Vlasta Peřinová

On a Non-Linear Integral Equation

Matematický časopis, Vol. 19 (1969), No. 1, 1--8,9--16

MATEMATICKÝ ČASOPIS

ROČNÍK 19 1969 ČÍSLO 1

ON A NON-LINEAR INTEGRAL EQUATION

VLASTA PEŘINOVÁ, Olomouc

In the present paper we shall deal with the homogeneous non-linear integral equation

$$u^2(x) = \lambda^2 \int_a^b \int_a^b L(x, y, z) u(y) u(z) \mathrm{d}y \mathrm{d}z \tag{1}$$

for the function $u(x)$, where $L(x, y, z)$ is a given function and λ is a real parameter.

For equation (1) we shall study:

1. The existence of a positive solution.
2. The branching of a solution which exists for a certain value of the parameter λ.
3. The continuation of the solution for an arbitrary value of the parameter λ.

To solve the first task we shall use the method which is analogical to the method used by W. Schmeidler and D. Morgenstern in [1]. The second task will be solved by the general method given for example in [2] on which is also based the process of continuation of the solution for an arbitrary value of λ.

In this paper we shall study only real solutions.

1. THE EXISTENCE OF A POSITIVE SOLUTION

Now we shall study the problem of the existence of a positive eigenvalue and a positive eigenfunction of equation (1). Under an eigenvalue of equation (1) such a value of λ must be understood for which equation (1) has a non-trivial normed solution. This solution is called an eigenfunction.

Theorem 1. *If the function $L(x, y, z)$ satisfies the assumptions:*

a) *$L(x, y, z)$ is a real function continuous in $\langle a, b\rangle \times \langle a, b\rangle \times \langle a, b\rangle$,*

b) *$L(x, y, z) > l > 0$ for all x, y, z,*

then there exist the positive eigenvalue λ_0 and the real continuous eigenfunction

$u_0(x) > k > 0$ satisfying equation (1) *and*

$$(1.1)\qquad \lambda_0 = \frac{1}{\sqrt{\int_a^b\int_a^b\int_a^b L(x, y, z)u_0(y)u_0(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z}}$$

is valid.

Proof. Let us make the substitution $u^2(x) = v(x)$ in (1) and write it in the operator form

$$(1.2)\qquad v = \lambda^2 A v$$

where

$$Av = \int_a^b\int_a^b L(x, y, z)\sqrt{v(y)}\sqrt{v(z)}\mathrm{d}y\mathrm{d}z.$$

Let us consider the set M of functions $v(x) > k > 0$ for which $\|v\| = \int_a^b v(x)\mathrm{d}x = 1$ in the space of the functions continuous in $\langle a, b\rangle$. Let the operator A be defined on the set M.

To prove the existence of a solution of (1.2) in M we shall use the Brouwer-Schauder fixed point theorem: If a continuous operator B maps a convex set T of the Banach space into a compact part of the set T, then there exists such a point $x \in T$ that $Bx = x$. First we shall prove the continuity of the operator A. Let us consider the sequence of such functions $v_n(x)$ from M that $\|v - v_n\| \to 0$ $(n = \overline{1, \infty})$ and denote $\max\limits_{x,y,z} L(x, y, z) = L$. Then for $\|Av - Av_n\|$ we obtain, using the Schwarz inequality,

$$\|Av - Av_n\| = \int_a^b |Av - Av_n|\,\mathrm{d}x \leqslant L(b-a)\int_a^b(\sqrt{v(y)} + \sqrt{v_n(y)})\,\mathrm{d}y\,.\int_a^b|\sqrt{v(z)} - \sqrt{v_n(z)}|\,\mathrm{d}z < \frac{L\sqrt{(b-a)^3}}{\sqrt{k}}\|v - v_n\|;$$

from this the continuity of A follows.

Let us prove that the set of images of A is compact. For this it is necessary and sufficient to prove that the image set is composed from functions uniformly bounded and equicontinuous. The functions Av are uniformly bounded as follows from the relation

$$\|Av\| \leqslant L(b-a)\,(\int_a^b\sqrt{v(y)}\,\mathrm{d}y)^2 \leqslant L(b-a)^2.$$

Let us prove the equicontinuity of functions Av. Let $\varepsilon > 0$ be given. Then in consequence of the uniform continuity of the kernel $L(x, y, z)$ there exists such $\delta > 0$ that

$$|L(x_1, y, z) - L(x_2, y, z)| < \frac{\varepsilon}{b - a}$$

is valid for $|x_1 - x_2| < \delta$ and for all y, z from $\langle a, b\rangle$. Then

$$|Av(x_1) - Av(x_2)| < \frac{\varepsilon}{b - a}\Big(\int_a^b \sqrt{v(y)}\,\mathrm{d}y\Big)^2 \leqslant \varepsilon$$

is valid for every function $v(x) \in M$. Hence, the functions Av are equicontinuous and the operator A is totally continuous on M. As follows from the inequality

$$||Av|| > lk(b - a)^3,$$

$||Av||$ is different from zero and so the operator $A/||A||$ is totally continuous on M and maps the set M into itself.

Now we shall show that M is convex, i.e. if functions $r(x)$ and $s(x)$ belong to M the function $t(x) = \lambda r(x) + \mu s(x)/\lambda + \mu$ for positive λ and μ belongs to M, too. As for $||r|| = ||s|| = 1$ the following is valid: $||t|| = (\lambda||r|| + \mu||s||)/\lambda + + \mu = 1$ and for the function $t(x)$ we have $t(x) > k$, hence $t(x)$ belongs to M and M is convex.

Hence, the assumptions of the above Theorem are satisfied and there exists such a function $v_0(x)$ that

$$v_0 = \frac{Av_0}{||Av_0||},$$

i.e.

$$v_0 = \lambda_0^2 A v_0$$

where

$$\lambda_0^2 = \frac{1}{||Av_0||}.$$

If we introduce the primary notation Theorem 1 is proved.

2. THE BRANCHING OF THE SOLUTION

The couple $(\lambda_0, u_0(x))$ which obeys equation (1) is called the branch point for this equation if for every $\varepsilon > 0$ there exists such λ that $|\lambda - \lambda_0| < \varepsilon$ and equation (1) has for this λ at least two solutions which lie in the ε-neighbourhood of the solution $u_0(x)$.

Theorem 2. *Let $L(x, y, z)$ be a real function continuous in $\langle a, b\rangle \times \langle a, b\rangle \times \times \langle a, b\rangle$ and symmetric according to the variables y, z. Let $u_0(x)$ be a real continuous eigenfunction of equation* (1) *corresponding to an eigenvalue $\lambda_0 \neq 0$ and let $u_0(x)$ be different from zero in $\langle a, b\rangle$. Then for a neighbourhood of the point $\lambda = \lambda_0$ the following assertions are valid:*

a) *If λ_0^2 is not an eigenvalue of the kernel*

$$G(x, z) = \frac{1}{u_0(x)} \int_a^b L(x, y, z) u_0(y) \mathrm{d}y,$$

then there exists the unique real solution of (1), *which can be expanded in the series*

$$\text{(d)} \qquad u(x) = \sum_{i=0}^{\infty} (\lambda - \lambda_0)^i u_i(x).$$

b) *If λ_0^2 is an n-multiple eigenvalue of the kernel $G(x, z)$ with the continuous associated eigenfunctions $\alpha_i(x)$ $(i = \overline{1, n})$ and if*

$$\text{(e)} \qquad \int_a^b u_0(x) \alpha_i(x) \mathrm{d}x = 0, \qquad i = \overline{1, n},$$

is valid, then there exist 2^n real solutions of (1) *in the form* (d).

c) *If λ_0^2 is the simple eigenvalue of the kernel $G(x, z)$ and the condition* (e) *for $i = 1$ is not fulfilled, then there exist two real solutions of* (1) *in the form*

$$u(x) = \sum_{i=0}^{\infty} (\lambda - \lambda_0)^{i/2} u_i(x),$$

which either finish at the point λ_0 or start from it.

All these solutions are continuous in $\langle a, b\rangle$.

Proof. If we denote

$$\text{(2.1)} \qquad \lambda - \lambda_0 = \mu, \qquad u(x) - u_0(x) = p(x),$$

equation (1) can be rewritten in the form

$$\text{(2.2)} \qquad p(x) - \lambda_0^2 \int_a^b G(x, z) p(z) \mathrm{d}z = g[u_0, p],$$

where

$$g[u_0, p] = \frac{1}{2u_0(x)} [(2\lambda_0\mu + \mu^2) \int_a^b \int_a^b L(x, y, z) (u_0(y) + p(y)) (u_0(z) + p(z)) \mathrm{d}y \mathrm{d}z + \\ + \lambda_0^2 \int_a^b \int_a^b L(x, y, z) p(y) p(z) \mathrm{d}y \mathrm{d}z - p^2(x)].$$

Let us seek the solution of (2.2) in the form

$$p(x) = \sum_{i=1}^{\infty} \mu^i u_i(x). \tag{2.3}$$

If we substitute (2.3) in (2.2) and compare the coefficients of the same powers of μ, we obtain for $u_i(x)$ the system of equations

$$u_i(x) = \lambda_0^2 \int_a^b G(x, z) u_i(z) \mathrm{d}z + f_i(x), \qquad i = \overline{1, \infty} \tag{2.4}$$

where

$$f_1(x) = \frac{u_0(x)}{\lambda_0},$$

$$f_2(x) = \frac{1}{2u_0(x)} [\int_a^b \int_a^b L(x, y, z) \, (4\lambda_0 u_0(y) u_1(z) + \lambda_0^2 u_1(y) u_1(z)) \mathrm{d}y \mathrm{d}z - - u_1^2(x) + M_2],$$

$$f_i(x) = \frac{1}{u_0(x)} [\int_a^b \int_a^b L(x, y, z) \, (2\lambda_0 u_0(y) u_{i-1}(z) + \lambda_0^2 u_1(y) u_{i-1}(z)) \mathrm{d}y \mathrm{d}z - - u_1(x) u_{i-1}(x) + M_i] = K[u_{i-1}] + \frac{M_i}{u_0(x)}, \qquad i = \overline{3, \infty} \tag{2.5}$$

and

$$M_i = M_i[x; u_0, u_1, \ldots, u_{i-2}], \qquad i = \overline{2, \infty}.$$

Equations (2.4) are non-homogeneous linear integral equations. Solving these equations it is necessary to distinguish whether λ_0^2 is or is not an eigenvalue of the kernel $G(x, z)$.

a) If λ_0^2 is not an eigenvalue of the kernel $G(x, z)$, then there exists the continuous resolving kernel $\Gamma(x, z; \lambda_0^2)$ and the unique solutions of equations (2.4) can be written in the form

$$u_i(x) = f_i(x) + \lambda_0^2 \int_a^b \Gamma(x, z; \lambda_0^2) f_i(z) \mathrm{d}z, \qquad i = \overline{1, \infty}. \tag{2.6}$$

Thus it is possible to determine all the functions $u_i(x)$ and to construct the series (2.3) formally.

Now we shall prove that the constructed series converges absolutely and uniformly according to x and μ in $\langle a, b\rangle$ for μ sufficiently small. Let us choose such numbers A, B, C, D that for $x \in \langle a, b\rangle$

$$1) \int_a^b \int_a^b |L(x, y, z)| \mathrm{d}y \mathrm{d}z < A,$$

$$(2.7)\qquad \begin{aligned} &2)\ 0 < D < |u_0(x)| &< B,\\ &3)\ \int_a^b |\Gamma(x, z; \lambda_0^2)|\mathrm{d}z &< C, \end{aligned}$$

are valid. Equation (2.2) may be considered as an equation of the same type as (2.4) with the resolving kernel $\Gamma(x, z; \lambda_0^2)$ and written in the form

$$(2.8)\qquad p(x) = g[u_0, p] + \lambda_0^2 \int_a^b \Gamma(x, z; \lambda_0^2) g[u_0, p]\mathrm{d}z.$$

If the restriction $|p(x)| < P(x \in \langle a, b\rangle)$ is valid, then we obtain for P from (2.8) and (2.7)

$$(2.9)\qquad P = E[(2|\lambda_0|\mu + \mu^2)(B + P)^2 A + (1 + \lambda_0^2 A)P^2]$$

where

$$E = \frac{1}{2D}(1 + \lambda_0^2 C).$$

Let us seek the solution of (2.9) in the form of the power series expansion

$$(2.10)\qquad P(\mu) = \sum_{i=1}^{\infty} \mu^i k_i.$$

Substituting (2.10) in (2.9) and comparing coefficients of the same powers of μ we obtain

$$2.1\ 1)\qquad \begin{aligned} k_1 &= 2|\lambda_0| A B^2 E,\\ k_2 &= (4|\lambda_0| A B k_1 + (1 + \lambda_0^2 A) k_1^2 + \overline{M}_2)E,\\ k_i &= (4|\lambda_0| A B k_{i-1} + 2(1 + \lambda_0^2 A) k_1 k_{i-1} + \overline{M}_i)E, \qquad i = \overline{3, \infty}, \end{aligned}$$

where $\overline{M}_i$ is a majorant for $M_i[x; u_0, \ldots, u_{i-2}]$. From the relations (2.11) and (2.4) it can be seen that

$$|u_i(x)| < k_i \quad \text{for} \quad x \in \langle a, b\rangle, \qquad i = \overline{1, \infty}.$$

From this it follows that the region of convergence of (2.10) will be the region of convergence of (2.3). From (2.9) and $P(0) = 0$ we obtain

$$(2.12)\qquad P(\mu) = \frac{-d - \sqrt{d^2 - 4ec}}{2c},$$

where

$$\begin{aligned} c &= E(A\mu(\mu + 2|\lambda_0|) + (1 + \lambda_0^2 A)),\\ d &= 2ABE\mu(\mu + 2|\lambda_0|) - 1,\\ e &= AB^2 E\mu(\mu + 2|\lambda_0|). \end{aligned}$$

From the condition $d^2 - 4ec \geqslant 0$ we derive the following relation

$$(2.13) \qquad 0 < \mu \leqslant \frac{-f|\lambda_0| + \sqrt{f^2|\lambda_0|^2 + f}}{f},$$

where

$$f = 4ABE(1 + (1 + \lambda_0^2 A)BE).$$

Hence, the series (2.10) converges in the interval (2.13) and because (2.10) is a majorant for the series (2.3), the series (2.3) converges absolutely and uniformly according to x and μ to a continuous function $p(x)$ in $\langle a, b\rangle$ and (2.13). Thus in the neighbourhood of the point $\lambda = \lambda_0$ there exists the unique solution of (1)

$$u(x) = u_0(x) + p(x),$$

which tends to $u_0(x)$ for $\lambda \to \lambda_0$.

b) Let λ_0^2 be an n-multiple eigenvalue of the kernel $G(x, z)$ with continuous eigenfunctions $\varphi_l(x)(l = \overline{1, n})$ and continuous associated eigenfunctions $\alpha_l(x)(l = \overline{1, n})$. If equations (2.4) are to have solutions it is necessary and sufficient that

$$(2.14) \qquad \int_a^b f_i(x)\alpha_l(x)\mathrm{d}x = 0, \qquad i = \overline{1, \infty},\ l = \overline{1, n},$$

is valid.

Let us consider the case when the condition (2.14) is valid for $i = 1$. Then the solutions of (2.4) can be written in the form

$$(2.15) \qquad u_1(x) = t_1(x) + \sum_{j=1}^{n} C_j^1 \varphi_j(x),$$

where

$$(2.16) \qquad t_i(x) = f_i(x) + \lambda_0^2 \int_a^b T(x, z; \lambda_0^2) f_i(z)\mathrm{d}z,$$

$$(i = \overline{1, \infty}, \text{ for } i = \overline{2, \infty} \text{ see further})$$

where $T(x, z; \lambda_0^2)$ is the continuous resolving kernel of the kernel

$$G(x, z) - \frac{1}{\lambda_0^2} \sum_{j=1}^{n} \varphi_j(x)\alpha_j(z).$$

For the determination of the constants C_j^1 we obtain the system of n non-linear equations

$$\int_a^b \frac{\alpha_l(x)}{u_0(x)}\Bigg\{\int_a^b\int_a^b L(x, y, z)(4\lambda_0 u_0(y)\left[t_1(z) + \sum_{j=1}^{n} C_j^1\varphi_j(z)\right] +$$

$$(2.17)\qquad + \lambda_0^2[t_1(y) + \sum_{j=1}^{n} C_j^1\varphi_j(y)]\cdot[t_1(z) + \sum_{j=1}^{n} C_j^1\varphi_j(z)])\mathrm{d}y\mathrm{d}z -$$

$$- (t_1(x) + \sum_{j=1}^{n} C_j^1\varphi_j(x))^2 + M_2\Bigg\}\,\mathrm{d}x = 0,\quad l = \overline{1, n}$$

from the conditions (2.14) for $i = 2$ after substitution (2.15) in $f_2(x)$. From the system (2.17) we obtain, in general, 2^n systems C_j^1 ($j = 1, \dots, n$). Thus we determine 2^n functions $u_1(x)$

$$(2.18)\qquad u_{1k}(x) = t_1(x) + \sum_{j=1}^{n} C_{jk}^1\varphi_j(x),\quad k = \overline{1, 2^n}.$$

The solution of the i-th ($i \geqslant 2$) equation of the system (2.4) can be written in the form

$$(2.19)\qquad u_i(x) = t_i(x) + \sum_{j=1}^{n} C_j^i\varphi_j(x).$$

From the conditions (2.14) of solving the $(i + 1)$-th equation from (2.4) we obtain the following system of n linear equations for C_j^i

$$(2.20)\qquad \sum_{j=1}^{n} C_j^i\int_a^b \alpha_l(x)K[\varphi_j]\mathrm{d}x = -\int_a^b \alpha_l(x)\left(K[t_i] + \frac{M_i}{u_0(x)}\right)\mathrm{d}x = m_l^i,\quad l = \overline{1, n}.$$

From this system it is possible to determine uniquely $C_j^i(j = \overline{1, n})$ in the form

$$(2.21)\qquad C_j^i = \sum_{l=1}^{n} a_{jl}m_l^i,$$

under the assumption that the determinant is different from zero and so the functions $u_i(x)$ are determined uniquely.

Therefore it is possible to construct 2^n series (2.3) formally. The convergence of these series may be proved in the following way. Let us consider two sequences of such numbers u_i, v_i, $\{u_i\}_0^\infty$, $\{v_i\}_0^\infty$ that

$$(2.22)\qquad \left.\begin{aligned} &|u_1(x)| \leqslant |t_1(x)| + \sum_{j=1}^{n} |C_j^1|\,|\varphi_j(x)| < u_0 + v_0 \\ &|u_{i+1}(x)| \leqslant |t_{i+1}(x)| + \sum_{j=1}^{n} |C_j^{i+1}|\,|\varphi_j(x)| < u_i + v_i,\ i = \overline{1, \infty} \end{aligned}\right\}\quad x \in \langle a, b\rangle$$

is valid.

To determine such numbers u_i, v_i let us consider the function

$$(2.23)\quad S(w) = \frac{1}{2D}\,[\mu^2AB^2 + 2AB\mu(2|\lambda_0| + \mu)w + (1 + A(|\lambda_0| + \mu)^2)w^2],$$

where the constants A, B, D are determined by (2.7) 1), 2). If we put instead of w

$$w = \sum_{i=0}^{\infty} \mu^{i+1}(u_i + v_i) \tag{2.24}$$

in (2.23) and if we expand the expression obtained in the powers of μ, then

$$S(w) = \sum_{i=2}^{\infty} \mu^i S_i, \tag{2.25}$$

where

$$\begin{aligned} S_2 &= \frac{1}{2D}\,[AB^2 + 4AB|\lambda_0|(u_0 + v_0) + (1 + \lambda_0^2 A)(u_0 + v_0)^2], \\ S_i &= \frac{1}{D}\,\{[2AB|\lambda_0| + (1 + \lambda_0^2 A)(u_0 + v_0)](u_{i-2} + v_{i-2}) + \overline{M}_i\},\quad i = \overline{3, \infty}; \end{aligned} \tag{2.26}$$

$\overline{M}_i$ is a majorant for M_i.

From the relations (2.5), (2.22) and from the assumptions (2.7) 1), 2) it follows that S_i are upper bounds for the functions $f_i(x)(i = \overline{2, \infty})$.

Let us further choose such numbers N, T, γ that

$$\begin{aligned} &1)\quad \max_{k,l} |a_{kl}| = N, \\ &2)\quad \lambda_0^2 \int_a^b |T(x, z; \lambda_0^2)|\mathrm{d}z < T,\quad x \in \langle a, b\rangle, \\ &3)\quad \max_{i,x} (|\varphi_i(x)|,\ |\alpha_i(x)|) < \gamma \end{aligned} \tag{2.27}$$

are valid and designate $\max(1, T) = M$. Then we determine u_i $(i = \overline{1, \infty})$ from the equation

$$u_i = 2MS_{i+1}. \tag{2.28}$$

If we take into account that

$$|t_{i+1}(x)| < \max_x |f_{i+1}(x)|(1 + \lambda_0^2 \int_a^b |T(x, z; \lambda_0^2)|\mathrm{d}z) < 2MS_{i+1}$$

then $|t_{i+1}(x)| < u_i$. The constants $v_i(i = \overline{1, \infty})$ can be determined from the equation

$$(2.29) \qquad \left[1 + \frac{d}{D}(2AB|\lambda_0| + (1 + \lambda_0^2 A)(u_0 + v_0))\right] v_i = dS_{i+2},$$

where

$$d = n^2\gamma^2(b-a)N.$$

As the following is valid

$$|C_j^{i+1}| < N \sum_{l=1}^{n} |m_l^i| < N\gamma n(b-a)\boldsymbol{Q}, \quad \sum_{j=1}^{n} |C_j^{i+1}|\,|\varphi_j(x)| < d\boldsymbol{Q},$$

where

$$\boldsymbol{Q} = \frac{1}{D}[(2AB|\lambda_0| + (1 + \lambda_0^2 A)(u_0 + v_0))u_i + \overline{M}_{i+2}]$$

and from (2.29) we obtain $v_i = d\boldsymbol{Q}$, then

$$\sum_{j=1}^{n} |C_j^{i+1}|\,|\varphi_j(x)| < v_i$$

and thus $|u_{i+1}(x)| < u_i + v_i$.

If we introduce the notation

$$(2.30) \qquad u = \sum_{i=1}^{\infty} \mu^{i+1} u_i, \quad v = \sum_{i=1}^{\infty} \mu^{i+1} v_i,$$

then

$$w = \mu(u_0 + v_0) + u + v$$

and the determination of u_i, v_i from (2.28) and (2.29) is equivalent to the solving of the following system for u, v

$$u = 2MS(\mu(u_0 + v_0) + u + v),$$

$$(2.31) \qquad \left[1 + \frac{d}{D}(2AB|\lambda_0| + (1 + \lambda_0^2 A)(u_0 + v_0))\right]\mu v =$$

$$= d[S(\mu(u_0 + v_0) + u + v) - \mu^2 S_2]$$

in the form (2.30). If we carry out the substitution

$$u = \mu U, \qquad v = \mu V$$

in (2.31) and devide the first equation by μ and the second by μ^2, we obtain for U, V the system

$$\Phi_1 \equiv U - \frac{M}{D}[\mu AB^2 + 2AB\mu(2|\lambda_0| + \mu)(u_0 + v_0 + U + V) +$$

$$+ (1 + (|\lambda_0| + \mu)^2 A)\mu(u_0 + v_0 + U + V)^2] = 0,$$

(2.32) $$\Phi_2 \equiv \left[1 + \frac{d}{D}(2AB|\lambda_0| + (1 + \lambda_0^2 A)(u_0 + v_0))\right] U -$$

$$- \frac{d}{2D}[2AB(2|\lambda_0| + \mu)(u_0 + v_0 + U + V) +$$

$$+ (1 + (|\lambda_0| + \mu)^2 A)(u_0 + v_0 + U + V)^2 -$$

$$- (4AB|\lambda_0| + (1 + \lambda_0^2 A)(u_0 + v_0))(u_0 + v_0)] = 0.$$

For (2.32) we shall use the implicit function theorem. If the system (2.32) is to determine unambiguous continuous functions $U(\mu)$, $V(\mu)$ in a neighbourhood of the point $\mu = 0$, it is necessary and sufficient that

$$\Delta = \frac{D(\Phi_1, \Phi_2)}{D(U, V)} \neq 0 \quad \text{for} \quad \mu = U = V = 0 \quad \text{is valid.}$$

As for $\mu = U = V = 0$ we have $\Delta = -1$, the assumptions of the above Theorem are fulfilled and the system (2.32) has only one solution U, V in the form of the series

$$U = \sum_{i=1}^{\infty} \mu^i u_i, \qquad V = \sum_{i=1}^{\infty} \mu^i v_i,$$

which have a finite radius of convergence in a neighbourhood of the point $\mu = 0$. The same is valid for the series (2.24). As this series is the majorant for (2.3), the series (2.3) converges absolutely and uniformly according to x and μ in $\langle a, b\rangle$ and in a neighbourhood of the point $\mu = 0$ and because of the continuity of the single terms the limit functions $u_k(x)$ are continuous.

Hence, in a neighbourhood of the point $\lambda = \lambda_0$ there exist 2^n real solutions of equation (1) in the form (2.3) which converge to $u_0(x)$ for $\lambda \to \lambda_0$.

c) If some of the conditions (2.14) for $i = 1$ are not fulfilled, it is not possible to solve (2.4) and the problem of determination of the number of solutions of equation (1) for λ from a neighbourhood of λ_0 becomes more complicated. Such solutions can be sometimes sought in the form

(2.33) $$u(x) = \sum_{i=0}^{\infty} (\lambda - \lambda_0)^{i/l} u_i(x),$$

where l is a natural number. The functions $u_i(x)$ can be determined from a system of linear integral equations obtained with the aid of substitution (2.33) in (1) and comparison of coefficients of the same powers of $\nu = (\lambda - \lambda_0)^{1/l}$. For example, for $l = 2$ we obtain, in the case where λ_0^2 is a simple eigenvalue of the kernel $G(x, z)$, the following system

$$u_1(x) = \lambda_0^2 \int_a^b G(x, z)u_1(z)\mathrm{d}z,$$

(2.34)

$$u_i(x) = \lambda_0^2 \int_a^b G(x, z)u_i(z)\mathrm{d}z + g_i(x), \quad i = \overline{2, \infty},$$

where

$$g_2(x) = \frac{1}{2\mu_0(x)} [\lambda_0^2 \int_a^b\int_a^b L(x, y, z)(u_0(y)u_0(z) + u_1(y)u_1(z))\mathrm{d}y\mathrm{d}z - u_1^2(x)]$$

(2.35)

$$g_i(x) = \frac{1}{u_0(x)} [\lambda_0^2 \int_a^b\int_a^b L(x, y, z)u_1(y)u_{i-1}(z)\mathrm{d}y\mathrm{d}z - u_1(x)u_{i-1}(x) +$$

$$+ N_i[x; u_0, \ldots, u_{i-2}]], \qquad i = \overline{3, \infty}.$$

The solution of the first equation from (2.34) can be written in the form

(2.36) $$u_1(x) = D_1\varphi_1(x);$$

the constant D_1 will be determined later.

If the other equations of the system (2.34) are to have solutions it is necessary and sufficient to fulfil the condition

(2.37) $$\int_a^b g_i(x)\alpha_1(x)\mathrm{d}x = 0, \qquad i = \overline{2, \infty}.$$

If we substitute (2.36) in (2.37) we obtain for $i = 2$

(2.38) $$D_1^2 = \frac{-2E_1}{E_2},$$

where

$$E_1 = \lambda_0 \int_a^b\int_a^b\int_a^b \frac{\alpha_1(x)}{u_0(x)} L(x, y, z)u_0(y)u_0(z)\mathrm{d}y\mathrm{d}z\mathrm{d}x,$$

(2.39)

$$E_2 = \int_a^b \frac{\alpha_1(x)}{u_0(x)} \left[\lambda_0^2 \int_a^b\int_a^b L(x, y, z)\varphi_1(y)\varphi_1(z)\mathrm{d}y\mathrm{d}z - \varphi_1^2(x)\right] \mathrm{d}x.$$

If $E_2 \neq 0$ we obtain two values for D_1 distinguished only by the sign and so we have two functions $u_1(x)$

$$u_{1k}(x) = D_{1k}\varphi_1(x), \qquad k = 1, 2.$$

In general the solution of the i-th equation from (2.34) can be written

$$u_i(x) = r_i(x) + D_i\varphi_1(x), \tag{2.40}$$

where

$$r_i(x) = g_i(x) + \lambda_0^2 \int_a^b \Phi(x, z; \lambda_0^2) g_i(z) \mathrm{d}z, \qquad i = \overline{2, \infty};$$

$\Phi(x, z; \lambda_0^2)$ has the same meaning as $T(x, z; \lambda_0^2)$ for $n = 1$ from section b). For D_i we obtain, on the basis of the condition (2.37) for $g_{i+1}(x)$, the linear equation

$$mD_i + n_i = 0, \tag{2.41}$$

where

$$\begin{aligned} m &= \int_a^b \frac{\alpha_1(x)}{u_0(x)} \left[\lambda_0^2 \int_a^b\int_a^b L(x, y, z) u_1(y)\varphi_1(z) \mathrm{d}y\mathrm{d}z - u_1(x)\varphi_1(x)\right] \mathrm{d}x, \\ n_i &= \int_a^b \frac{\alpha_1(x)}{u_0(x)} \left[\lambda_0^2 \int_a^b\int_a^b L(x, y, z) u_1(y)v_i(z) \mathrm{d}y\mathrm{d}z - u_1(x) r_i(x) + \right. \\ &\quad \left. + N_{i+1}[x; u_0, \ldots, u_{i-1}]\right] \mathrm{d}x, \qquad i = \overline{2, \infty}. \end{aligned} \tag{2.42}$$

From this equation we shall determine D_i unambiguously under the assumption $m \neq 0$ and thus $u_i(x)$ are determined.

Hence if $l = 2$, two series (2.33) can be formally constructed. The proof of convergence of these series in a neighbourhood of the point $\nu = 0$ will be carried out analogically as the one in section b). Let us choose such a constant u_0 that

$$|u_1(x)| < u_0 \quad \text{for} \quad x \in \langle a, b \rangle \tag{2.43}$$

is valid. Further, let us consider the function

$$F(w) = \frac{1}{2D} [(2|\lambda_0| + \nu^2)(B + 2w)AB\nu^2 + (1 + (|\lambda_0| + \nu^2)^2 A)w^2], \tag{2.44}$$

where A, B, D are determined by (2.7) 1), 2). If we put instead of w

$$w = \nu u_0 + \sum_{i=1}^{\infty} \nu^{i+1} (u_i + v_i) \tag{2.45}$$

in $F(w)$ and if we expand the expression obtained in the powers of ν we obtain

$$F(w) = \sum_{i=2}^{\infty} \nu^i F_i, \tag{2.46}$$

where F_i are upper bounds for the functions $g_i(x)$ if $(u_{i-1} + v_{i-1})$ are majorants for $u_i(x)$ $(i = \overline{2, \infty})$.

Let us choose such numbers Φ and β that

$$\begin{aligned} &1) \quad \lambda_0^2 \int_a^b |\Phi(x, z; \lambda_0^2)| \mathrm{d}z < \Phi, \qquad x \in \langle a, b \rangle, \\ &2) \quad \max_x (|\varphi_1(x)|, |\alpha_1(x)|) < \beta \end{aligned} \tag{2.47}$$

are valid and designate $\max(1, \Phi) = L$. Then we shall determine u_i, v_i from the equations

$$u_i = 2LF_{i+1},$$

$$\left(|m| + \frac{\beta^2(b-a)}{D}(1 + \lambda_0^2 A)u_0\right)v_i = \beta^2(b-a)F_{i+2}, \qquad i = \overline{1, \infty}. \tag{2.48}$$

We can easily see that

$$|r_{i+1}(x)| < u_i, \quad |D_{i+1}\varphi_1(x)| < v_i \quad \text{for} \quad x \in \langle a, b \rangle.$$

If we introduce the notation

$$U = \sum_{i=1}^{\infty} \nu^i u_i, \qquad V = \sum_{i=1}^{\infty} \nu^i v_i, \tag{2.49}$$

then the determination of u_i, v_i $(i = \overline{1, \infty})$ from equations (2.48) is equivalent to the solving of the equations for U, V,

$$\begin{aligned} \Phi_1 \equiv U - \frac{L}{D}[(2|\lambda_0| + \nu^2)(B + 2\nu(u_0 + U + V))\nu AB + \\ + (1 + (|\lambda_0| + \nu^2)^2 A)\nu(u_0 + U + V)^2] = 0, \end{aligned} \tag{2.50}$$

$$\begin{aligned} \Phi_2 \equiv \left[|m| + \frac{\beta^2(b-a)}{D}(1 + \lambda_0^2 A)u_0\right]U - \frac{\beta^2(b-a)}{2D} \times \\ \times [\nu^2 AB^2 + 2AB\nu(2|\lambda_0| + \nu^2)(u_0 + U + V) + \\ + (1 + (|\lambda_0| + \nu^2)^2 A)(u_0 + U + V)^2] = 0 \end{aligned}$$

in the form (2.49).

For the system (2.50) we use again the implicit function Theorem. As for $\nu = U = V = 0$ the following is valid

$$\frac{\partial \Phi_1}{\partial U} = 0, \quad \frac{\partial \Phi_1}{\partial V} = 1, \quad \frac{\partial \Phi_2}{\partial U} = |m|, \quad \frac{\partial \Phi_2}{\partial V} = \frac{-\beta^2(b-a)}{D} (1 + \lambda_0^2 A) u_0,$$

we have $\Delta = -|m| \neq 0$. Hence, it is possible to determine U and V from (2.50) as unambiguous and continuous functions of ν. That means that the series (2.45) has a finite radius of convergence. As (2.45) is a majorant for the function $(u(x) - u_0(x))$, the series (2.33), if $l = 2$, converges absolutely and uniformly according to x and ν to the continuous functions $u_k(x)$ in $\langle a, b \rangle$ and in a neighbourhood of the point $\nu = 0$.

This proves that in a neighbourhood of $\lambda = \lambda_0$ there exist two solutions of equation (1) in the form (2.33) if $l = 2$. From the relation (2.38) it is obvious that if the quantities (2.39) have different signs then for $\lambda > \lambda_0$ there exist two real solutions and for $\lambda < \lambda_0$ there exists no real solution. Two branches of solution start from the point λ_0. If the quantities (2.39) have the same signs, then for $\lambda < \lambda_0$ there exist two real solutions and for $\lambda > \lambda_0$ there exists no real solution. Two branches of the solution finish at the point λ_0. To prove the assertion for $\lambda < \lambda_0$ we must seek the solution $u(x)$ in the form of the series in the powers of $\sqrt{\lambda_0 - \lambda}$ and apply the above considerations.

3. THE CONTINUATION OF THE SOLUTION

On the basis of results from the second section it is possible to continue the solution $u_0(x, \lambda_0)$ corresponding to the value $\lambda = \lambda_0$ as follows: For λ from a neighbourhood of λ_0 it is possible to construct a solution $u(x, \lambda)$ in the form of a power series of $(\lambda - \lambda_0)$. This solution will be only one (i. e. $(\lambda_0, u_0(x, \lambda_0))$ is not a branch point) if λ_0^2 is not an eigenvalue of the kernel

$$\frac{1}{u_0(x, \lambda_0)} \int_a^b L(x, y, z) u_0(y, \lambda_0) \mathrm{d}y.$$

If λ_0^2 is an eigenvalue of the above mentioned kernel, we can obtain more than one solution. For further continuation it is necessary to continue each of these solutions. If we continue successively the solution $u(x, \lambda)$ for the whole real axis λ, we obtain with the aid of this method the solution of (1) for an arbitrary value of the parameter λ.

REFERENCES

[1] Schmeidler W., Morgenstern D., *Algebraische Integralgleichungen II*, Archiv der Mathematiker *4* (1954), 452—457.

[2] Vilenkin N. J. i dr., *Funkcionalnyj analiz*, Moskva 1964.

Received January 12, 1967.

Katedra algebry a geometrie
přírodovědecké fakulty
University Palackého, Olomouc