Andrew M. Bruckner; Jaroslav Smítal

The structure of ω-limit sets for continuous maps of the interval

Mathematica Bohemica, Vol. 117 (1992), No. 1, 42–47

Persistent URL: http://dml.cz/dmlcz/126240

THE STRUCTURE OF ω-LIMIT SETS FOR CONTINUOUS MAPS OF THE INTERVAL

Andrew M. Bruckner, Santa Barbara, Jaroslav Smítal, Bratislava

(Received November 22, 1989)

Summary. We prove that every infinite nowhere dense compact subset of the interval I is an ω-limit set of homoclinic type for a continuous function from I to I.

Keywords: discrete dynamical systems, ω-limit sets.

AMS classification: 58F12, 26A18, 54H20

Let f be a continuous map of the interval $I = [0, 1]$ to itself. For any $x \in I$, let $\omega_f(x)$ be the ω-limit set of x, i.e., the limit set of the sequence $\{f^n(x)\}_{n=0}^{\infty}$, where f^n denotes the n-th iterate of f. There is a natural problem to describe the sets $\omega_f(x)$ for all f.

If f has zero topological entropy, i.e., if every periodic point has period 2^n for some n, then every ω-limit set $\omega_f(x)$ is either finite or uncountable. In the latter case, $\omega_f(x) = A \cup B$, where $A \neq \emptyset$ is a nowhere dense perfect set and B is either empty or an infinite countable set of isolated points such that each interval J contiguous to A contains at most two points from B (cf. [4] for more details).

If there is no restriction on periods of the periodic points, the situation is more complicated (cf. also [3]): An ω-limit set $\omega_f(x)$ can have a non-empty interior. In theis case $\omega_f(x)$ has the form

$$J_0 \cup \ldots \cup J_{n-1} \tag{1}$$

where J_i are compact periodic intervals of period n, satisfying $f^j(J_i) = J_{i+j(\bmod n)}$ for every $i, j \geqslant 0$, cf. [2]. Thus every $\omega_f(x)$ either is nowhere dense, or has the form (1) (and, of course, must be compact). These conditions characterize the ω-limit sets. It is easy to see that any set of the form (1) is an ω-limit set for some continuous f, and in [1] it is proved that any compact nowhere dense set $A \neq \emptyset$ is an ω-limit set for a continuous map (note that this result seems to be surprising, compare lines 3–4 on p. 270 in [2]).

The main aim of the present note is to prove the result from [1] in a much simpler way. It turns out that we prove a slight variant of the result in [1]. First we introduce our basic notion.

Definition. Let M be a nowhere dense compact set, $A = \{a_0, \dots, a_{k-1}\} \neq \emptyset$ a set of limit points of M. Assume there is a system $\{M_n^i\}_{n=0}^{\infty}$, $i = 0, \dots, k-1$, of non-empty pairwise disjoint compact subsets of M such that $M \setminus \bigcup_{i,n} M_n^i = A$ and $\lim_{n\to\infty} M_n^i = a_i$ for any i. Let $f\colon M \to M$ be a continuous map and let A be a k-cycle of f such that $f(a_i) = a_{i-1}$ for $i > 0$ and $f(a_0) = a_{k-1}$. If $f(M_n^i) = M_n^{i-1}$ for $i > 0$ and any n, $f(M_n^0) = M_{n-1}^{k-1}$ for $n > 0$, and $f(M_0^0) = a_{k-1}$, then M is called a *homoclinic set* (of order k) *with respect to f*.

Remark. If M is homoclinic of order k with respect to f, then for each i, the set $M^i = \{a_i\} \cup \bigcup_{n=0}^{\infty} M_n^i$ is homoclinic of order 1 with respect to $g = f^k$; more precisely, we have $g(M_n^i) = M_{n-1}^i$ for $n > 0$ and $g(M_0^i) = g(a_i) = a_i$.

Theorem 1. *Every infinite nowhere dense compact set $M \subset I$ is an ω-limit set of homoclinic type for a continuous map $I \to I$.*

This result is a simple consequense of Theorems 2 and 3 stated below; their proofs are based on a sequence of lemmas.

Theorem 2. *Let $M \subset I$ be a nowhere dense compact set, let $f\colon M \to M$ be continuous and suppose M is homoclinic with respect to f. Then there is a continuous extension $F\colon I \to I$ of f such that $M = \omega_F(x)$ for some $x \in I$.*

The assumptions in the next two lemmas are the same as in Theorem 2.

Lemma 1. *There is a continuous extension $F\colon I \to I$ of f such that for any $u \in M$ and any neighborhood U of u (in the relative topology with respect to I), the set $F(U)$ is a neighboorhood of $F(u) = f(u)$.*

Proof. Let $\{J_n\}_{n=0}^{\infty}$ be open intervals such that $\operatorname{diam} J_n < 1/n$ and $J_n \cap M \neq \emptyset$ for any n, and such that every open set intersecting M contains some J_n. Define by induction continuous extensions $\{F_n\}_{n=0}^{\infty}$ of f to I as follows: Let F_0 be linear on every interval contiguous to M, and put $K_0 = \emptyset$. If F_m and K_m are defined, let K_{m+1} be a compact subinterval of $J_m \setminus \{M \cup K_0 \cup \dots \cup K_m\}$. Let $F_{m+1}(x) = F_m(x)$ if $x \notin K_{m+1}$ and let $F_{m+1}(K_{m+1})$ be the neighborhood of $F_m(J_m \cap M) = f(J_m \cap M)$ such that $\operatorname{diam} F_{m+1}(K_{m+1}) < \delta_m = 2 \operatorname{diam} f(J_m \cap M)$. Then $\|F_{m+k} - F_m\| < \max\{\delta_{m+1}, \dots, \delta_{m+k}\}$ and by the continuity of f, $\lim_{n\to\infty} \delta_n = 0$. Thus $\lim_{n\to\infty} F_n = F$ uniformly. Clearly, F has the required properties. □

Lemma 2. *Let F be as in Lemma 1. Then for any $u, v \in M$, any neighborhood U of u, and any $\delta > 0$ there is a compact set $K \subset U$ and an $n > 0$ such that $F^n(K)$ is a neighborhood of v, and $F^i(K)$ is in the δ-neighborhood $S(M, \delta)$ of M whenever $0 \leqslant i \leqslant n$.*

P r o o f. For some s, r we have $F^s(u) = f^s(u) = a_r$, where a_r is as in Definition. Let $L \subset U$ be a compact neighborhood of u such that $\operatorname{diam} F^i(L) < \delta$ for $0 \leqslant i \leqslant s$. By Lemma 1, $F^s(L)$ is a neighborhood of a_r, hence it contains sets M_j^r with arbitrarily large j since $\lim_{j \to \infty} M_j^r = a_r$. Since $v \in M_m^t$ or $v = a_t$ for some m and t, there are $w \in M \cap F^s(L)$ and $p > 0$ such that $F^p(w) = v$. Let $T \subset F^s(L)$ be a compact neighborhood of w such that $\operatorname{diam} F^i(T) < \delta$ for $0 \leqslant i \leqslant p$. Take $K = F^{-s}(T) \cap L$ and $n = s + p$. □

P r o o f of Theorem 2. Let $\{V_i\}_{i=0}^{\infty}$ be open intervals such that any open set intersecting M contains some V_i, and let $b_i \in V_i \cap M \neq \emptyset$ for any i. Using Lemma 2 we can define by induction a decreasing sequence of compact sets $\{K_i\}_{i=0}^{\infty}$ and an increasing sequence $\{n(i)\}_{i=0}^{\infty}$ of positive integers such that for every i, $F^{n(i)}(K_i) \subset V_i$ is a neighborhood of b_i, and $F^j(K_i) \subset S(M, 1/i)$ whenever $n(i) \leqslant j \leqslant n(i+1)$. Let $x \in \bigcap_{i=0}^{\infty} K_i \neq \emptyset$. Since $F^j(x) \in S(M, 1/i)$ for $j \geqslant n(i)$, we have $\omega_F(x) \subset M$. On the other hand, since the trajectory $\{F^j(x)\}_{j=0}^{\infty}$ visits infinitely many times every V_i, and hence every open set intersecting M, we have $\omega_F(x) \supset M$. □

In the sequel, for any two subsets A, B of I, $A \succ B$ means that there is a continuous map of A onto B.

Lemma 3. *Let $A, B \subset I$ be nowhere dense compact sets, A uncountable and $B \neq \emptyset$. Then $A \succ B$.*

P r o o f. Let $P \subset A$ be a non-empty perfect set. Since $P \succ I$ there is a compact subset $Q \subset P$ such that $Q \succ B$. It suffices to show that $A \succ Q$. Let $\{J_n\}_{n=0}^{\infty}$ be a system of compact intervals such that for any $m \neq n$, $J_n \cap Q = \{q_n\}$, $J_m \cap J_n \subset Q$, $J_n \cap (A \setminus Q) \neq \emptyset$, and $(A \setminus Q) \subset \bigcup_{n=0}^{\infty} J_n$. Now let φ be the identity map on Q, and let φ be constant on every $J_n \cap A$. Clearly φ is continuous and $\varphi(A) = Q$. □

Lemma 4. *Every uncountable nowhere dense compact subset M of I is homoclinic with respect to a continuous map f.*

P r o o f. Let $\{I_n\}_{n=0}^{\infty}$ be a sequence of pairwise disjoint compact intervals such that $I_n \cap M = M_n$ is uncountable for every n, $M \setminus \bigcup_{n=0}^{\infty} I_n = \{a\}$ and $\lim_{n \to \infty} I_n = a$. By Lemma 3 there is a continuous function $f \colon M \to M$ such that $f(x) = a$ if $x = a$ or $x \in M_0$, and $f(M_{n+1}) = M_n$ for every $n \geqslant 0$. □

Before stating the next lemmas we need some notation. Let $A \subset I$ be a countable compact set. Define a transfinite sequence $\{A_\alpha\}_{\alpha \in \Omega}$ of subsets of A as follows: $A_0 = A$, $A_\gamma = \bigcap_{\alpha<\gamma} A_\alpha$ if γ is a limit ordinal, and A_γ is the derivative (i.e., the set of limit points) of $A_{\gamma-1}$ otherwise. Clearly, for any such A there is an ordinal $\beta < \Omega$ such that A_β is non-empty and finite, and $A_{\beta+1} \neq \emptyset$. Denote such β by $T(A)$.

Lemma 5. *Let D be a non-empty countable compact set and let $T(D) = \gamma$. Assume $D_\gamma = \{d\}$. Then there is a sequence $\{D_n\}_{n=1}^\infty$ of pairwise disjoint compact subsets of D such that $D \setminus \bigcup_{n=1}^\infty D_n = \{d\}$ and $\lim_{n\to\infty} D_n = d$. Moreover, for each n, $\gamma > \delta(n+1) \geqslant \delta(n)$ where $\delta(n) = T(D_n)$ and $(D_n)_{\delta(n)} = \{d_n\}$.*

Proof. Assume $d \in (0,1)$; for $d = 0$ or $d = 1$ the proof is similar. Let $\{I_n\}_{n=1}^\infty$ be a strictly decreasing sequence of compact intervals with end-points in $I \setminus D$ and such that $\bigcap_{n=1}^\infty I_n = \{d\}$. Take $n(1)$ such that $D \setminus I_{n(1)} \neq \emptyset$. Since $\lim_{n\to\infty} T(D \setminus I_n)$ is γ if γ is a limit ordinal, and is $\gamma - 1$ otherwise, there exists $n(2) > n(1)$ such that $\gamma > T(D \cap (I_{n(1)} \setminus I_{n(2)})) \geqslant T(D \setminus I_{n(1)})$, etc. By induction we get an increasing sequence $\{n(k)\}_{k=1}^\infty$ of positive integers such that, for any $k > 0$, $\gamma > T\big(D \cap (I_{n(k)} \setminus I_{n(k+1)})\big) \geqslant T\big(D \cap (I_{n(k-1)} \setminus I_{n(k)})\big)$, where $I_{n(0)} = I$. Put $D_k = D \cap (I_{n(k-1)} \setminus I_{n(k)})$.

If for some k, $(D_k)_{\delta(k)}$ has more than one element, replace D_k in the sequence by a string $D_k^1, \ldots, D_k^{m(k)}$, where D_k^i are portions of D_k such that $T(D_k^i) = \delta(k)$ and $(D_k^i)_{\delta(k)} = \{b_k^i\}$ for each i. □

The following lemma should be known but we are not able to give a reference.

Lemma 6. *Let $A, B \subset I$ be non-empty countable compact sets with $\alpha = T(A) \geqslant T(B) = \beta$, and let $B_\beta = \{b\}$. Then $A \succ B$.*

Proof. We use transfinite induction. The result is true for $\alpha = 0$. Assume it is true for any $\alpha < \alpha(0)$, and let $T(A) = \alpha(0)$. First consider the case $A_{\alpha(0)} = \{a\}$. Apply Lemma 5 to $D = B$; let $\{B_n\}_{n=1}^\infty$ be the corresponding sequence of subsets of B. For each n denote $T(B_n) = \beta(n)$. Then apply Lemma 5 to $D = A$ and let $\{D_n\}_{n=1}^\infty$ be the corresponding sequence of compact subsets of A. For each k, $\lim_{n\to\infty} T(D_n) \geqslant \beta(k)$, hence there is $n(k)$ such that $T(D_{n(k)}) \geqslant \beta(k)$. We may assume that $\{n(k)\}_{k=1}^\infty$ is an increasing sequence and put $A_k = D_{n(k-1)+1} \cup \ldots \cup D_{n(k)}$. Then $\alpha(0) > T(A_n) \geqslant \beta(n)$ for each n, hence by the hypothesis, $A_n \succ B_n$ (note that by Lemma 5, $(B_n)_{\beta(n)} = \{b_n\}$ for each n); let φ_n be the corresponding map. Define φ by $\varphi(x) = \varphi_n(x)$ if $x \in A_n$, and $\varphi(a) = b$. Since $A = \bigcup A_n \cup \{a\}$ and $B = \bigcup B_n \cup \{b\}$, φ is a map from A onto B, and since $\lim A_n = a$ and $\lim B_n = b$, φ is continuous.

Finally, if $A_{\alpha(0)} = \{a_0, \dots, a_{k-1}\}$ with $k > 1$, divide $A_{\alpha(0)}$ into compact portions $A^0, \dots, A^{k-1}$ such that $A^i_{\alpha(0)} = \{a_i\}$ for each i. Since $A \supset A_i \succ B$ for any i, we have $A \succ B$. □

Theorem 3. *Every infinite nowhere dense compact set M is homoclinic with respect to a continuous map f.*

Proof. If M is uncountable, the result follows from Lemma 4. So assume that M is countable. Let $T(M) = \alpha$, $M_\alpha = \{a_0, \dots, a_{k-1}\}$. Let $I^0, \dots, I^{k-1}$ be pairwise disjoint compact intervals covering M and such that I^i is a neighborhood of a_i for any i. Denote $M^i = M \cap I^i$. Then $T(M^i) = \alpha$ and $M^i_\alpha = \{a_i\}$. Apply Lemma 5 to every M^i. Let $\{D^i_n\}_{n=1}^\infty$ be the corresponding compact subsets of M^i for any i. We may assume that $D^i_n \neq \emptyset$ for any i and n. To finish the proof it suffices to define sets M^i_n with properties described in Definition. Towards this, every M^i_n will have the form

$$M^i_n = D^i_{k(i,n-1)+1} \cup \dots \cup D^i_{k(i,n)} \tag{2}$$

where $k(i,-1) = 0$ and $k(i, n-1) < k(i,n)$ for any i and n. Put $M^0_0 = D^0_1$; then clearly $M^0_0 \succ \{a_{k-1}\}$. Next assume by induction that M^i_n is defined. Let $(j,m) = (i+1,n)$ if $i \neq k-1$, and let $(j,m) = (0, n+1)$ otherwise. We need M^j_m such that $M^j_m \succ M^i_n$ (cf. Definition). By Lemma 5 there is $s > k(j, m-1)$ such that $T(D^j_s) \geqslant T(D^i_{k(i,n)})$, and hence (cf. Lemma 6), $D^j_t \succ D^i_r$ for any $t \geqslant s$ and any $r \leqslant k(i,n)$. Then by (2), $D^j_s \cup \dots \cup D^j_{s+k(i,n)-k(i,n-1)} \succ M^i_n$ so if we take $k(j,m) = s + k(i,n) - k(i,n-1)$, we get $M^j_m = D^j_{k(j,m-1)+1} \cup \dots \cup D^j_{k(j,m)} \succ M^i_n$ as required. Since all M^i_n are pairwise disjoint and compact and cover $M \setminus \{a_1, \dots, a_{k-1}\}$, and since $\lim\limits_{n\to\infty} M^i_n = a_i$ for each i, we see that there is a continuous map f such that M is homoclinic. □

Remark. It would be interesting to know which sets can be ω-limit sets for smooth maps. If, for example, M in Theorem 1 is countable then there is a map f even in $C^1(I,I)$ with $\omega_f(x) = M$ for some $x \in I$. But if $M = P \cup Q$, where P, Q are disjoint non-empty perfect sets such that every portion of P has a positive Lebesgue measure, while the measure of Q is zero, then f cannot satisfy Luzin's condition (N) and hence cannot be differentiable. To see this, note that if $M = \omega_f(x)$, there must be a portion P_0 of P such that $f^{-1}(P_0) \cap M \subset Q$.

Acknowledgement. The second author is thankful to the University of California at Santa Barbara for their kind hospitality.

References

[1] *S. J. Agronsky, A. M. Bruckner, J. G. Ceder, T. L. Pearson*: The structure of ω-limit sets for continuous functions, Real Analysis Exchange *15* (1989–1990), 483–510.

[2] *A. N. Šarkovskii*: Attracting and attracted sets, Soviet Math. Dokl. *6* (1965), 268–270.
[3] *A. N. Šarkovskii*: The partially ordered system of attracting sets, Soviet Math. Dokl. *7* (1966), 1384–1386.
[4] *A. N. Šarkovskii*: Attracting sets containing no cycles, Ukrain. Mat. Ž. *20* (1968), 136–142. (In Russian.)

Souhrn

STRUKTURA ω-LIMITNÍCH MNOŽIN SPOJITÝCH ZOBRAZENÍ INTERVALU

ANDREW M. BRUCKNER, JAROSLAV SMÍTAL

Dokazujeme, že každá nekonečná řídká kompaktí podmnožina intervalu I je ω-limitní množinou homoklinického typu pro vhodnou spojitou funkci f z I do I.

Author's address: *A. M. Bruckner*, Department of Mathematics, University of California, Santa Barbara, CA 93106, USA; *J. Smítal*, Departement of Mathematics, Comenius University, 842 15 Bratislava, Czechoslovakia.