groupoid of $\Gamma(G)$ by T in the usual sense.

Note that conservative groupoids do not form a variety; the direct product of two conservative groupoids need not be conservative.

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