Mat. čas. 25, 1975, No. 4, 361—368

TRANSLATIONS IN IDEMPOTENT GROUPOIDS

JOZEF PÓCS

A mapping λ of a semilattice H into itself is called a translation if $\lambda(xy) = \lambda(x)y$ for any elements x, y of H.

The papers [1]—[4] deal with translations in semilattices or lattices. G. Szász [1] has proved that the image of a semilattice H in a translation λ is an ideal in H; a translation of a semilattice is an idempotent endomorphism (i.e. for any $x, y \in H$, $\lambda(x) = \lambda[\lambda(x)]$ and $\lambda(xy) = \lambda(x)\lambda(y)$ hold true).

Szász and Szendrei [2] have proved that the set of all translations of a semilattice forms a semilattice with respect to the superposition of mappings.

Szász [3] has investigated mappings of a lattice into itself which are translations with respect to the lattice operation of join. He found that the image of a lattice in such a mapping is a dual ideal; further, he showed that the equality of two dual ideals corresponding to two translations implies the equality of these two translations.

Kolibiar [4] has investigated a connection between translations in a lattice and relations of congruence on a lattice which preserve the operation of join. He found also a necessary and sufficient condition under which to a dual ideal D of a lattice L there exists a translation λ on L such that $\lambda(L) = D$.

Szász [5] has defined a translation in a general groupoid as follows: a translation λ in the groupoid is a mapping of the groupoid into itself with the property $\lambda(xy) = \lambda(x)y = x\lambda(y)$ for any elements x, y of the groupoid. He also investigated a connection between translations and endomorphisms.

Some analogical questions concerning translations in universal algebras were investigated in [6].

In this note the notion of translation will be used in the sense of the paper [5]. We shall show that the results from the papers [1]—[4] cited above can be generalized for idempotent groupoids. Further, there is investigated the connection between ideals and translations, or translations and congruence relations, respectively.

1. Definitions and notations

Definition 1. *A mapping λ of a groupoid G into itself with the property $\lambda(xy) = \lambda(x)y = x\lambda(y)$ for any $x, y \in G$ is called a translation of the groupoid G.*

Definition 2. *A mapping λ with the property $\lambda[\lambda(x)] = \lambda(x)$ for each $x \in G$ is called idempotent.*

Definition 3. *A nonvoid subset I of a groupoid G is said to be an ideal of the groupoid G if $xa \in I$, $ax \in I$ for any $a \in I$ and $x \in G$.*

Definition 4. *A groupoid G will be called idempotent if $x^2 = xx = x$ for each $x \in G$.*

The superposition of two mappings λ and μ will be considered in the usual sense.

Clearly, the identical mapping ε is a translation too. We denote the set of all translations of a groupoid G by $T(G)$. The set of all ideals and congruence relations of a groupoid G are denoted by $I(G)$ and $\Theta(G)$, respectively. Let c be a fixed element of a semilattice S; the mapping $x \to xc$ is a translation and is called a special translation of S.

Let A and B be nonvoid subsets of a groupoid G, then we put $AB = \{ab : a \in A, b \in B\}$ and if $\lambda \in T(G)$ then $\lambda(A) = \{\lambda(a) : a \in A\}$.

The set theoretical intersection and inclusion are denoted by $\cap$ and $\subseteq$, respectively. The join operation in the lattice $\Theta(G)$ will be designated by $\vee$. A principal ideal of a groupoid G generated by an element a is denoted by $I(a)$.

2. Properties of translations

The following Theorem 1 and its Corollary generalize the part of the necessary condition of Thm. 2 from [1] and the first part of Thm. 3 from [2].

Theorem 1. *Let G be an idempotent groupoid. Then for any λ, $\mu \in T(G)$ and for all $x, y \in G$ there holds:*

(A) $\lambda^2 = \lambda$;

(B) $\lambda(xy) = \lambda(x)\lambda(y)$;

(C) $\lambda\mu = \mu\lambda$;

(D) $\lambda\mu(x) = \lambda(x)\mu(x)$;

(E) $\lambda(x) = \lambda(x)x$;

(F) $\lambda\mu \in T(G)$.

Proof. First we prove the property (D). Let $\lambda, \mu \in T(G)$ and $x \in G$, then we have

$$\lambda\mu(x) = \lambda\mu(x^2) = \lambda[\mu(xx)] = \lambda[x\mu(x)] = \lambda(x)\mu(x) .$$

The condition (C) follows from (D) and from the following identity

$$\mu\lambda(x) = \mu\lambda(x^2) = \mu[\lambda(xx)] = \mu[\lambda(x)x] = \lambda(x)\mu(x)\ .$$

Put $\lambda = \mu$ or $\mu = \varepsilon$ in (D); then we obtain both properties (A) and (E).
Let $x, y \in G$ and $\lambda \in T(G)$. From (A) we have

$$\lambda(xy) = \lambda^2(xy) = \lambda[x\lambda(y)] = \lambda(x)\lambda(y)\ ,$$

i.e. the property (B).

Now we prove the condition (F). Assume that λ, $\mu \in T(G)$ and $x, y \in G$. Then we have

$$\lambda\mu(xy) = \lambda[\mu(xy)] = \lambda[\mu(x)y] = \lambda\mu(x)y$$

and similarly $\lambda\mu(xy) = x\lambda\mu(y)$; therefore $\lambda\mu \in T(G)$.

Corollary. *Let G be an idempotent groupoid. Then $T(G)$ is a semilattice, where the semilattice operation is a superposition of translations.*

The properties (A), (B) and (E) are necessary conditions for a mapping λ of an idempotent groupoid into itself to be a translation. Now we investigate one necessary and sufficient condition.

Theorem 2. *Let λ be a mapping of an idempotent groupoid G into itself. Then $\lambda \in T(G)$ if and only if the following three conditions are satisfied:*

(A) $\lambda^2 = \lambda$;

(B) $\lambda(xy) = \lambda(x)\lambda(y)$;

(A′) $\lambda(x)y = x\lambda(y)$;

for all $x, y \in G$.

Proof. The necessary condition follows from Theorem 1.

Conversely, assume that for some mapping λ of the idempotent groupoid G into itself the conditions (A), (B) and (A′) are satisfied. Then by the consecutive application of the properties (A), (A′) and (B) we obtain

$$x\lambda(y) = x\lambda^2(y) = x\lambda[\lambda(y)] = \lambda(x)\lambda(y) = \lambda(xy)$$

for all $x, y \in G$ and analogously $\lambda(x)y = \lambda(xy)$. The proof is complete.

Remark. In the proof of the sufficient condition it could be easily seen that the assumption of the idempotency of the groupoid G was not used. We may put the question whether one of the assumptions (A), (B), (A′) can be omitted in the proof of the sufficient condition. The following three examples show that the conditions (A), (B), (A′) are independent.

Example 1. Let us consider the set of reals $G = \{x : 0 < x < \infty\}$ with the operation $\circ$ defined by the formula $x \circ y = \sqrt{xy}$. Then evidently the algebra $(G; \circ)$ is an idempotent groupoid. Put $\lambda(x) = cx$ for some $c \neq 1$, $c \in G$. Then the mapping λ fulfills (B), (A′) but (A) is not satisfied.

Example 2. Let us take the set of reals $G = \{x : -1 < x < 1\}$ with the operation $*$ defined in the following way:

$$x * y = \begin{cases} xy & \text{for } |x| \neq |y|\,, \\ x & \text{for } x = y\,, \\ |x| & \text{for } x = -y\,. \end{cases}$$

If we put $\lambda(x) = |x|$, then it can be verified that the conditions (A), (B) are satisfied but (A′) does not hold.

Example 3. Let $G = \{a, b, a', b'\}$ with the multiplication defined by the following table:

	a	b	a'	b'
a	a	a	a'	b'
b	a	b	b'	b'
a'	a'	b'	a'	b'
b'	b'	b'	b'	b'

Let us define a mapping λ in the following way: $\lambda(a) = \lambda(a') = a'$ and $\lambda(b) = \lambda(b') = b'$. Then it can be verified that the conditions (A), (A′) are satisfied but (B) does not hold.

3. Translations and ideals

In paper [5] it is proved that the image of any groupoid G under a mapping λ, where $\lambda \in T(G)$, is an ideal in G. The ideal $\lambda(G)$ for $\lambda \in T(G)$ is called the ideal corresponding to the translation λ. Denote $\lambda(G) = I_\lambda$ for $\lambda \in T(G)$.

The following theorem generalizes Thm. 3 from [3].

Theorem 3. *Let G be an idempotent groupoid and let $I_\lambda = I_\mu$ for some $\lambda, \mu \in T(G)$. Then $\lambda = \mu$.*

Proof. It follows from the property (A) of Thm. 1 that $\lambda(x) = x$ for all $x \in I_\lambda$. Choose any element x from G. Then $\lambda\mu(x) = \mu(x)$, because of $\mu(x) \in I_\lambda$. Analogously we obtain that $\mu\lambda(x) = \lambda(x)$. Then according to (C) of Thm. 1 we have $\lambda(x) = \mu(x)$; i.e. $\lambda = \mu$.

Remark. The set $I(G)$ of all ideals of an idempotent groupoid G can be partially ordered by inclusion. Clearly, this partially ordered set is a semilattice with respect to the intersection. Moreover $I_1 \cap I_2 = I_1 I_2$ holds for any two ideals $I_1, I_2 \in I(G)$.

The following theorem gives the answer to the question of the connection between the superposition of translations and the meet operation in the semilattice $I(G)$.

Theorem 4. *Let G be an idempotent groupoid. Then $I_{\lambda\mu} = I_\lambda \cap I_\mu$ for any $\lambda, \mu \in T(G)$.*

Proof. It is easy to show that $\lambda(AB) = A\lambda(B) = \lambda(A)B$ holds for any nonvoid subsets A, B of the groupoid G and $G = GG$. Then we have

$$I_{\lambda\mu} = \lambda\mu(G) = \lambda\mu(GG) = \lambda[G\mu(G)] = \lambda(G)\mu(G) = I_\lambda I_\mu = I_\lambda \cap I_\mu .$$

Corollary 1. *The set of all ideals of an idempotent groupoid G corresponding to all its translations is a subsemilattice of the semilattice $I(G)$.*

In the semilattice $T(G)$ for any λ and μ put $\lambda \leqq \mu$ if and only if $\lambda\mu = \lambda$.

Corollary 2. *Let G be an idempotent groupoid. Then $\lambda \leqq \mu$ for $\lambda, \mu \in T(G)$ if and only if $I_\lambda \subseteqq I_\mu$.*

Proof. The statement follows from Thm. 3 and Thm. 4.

Corollary 3. *If G is an idempotent groupoid, then the mapping $\lambda \to I_\lambda$ is a monomorphism of the semilattice $T(G)$ into the semilattice $I(G)$.*

The following theorem concerns a representation of translations by special translations.

Theorem 5. *Let G be an idempotent groupoid. Then there exists a semilattice H such that the semilattice $T(G)$ can be embedded into $T(H)$ so that to each $\lambda \in T(G)$ there corresponds some special translation of the semilattice H.*

Proof. Put $H = I(G)$. Now we define a mapping F of the semilattice $T(G)$ into $T[I(G)]$ as follows: $F(\lambda) = \bar{\lambda}$, where $\bar{\lambda}(I) = I \cap I_\lambda$ for any $I \in I(G)$. Clearly $\bar{\lambda}$ is a special translation of the semilattice $I(G)$. Next we prove that F is one-one and a morphism.

Let $\bar{\lambda} = \bar{\mu}$ i.e. $\bar{\lambda}(I) = \bar{\mu}(I)$ for any $I \in I(G)$. Hence $I \cap I_\lambda = I \cap I_\mu$ for all $I \in I(G)$. Put $I = I_\lambda$ and $I = I_\mu$. Then we obtain $I_\lambda = I_\lambda \cap I_\mu = I_\mu$. An application of Thm. 3 yields $\lambda = \mu$.

According to Thm. 4 we have

$$\lambda\mu(I) = \mathrm{I} \cap I_{\lambda\mu} = I \cap I_\lambda \cap I_\mu = \bar{\lambda}(I \cap I_\mu) = \bar{\lambda}(\bar{\mu}(I)) = \bar{\lambda}\bar{\mu}(I) ,$$

hence the mapping F is a morphism.

The following Theorem generalizes the necessary condition from Thm. 4 [4].

Theorem 6. *Let G be an idempotent groupoid and $\lambda \in T(G)$. Then the intersection of any principal ideal of G with the ideal I_λ is a principal ideal of the groupoid G.*

Proof. We shall prove that for any $a \in G$, $I(a) \cap I_\lambda = I[\lambda(a)]$ holds. According to (E) Thm. 1 $\lambda(a) = \lambda(a)a$ is true. Since $\lambda(a)a \in I(a)$ and $\lambda(a) \in I_\lambda$, $\lambda(a) \in I(a) \cap I_\lambda$. Further $I[\lambda(a)] \subseteqq I(a) \cap I_\lambda$, because $I(a) \cap I_\lambda$ is an ideal. Now let $x \in I(a) \cap I_\lambda$. Then $x = p(a_1, \ldots, a_{i-1}, a, a_i, \ldots, a_n)$ where p is some word constructed from the elements $a_1, \ldots, a_i, a, \ldots, a_n \in G$ and $\lambda(x) = x$ because of $x \in I_\lambda$. We may write $x = \lambda(x) = \lambda[p(a_1 . \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_n)] = p(a_1, \ldots, a_{i-1}, \lambda(a), a_i, \ldots, a_n) \in I[\lambda(a)]$. The last equality is easy to prove using the definition of the translation.

Remark. In the case of lattices the ideals corresponding to translations are dual ideals. In this case the following theorem is valid: If D is a dual ideal of a lattice V such that the intersection of any dual principal ideal with D is a dual principal ideal, then there exists a translation λ of a lattice V such that $\lambda(V) = D$ (see [4]). For an idempotent groupoid we cannot prove an analogical assertion; this is shown by the following example.

Example 4. Let $G = \{a, b, c\}$. Define the multiplication in G by the following table:

	a	b	c
a	a	a	a
b	b	b	a
c	b	a	c

Consider the ideal $I = \{a, b\}$. This ideal has a property that the intersection of any principal ideal with I is a principal ideal, but there does not exist any translation λ satisfying the condition $\lambda(G) = I$.

4. Translations and congruence relations

Using (B) of Thm. 1 we get that any translation in an idempotent groupoid is an endomorphism. Therefore to any $\lambda \in T(G)$ there corresponds some congruence relation Θ_λ which is defined as follows:

$x\Theta_\lambda y$ if and only if $\lambda(x) = \lambda(y)$.

Remark. It is well known that the congruence relations in any algebra A form a complete lattice, where $\Theta \leqq \Phi$ if and only if $x \,\Theta\, y$ implies $x\Phi y$ for each $x, y \in A$. The following lemma is analogous to Thm. 2 [4].

Lemma 1. *Let G be an idempotent groupoid and $\lambda, \mu \in T(G)$. Then $\lambda \leqq \mu$ if and only if $\Theta_\mu \leqq \Theta_\lambda$.*

Proof. Let $\lambda \leqq \mu$; i.e. $\lambda\mu = \lambda$, and let $x\Theta_\mu y$ for some $x, y \in G$. Then $\mu(x) = \mu(y)$. Hence $\lambda(x) = \lambda\mu(x) = \lambda\mu(y) = \lambda(y)$. Therefore $x\Theta_\lambda y$.

Conversely, assume that $\Theta_\mu \leqq \Theta_\lambda$. Choose an arbitrary element $x \in G$. The idempotency of the translation μ implies $x\Theta_\mu\mu(x)$. Since $\Theta_\mu \leqq \Theta_\lambda$, we have $x \,\Theta_\lambda\mu(x)$. From the definition of the congruence relation Θ_λ we obtain $\lambda(x) = \lambda\mu(x)$, i.e. $\lambda \leqq \mu$.

Theorem 7. *Let G be an idempotent groupoid. Then $\Theta_\lambda \vee \Theta_\mu = \Theta_{\lambda\mu}$ for any $\lambda, \mu \in T(G)$.*

Proof. Since $\lambda\mu \leqq \lambda$ and $\lambda\mu \leqq \mu$, from Lemma 1 we obtain $\Theta_\lambda \leqq \Theta_{\lambda\mu}$ and $\Theta_\mu \leqq \Theta_{\lambda\mu}$, respectively. Thus $\Theta_\lambda \vee \Theta_\mu \leqq \Theta_{\lambda\mu}$. Assume that $x\Theta_{\lambda\mu}y$. The element $t = \lambda\mu(x) = \lambda\mu(y)$ belongs to $I_{\lambda\mu}$. Further, in view of Thm. 4, $t \in I_\lambda$and $t \in I_\mu$. By the idempotency of translations we have $\lambda(t) = t$, i.e. $\lambda(t) = \lambda\mu(x)$. Then $t\Theta_\lambda\mu(x)$. Because of $\mu(x)\Theta_\mu x$ we get $t\Theta_\lambda \vee \Theta_\mu x$. Analogously $t\Theta_\lambda \vee \Theta_\mu y$. Then by the transitivity of congruence relations we obtain $x\Theta_\lambda \vee \Theta_\mu y$.

Corollary. *The set of all congruence relations corresponding to all translations of an idempotent groupoid is a subsemilattice with respect to the join operation of the lattice $\Theta(G)$.*

Lemma 2. *Let G be an idempotent groupoid. If $x\Theta_\lambda y$ and $x, y \in I_\lambda$, then $x = y$ for any $\lambda \in T(G)$.*

Proof. Assume that $x, y \in I_\lambda$ and $x\Theta_\lambda y$. Then $\lambda(x) = x$ and $\lambda(y) = y$. Since $\lambda(x) = \lambda(y)$, we have $x = y$.

Let $\Theta \in \Theta(G)$. Denote $[x]\Theta = \{y : y \in G,\ x\Theta y\}$.

The following Theorem generalizes Thm. 1 [4].

Theorem 8. *Let G be an idempotent groupoid and $I \in I(G)$. Then there exists a translation of the groupoid G such that $\lambda(G) = I$ if and only if the following condition is fulfilled:* (1) *There exists a congruence relation Θ on G such that $I \cap [x]\Theta$ is a one-element set for every $x \in G$.*

Proof. Let λ be the translation of the groupoid G such that $\lambda(G) = I$. We shall prove that $[x]\Theta_\lambda \cap I \neq \emptyset$ for any $x \in G$. From the idempotency of the translation λ we get $\lambda(x) = \lambda^2(x)$. Hence $x\Theta_\lambda\lambda(x)$, i.e. $\lambda(x) \in [x]\Theta_\lambda$. Therefore $\lambda(x) \in [x]\Theta_\lambda \cap I$. According to Lemma 2 we obtain that $[x]\Theta_\lambda \cap I$ contains only one element.

Conversely, define a mapping λ of G into G as follows: $\lambda(x) = x' \in I \cap [x]\Theta$. Let x, y belong to G. Then $\lambda(xy) \in I \cap [xy]\Theta$. Further $\lambda(y)\Theta y$, in this case $x\lambda(y)\Theta xy$. But $x\lambda(y) \in I$ because I is an ideal in G. Hence $x\lambda(y) \in I \cap [xy]\Theta$. From condition (1) we get $\lambda(xy) = x\lambda(y)$. Similarly it can be proved that $\lambda(xy) = \lambda(x)y$. The proof is complete.

REFERENCES

[1] SZÁSZ, G.: Die Translationen der Halbverbände. Acta Sci. Math., 17, 1956, 165—169.

[2] SZÁSZ, G., SZENDREI, J.: Über die Translationen der Halbverbände. Acta Sci. Math., 18, 1957, 44—47.

[3] SZÁSZ, G.: Translationen der Verbände. Acta fac. rer. nat. Univ. Comen. Math., 5, 1961, 449—453.

[4] KOLIBIAR, M.: Bemerkungen über Translationen der Verbände. Acta fac. rer. nat. Univ. Comen. Math., 5, 1961, 455—458.
[5] SZÁSZ, G.: Grupoid transzlácioi és endomorfizmusai. Acta Acad. Pedagogicae Nyiregyháziensis Tom 2, 1968, 143—146.
[6] PÓCS, J.: Translations and endomorphisms in universal algebras. Acta fac. rer. nat. Univ. Comen. Math., 29, 1974, 95—99.

Received April 8, 1974

Katedra matematiky
Strojníckej fakulty Vysokej školy technickej
Švermova 5
040 01 Košice