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ON A CERTAIN TYPE OF DECOMPOSITIONS OF COMPLETE GRAPHS INTO FACTORS WITH EQUAL DIAMETERS

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Paper [2] deals with the existence of a decomposition of the complete graph into factors with given diameters.

In the present paper we shall study the existence of a decomposition of the complete graph into factors with equal diameters if both, the number h of factors and the difference s between the number of vertices and the diameter of factors are given. We shall solve this problem for $s = 1, 2$ if $h \geqq 2$ and for $h = 2$ if $s \geqq 1$; we give some results in the case of $s = 3$, too.

Lemma 4 was proved by Š. Znám (unpublished). I wish to thank Š. Znám for his kind permission to publish Lemma 4 in the present paper, as well as for his suggestions used in it.

*

All graphs considered in the present paper are undirected, finite, connected, without loops and multiple edges. The complete graph with n vertices will be (like in [2]) denoted by $\langle n \rangle$. By a factor of a graph G we mean a subgraph of G containing all vertices of G. By a decomposition of a graph G into factors we mean such a system of factors of G that every edge of G is contained in exactly one factor of the system. The diameter d of G is the maximum of the set of all distances $\varrho_G(x, y)$ between the pairs of vertices (x, y) of G.

For our further considerations we shall need some results of [2]. In [2] the symbol $F(d_1, d_2, \ldots, d_h)$ means the smallest natural number n such that the graph $\langle n \rangle$ can be decomposed into h factors with diameters $d_1, d_2, \ldots, d_h$; if such a natural number does not exist, then $F(d_1, d_2, \ldots, d_h) = \infty$.(1) In [2] the following statements were proved:

(1) In this place in [2], the cardinal number is considered. In [2] the problem of a decomposition of the complete graph into two factors was solved completely. In [3] it was proved that if $h \geqq 3$ and $d_i \geqq 2$ for $i = 1, 2, \ldots, h$, then $F(d_1, d_2, \ldots, d_h) \neq \infty$. For this reason it is sufficient to consider the case of the natural number only.

(a) *(Theorem 1.) If the complete graph $\langle n\rangle$ $(n > 1)$ is decomposable into h factors with diameters $d_1, d_2, \ldots, d_h$, then for $N > n$ the complete graph $\langle N\rangle$ is also decomposable into h factors with the diameters $d_1, d_2, \ldots, d_h$.*

(b) *(The second part of Theorem 2.) Let the natural numbers $h, n, d_1, d_2, \ldots, d_h$ be given. If the complete graph $\langle n\rangle$ is decomposable into h factors with the diameters $d_1, d_2, \ldots, d_h$, then*

$$2h \leqq n. \tag{1}$$

(c) *(Corollary 2 of Theorem 2.) Let h, n and d be natural numbers. If $\langle n\rangle$ is decomposable into h factors with equal diameters d, then*

$$n^2 - (2h + 1)n \leqq h(s^2 + s - 4), \tag{2}$$

where $s = n - d$.

(d) *(A special case of Theorem 5.)* $F(2, 2) = 5, F(3, 3) = 4$ and $F(d, d) \quad \infty$ *otherwise.*

*

Let the natural numbers $s \geqq 1$, $h \geqq 2$ be given. Our aim is to determine all natural numbers d such that the complete graph $\langle d + s\rangle$ is decomposable into h factors with equal diameters d, for every pair (s, h). Let us denote by the symbol $D_{s,h}$ the set of natural numbers d assigned to a certain pair (s, h) in this way.(2) First (Theorem 1) we prove that for every pair (s, h) there exists such a natural number d, i. e. the set $D_{s,h}$ is not empty.

Theorem 1. *Let natural numbers s, h, d be given such that $h \geqq 2$ and $d = 2h - 1$. Then the complete graph $\langle d + s\rangle$ is decomposable into h factors with equal diameters d.*

Proof. According to [1], p. 91, every complete graph $\langle 2h\rangle$ can be decomposed into h factors with equal diameters $2h - 1$. From Theorem 1 of [2] it follows that an arbitrary complete graph with a greater number of vertices can be decomposed in this way.

In the following we prove five lemmas.

Lemma 1. *Let a natural number $h \geqq 3$ be given. Then every complete graph $\langle 2h\rangle$ is decomposable into h factors with equal diameters $2h - 2$.*

Proof. The decomposition of $\langle 6\rangle$ into 3 factors with the diameter 4 is shown in Fig. 1.

(a) Let $h \geqq 4$ be an even number, i. e. $h = 2r$, where $r \geqq 2$, then the factor

(2) Thus $\langle n\rangle$ can be decomposed into h factors with equal diameters d obviously if and only if $d \in D_{n-d,h}$.

F_1 (F_{r+1}) has the form shown in Fig. 2 (Fig. 3). The factors F_j (F_{r+j}), $2 \leqq j \leqq r$ can be obtained from F_1 (F_{r+1}) by replacing each vertex i by the vertex $i + j - 1 \pmod{2h}$.

(b) Let $h > 4$ be an odd number, i. e. $h = 2r - 1$, where $r \geqq 3$, then the factor F_1 (F_{r+1}) has the form shown in Fig. 4 (Fig. 5). The factors F_j (F_{r+j}), $2 \leqq j \leqq r$ $(2 \leqq j \leqq r - 1)$ can be obtained from F_1 (F_{r+1}) by replacing each vertex i by the vertex $i + j - 1 \pmod{2h}$.

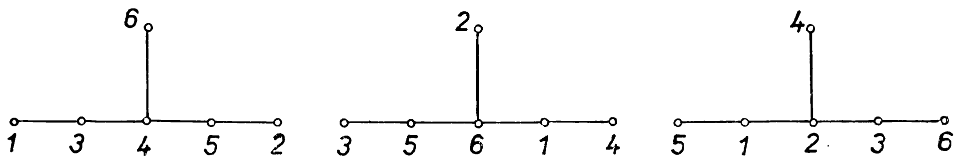

Fig. 1

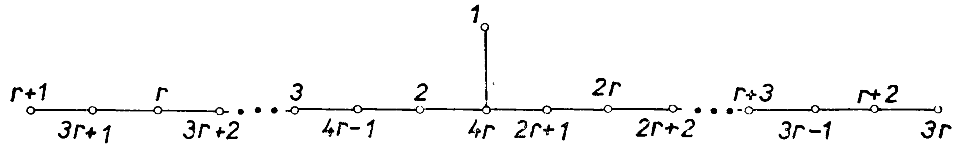

Fig. 2

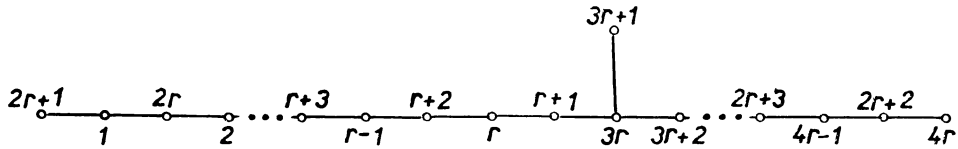

Fig. 3

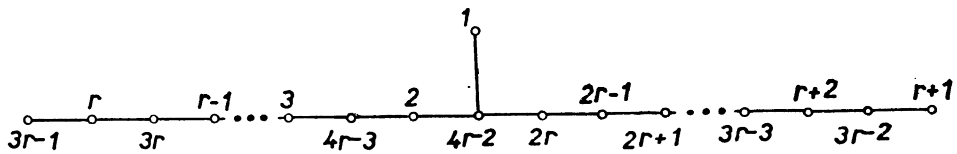

Fig. 4

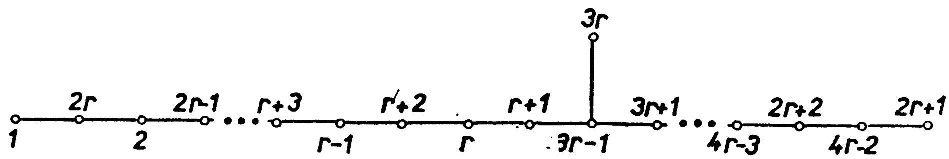

Fig. 5

We can easily check that every edge of the graph $\langle 2h \rangle$ is contained in exactly one of the factors F_i ($i = 1, 2, \dots, h$) and consequently the system of subgraphs F_i forms a decomposition of $\langle 2h \rangle$ into h factors. E. g. in case (a) this follows from the fact that if we draw $\langle 2h \rangle$ in the form of a regular polygon with all its diagonals, then the union of the factors F_1 and F_{r+1} consists just of the edges (diagonals) parallel to $(1, 4r)$, $(2, 4r)$, $(1, 2r)$ and $(1, 2r + 1)$, and cyclic permutations of vertices correspond to rotations of F_1 and F_{r+1}. It is also evident that each of the factors F_i has the diameter $2h - 2$.

Lemma 2. *Let a natural number $h \geqq 3$ be given. Then every complete graph $\langle 2h \rangle$ is decomposable into h factors with equal diameters $2h - 3$.*

Proof. The factor F_1 has the form shown in Fig. 6. The remaining factors F_i ($i = 2, 3, \dots, h$) can be obtained from F_1 by cyclic permutations of vertices. The rest of the proof is similar to that in Lemma 1.

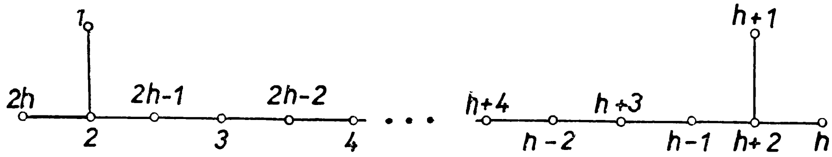

Fig. 6

Lemma 3. *Let natural numbers $s \geqq 1$, $h \geqq 2$ be given. Then for an arbitrary $d \in D_{s,h}$ we have:*

(3) $$d \leqq \tfrac{1}{2}\left(2h + 1 - 2s + \sqrt{[4h^2 + 4h(s^2 + s - 3) + 1]}\right).$$

Proof. The substitution $n = d + s$ in the inequality (2) gives:

(4) $$d^2 + d(2s - 2h - 1) + h(4 - 3s - s^2) + s^2 - s \leqq 0 .$$

The left side of (4) is a quadratic function of the variable d. As this function is convex, the solution of the inequality (4) with respect to d is

$$d_2 \leqq d \leqq d_1 ,$$

where

$$d_{1,2} = \tfrac{1}{2}(2h + 1 - 2s \pm \sqrt{[4h^2 + 4h(s^2 + s - 3) + 1]})$$

are the roots of the left-hand side of (4). For natural numbers $s \geqq 1$, $h \geqq 2$, the expression $4h^2 + 4h(s^2 + s - 3) + 1$ is positive and so the roots $d_{1,2}$ are real. Thus we have (3).

Lemma 4. *Let natural numbers $s \geqq 1$, $h \geqq 2$ be given. Then for an arbitrary $d \in D_{s,h}$* we have:

$$d \leqq 2h - 1 \qquad \text{if} \quad s = 1,$$

$$d \leqq 2h - 2 + \binom{s}{2} \qquad \text{if} \quad s \geqq 2.$$

Proof. If $s = 1$, the first relation follows from (3). Now we assume that $s \geqq 2$. If under the square root on the right-hand side of the inequality (3) we write 4 instead of 1, we have

$$(5) \qquad d < \tfrac{1}{2}(2h + 1 - 2s + 2\sqrt{[h^2 + h(s^2 + s - 3) + 1]}) .$$

For $s \geqq 2$ we have $(s^2 + s - 3)/2 > 1$ and therefore we can replace the unit under the square root on the right-hand side of (5) by the expression $[(s^2 + s - 3)/2]^2$. Then we have

$$(6) \qquad d < 2h - 1 + \tfrac{1}{2}(s^2 - s) .$$

The right-hand side of the inequality (6) being an integer, we can write

$$d \leqq 2h - 2 + \binom{s}{2} .$$

This completes the proof of the lemma.

Lemma 5. *Let natural numbers $s \geqq 1$, $h \geqq 2$ be given. Then for an arbitrary $d \in D_{s,h}$ we have:*

$$d \geqq \max[2, (2h - s)] .$$

Proof. The substitution $n = d + s$ in the inequality (1) gives $d \geqq 2h - s$. To complete the proof of the lemma it is sufficient to take into account that $d \geqq 2$.

Let us denote

$$\max D_{s,h} = G(s, h),$$

$$\min D_{s,h} = g(s, h).$$

From Theorem 1 and Lemmas 4 and 5 it follows that $G(s, h)$ and $g(s, h)$ are defined for arbitrary integers $s \geqq 1$ and $h \geqq 2$. The following two theorems are dealing with the functions G and g.

Theorem 2. *Let $s \geqq 1$ and $h \geqq 2$ be integers. Then we have:*

$$\begin{aligned} \text{I.} \quad & G(1, h) = 2h - 1, \\ \text{II.} \quad & G(2, h) = 2h - 1, \\ \text{III.} \quad & G(3, h) \begin{cases} = 2h & \text{if} \quad 3 \leqq h \leqq 5, \\ \leqq 2h + 1 & \text{if} \quad h \geqq 6, \end{cases} \\ \text{IV.} \quad & G(s, 2) = 3. \end{aligned}$$

Proof. I. From Lemma 4 it follows that $G(1, h) \leqq 2h - 1$. Therefore it is sufficient to prove that the graph $\langle 2h - 1 + 1 \rangle = \langle 2h \rangle$ is decomposable into h factors with equal diameters $2h - 1$. According to Theorem 1 such a decomposition evidently exists.

II. The proof is analogous.

III. For $s = 3$ from Lemma 4 we have

$$G(3, h) \leqq 2h + 1 . \tag{7}$$

The substitution $s = 3$ and $h = 3, 4, 5$ in (3) successively gives: $G(3, 3) \leqq 6$, $G(3, 4) \leqq 8$, $G(3, 5) \leqq 10$, i. e.

$$G(3, h) \leqq 2h \quad \text{if} \quad 3 \leqq h \leqq 5 . \tag{8}$$

(a) As (8) holds, to prove that $G(3, h) = 2h$ if $h = 3, 4, 5$ it is sufficient to decompose the graph $\langle 2h + 3 \rangle$ into h factors with equal diameters $2h$, when $h = 3, 4, 5$. These decompositions are shown in Fig. 7 (see also Fig. 6 of [2]), Figs. 8 and 9.

(b) If $h \geqq 6$, the inequality (7) holds.

IV. According to Theorem 5 of [2] we have $F(3, 3) = 4$ and $F(d, d) = \infty$ if $d > 3$. From this and from Theorem 1 of [2] it follows that 3 is the greatest natural number d such that the graph $\langle d + s \rangle$ is decomposable into two factors with equal diameters d. Thus $G(s, 2) = 3$.

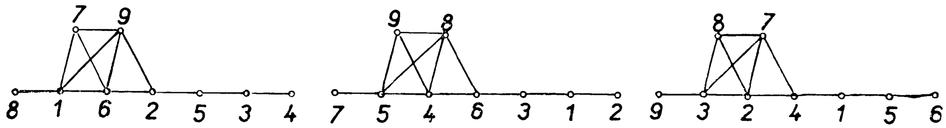

Fig. 7

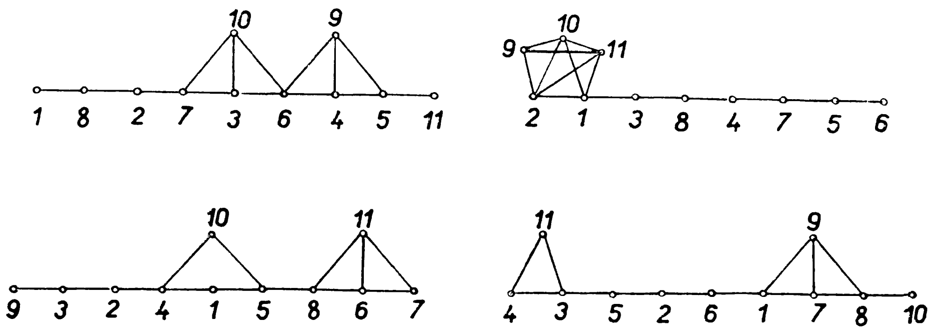

Fig. 8

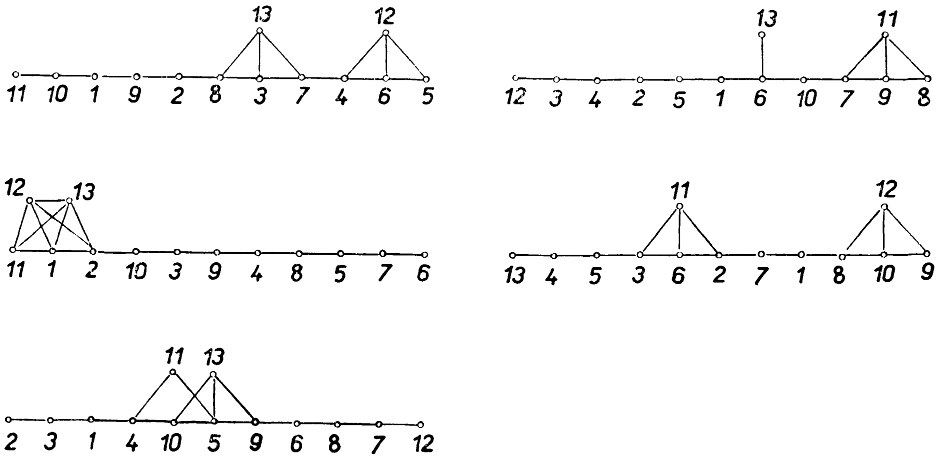

Fig. 9

Theorem 3. *Let* $s \geqq 1$ *and* $h \geqq 2$ *be integers. Then we have:*

$$\text{I.}\quad g(1, h) = 2h - 1,$$

$$\text{II.}\quad g(2, h) = \begin{cases} 3 & \text{if } h = 2, \\ 2h - 2 & \text{if } h \geqq 3, \end{cases}$$

$$\text{III.}\quad g(3, h) = 2h - 3 \quad \text{if } h \geqq 3,$$

$$\text{IV.}\quad g(s, 2) = 2 \quad \text{if } s \geqq 3.$$

Proof. The proof of I follows from Lemma 5 and Theorem 1.

II. (a) With respect to Lemma 5 $g(2, 2) \geqq 2$ holds, but $g(2, 2) \neq 2$, which follows from [2] (Theorem 5). In order to prove that $g(2, 2) = 3$, it is sufficient to decompose the graph $\langle 3 + 2 \rangle = \langle 5 \rangle$ into two factors with diameter 3. According to Theorems 5 and 1 of [2] such a decomposition evidently exists.

(b) If $h \geqq 3$, from Lemma 5 it follows that $g(2, h) \geqq 2h - 2$. Lemma 1 shows that the lower bound gives the exact value.

III. If $h \geqq 3$, from Lemma 5 it follows that $g(3, h) \geqq 2h - 3$. Lemma 2 shows that the lower bound gives the exact value.

IV. With respect to Theorem 5 of [2] $F(2, 2) = 5$ holds. From Theorem 1 of [2] it follows that the graph $\langle 2 + s \rangle$ (where $s \geqq 3$) is decomposable into two factors with diameter 2. Thus $g(s, 2) = 2$ if $s \geqq 3$.

Remark. Theorems 2 and 3 allow us to determine the set $D_{s,h}$ for these cases: $D_{1,h} = \{2h - 1\}$, $D_{2,2} = \{3\}$, $D_{s,2} = \{2, 3\}$ if $s \geqq 3$ and $D_{2,h} = \{2h - 2, 2h - 1\}$ if $h \geqq 3$. In cases $D_{3,3}$, $D_{3,4}$ and $D_{3,5}$ Theorems 2 and 3 determine only the maximum and minimum of the given sets, but by systematic decompositions of the corresponding graphs we can see that $D_{3,3} = \{3, 4, 5, 6\}$, $D_{3,4} = \{5, 6, 7, 8\}$ and $D_{3,5} = \{7, 8, 9, 10\}$.

REFERENCES

[1] HARARY F.: Graph theory. Addison-Wesley, Reading, Massachusetts 1969.

[2] BOSÁK J.—ROSA A.—ZNÁM Š.: On decompositions of complete graphs into factors with given diameters. In: Theory of graphs. Proceedings of the Colloquium held at Tihany, Hungary, September 1966, Akadémiai Kiadó, Budapest, 1968, 37—56.

[3] BOSÁK J.—ERDŐS P.—ROSA A.: Decompositions of complete graphs into factors with diameter two. Mat. Čas. *21*, 1971, 14—28.

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