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CHROMATIC INDEX OF HAMILTONIAN GRAPHS

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Abstract. Let G be a loopless graph of a finite degree d such that (i) G does not contain any double triangle; (ii) G contains at least $\left[\frac{d}{6}\right]$ edge-disjoint hamiltonian lines; (iii) the number of vertices of G is not 5. It is proved that then the chromatic index of G is at most $3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right]$.

Notation and terminology. We consider only loopless graphs (graphs without loops). However, multiple edges and infinite graphs are allowed.

Let G be a graph and let C be a set. By a *regular partial edge-colouring* of G by *colours* of C, or briefly, by a *partial C-colouring* of G, we mean a mapping φ from a subset of the edge set of G into C such that for any two adjacent edges e and e' we have $\varphi(e) \neq \varphi(e')$ provided that $\varphi(e)$ and $\varphi(e')$ are defined. If an edge e is assigned $\varphi(e) \in C$, we say that $\varphi(e)$ is the *colour* of e, and that the edge e is *coloured* by $\varphi(e)$. If all the edges of G are coloured by elements of C, we say that φ is a *C-colouring* of G. The minimal cardinal number $q(G)$ such that there exist a set C of cardinality $q(G)$ and a C-colouring of G, is called the *chromatic index* of G.

Suppose that there are given a graph G, a vertex v of G, a set C, a partial C-colouring φ of G and two colours $\sigma, \tau \in C$. By a *$\sigma\tau$-alternation* of φ in v we understand the partial C-colouring ψ of G defined as follows. Let $P(\sigma, \tau, v)$ be the maximal connected subgraph of G containing v such that every edge of $P(\sigma, \tau, v)$ is coloured by σ or by τ. Evidently, $P(\sigma, \tau, v)$ is generated by a circuit or by a (possibly one-way or two-way infinite) path. Now we change the colours of the edges in $P(\sigma, \tau, v)$. Those coloured by σ are now coloured by τ and conversely. The colours of the other edges of G remain unchanged. Thus a new partial C-colouring ψ of G is obtained. If no edge incident with v is coloured by σ or τ, i. e. if $P(\sigma, \tau, v)$ is generated by a path of length 0, we put $\psi = \varphi$.

By the *degree* of G we mean the supremum $d(G)$ of degrees of vertices of G. If $d(G)$ is infinite, then $q(G) = d(G)$ (see [2], Theorem 1). Therefore we may restrict ourselves into graphs of finite degrees. In such a case the degree $d(G)$ of a graph G is the maximal degree of its vertices.

By a *factor* of G we understand a subgraph of G containing all the vertices of G. A *hamiltonian line* of G is defined as a connected factor of G whose vertices are all of degree 2, i. e. a connected (regular) quadratic factor of G.

If $F_1, F_2, \ldots$ are factors of G, the symbols $F_1 - F_2$ and $F_1 \cup F_2 \cup \ldots$ are used in their usual sense.

By a *double triangle* we mean a graph of degree 4 with 3 vertices and 6 edges (any two vertices joined by 2 edges — see Fig. 1). This graph has been denoted in [1] and [2] by the symbol G_4, in [3] by G^*.

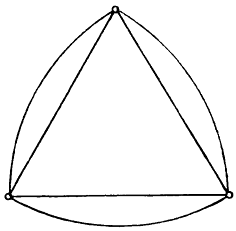

Fig. 1. The double triangle.

If x is a real number, then $[x]$ denotes the greatest integer $\leq x$ and $[x]^*$ is the smallest integer $\geq x$.

Auxiliary results. The starting point of our present considerations is the following

Lemma 1. *Let G be a graph of a finite degree d without double triangles* (Fig. 1). *Then we have:*

$$q(G) \leq 3\left[\frac{d+1}{2}\right] - \left[\frac{d+1}{4}\right]. \tag{1}$$

Proof. For finite graphs this assertion has been proved in [3]; see also [1], Chapter 12, Corollary to Theorem 8; for infinite graphs it has been established in [2], Corollary 2 to Theorem 5.

We shall show that if G has a certain number of hamiltonian lines and it is not a 5-vertex graph, then the estimation (1) may be improved (see Theorem below). At first we need two following lemmas.

Lemma 2. *Let l be a positive integer and let G be a graph of a degree $\leq 2l$. Then G is decomposable into l factors of degrees ≤ 2.*

Proof. See [4] (for finite graphs — Chapter 11, Theorem 6; for infinite graphs — Chapter 13).

Lemma 3. *Let k be a positive integer and let G be a graph with a hamiltonian line H such that*

(2) $$d(G) \leq k;$$

(3) $$q(G - H) \leq k - 1;$$

(4) *G has an even number or at least k vertices.*

Then we have:

(5) $$q(G) \leq k + 1.$$

Proof. If G has an infinite or an even number of vertices, then we evidently have $q(H) = 2$, so that $q(G) \leq q(G - H) + G(H) \leq (k - 1) + 2 = k + 1$. Therefore we may suppose that G is a finite graph with an odd number ($\geq k$) of vertices.

Let C be a set of cardinality $k - 1$ and let φ be a C-colouring of $G - H$. (Thus φ is a partial C-colouring of G.) If x is a vertex of G, denote by $f(x)$ the set of all colours of C absent in x, i. e. such that no edge incident with x is coloured by any of them. From (2) it follows that $d(G - H) \leq k - 2$ so that $f(x) \neq \emptyset$ for any x.

As we have only $k - 1$ colours and at least k vertices, there exist vertices u and v $(u \neq v)$ such that

(6) $$f(u) \cap f(v) \neq \emptyset.$$

Put $m = \min \varrho_H(u, v)$, where ϱ_H is the usual graph metric with respect to H and the minimum is taken through all pairs of different vertices u and v of G such that (6) holds.

We shall prove that by some alternations the sets $f(x)$ can be changed in such a way that $m = 1$ will be valid. Therefore suppose that $m > 1$. Let u and v be such vertices that $\varrho_H(u, v) = m$ and (6) holds. Pick $\alpha \in f(u) \cap f(v)$ and a vertex $w \neq u, v$ of the shortest path joining u and v in H. Evidently, $\alpha \notin f(w)$. Choose $\beta \in f(w)$; let φ' be the $\alpha\beta$-alternation of φ in w and let $f'(x)$ be the corresponding sets of colours absent in x at φ'. Distinguish two cases:

I. The maximal path $P(\alpha, \beta, w)$ containing w whose edges are coloured by α or β ends in u. Then put $u' = w$, $v' = v$.

II. $P(\alpha, \beta, w)$ does not end in u. Then put $u' = u$, $v' = w$.

It is easy to show that

$$u' \neq v',$$

$$\varrho_H(u', v') < m,$$

$$\alpha \in f'(u') \cap f'(v').$$

Obviously, this process can be iterated until (by less than m steps) we arrive at vertices U and V such that

$$U \neq V,$$

$$\varrho_H(U, V) = 1,$$

$$\alpha \in F(U) \cap F(V),$$

where $F(x)$ denotes the set of colours absent in a vertex x at the last C-colouring of $G - H$. Denote by e the edge of H joining U and V and by E the subgraph of H generated by U, V and e. Evidently, the edge e may be coloured by the colour α. The remaining edges of H form a path so that they can be coloured by another two colours. Thus we have: $q(G) \leq q((G - H) \cup E) + q(H - E) \leq (k - 1) + 2 = k + 1$. Q. E. D.

Main results.

Theorem. *Let G be a graph of a finite degree d such that*

(i) *G does not contain as a subgraph any double triangle* (Fig. 1);

(ii) *G contains at least $s = \left[\frac{d}{6}\right]$ edge-disjoint hamiltonian lines*;

(iii) *the number of vertices of G is different from* 5.

Then we have:

$$q(G) \leq 3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right]. \tag{7}$$

Proof. Let G fulfil the suppositions of Theorem. Denote by F the factor of G generated by edge-disjoint hamiltonian lines $H_1, H_2, \ldots, H_s$ of G. The graph $G - F$ has degree $d - 2s$. By Lemma 2 $G - F$ can be decomposed into $t = \left[\frac{d}{2} - s\right]^*$ factors of degrees ≤ 2; denote them by $F_1, F_2, \ldots, F_t$. For $i = 1, 2, \ldots, s$ construct graphs $G_i = H_i \cup F_{2i-1} \cup F_{2i}$ of degrees ≤ 6. Every graph $G_i - H_i$ has a degree ≤ 4. According to Lemma 1 we have $q(G_i - H_i) \leq 5$.

Denote the cardinality of the vertex set of G by n. If $n = 1$, or if $n = 3$ and $d = 0$, then the assertion of Theorem evidently holds. If $n = 3$ and $d > 0$, then from (i) it follows that

$$q(G) \leq d + 1 \leq 3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right].$$

Therefore by (iii) we may suppose that n is even, or $n \geq 6$. Using Lemma 3 for $k = 6$, we obtain

$$q(G_i) \leq 7.$$

Put $I = G - (G_1 \cup G_2 \cup \ldots \cup G_s)$. Obviously,

$$q(G) \leq q(G_1) + q(G_2) + \ldots + q(G_s) + q(I) \leq 7s + q(I).$$

Evidently, $d(I) \leq 2(t - 2s)$. Put

$$u = 2(t - 2s).$$

According to Lemma 1 we get

$$q(I) \leq 3\left[\frac{u+1}{2}\right] - \left[\frac{u+1}{4}\right].$$

Distinguish four cases:

Case A. $d = 6s$. Then $t = 2s$, $u = 0$, $q(I) = 0$,

$$q(G) \leq 7s = 3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right].$$

Case B. $d = 6s + 1$ or $6s + 2$. Then $t = 2s + 1$, $u = 2$, $q(I) \leq 3$,

$$q(G) \leq 7s + 3 = 3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right].$$

Case C. $d = 6s + 3$ or $6s + 4$. Then $t = 2s + 2$, $u = 4$, $q(I) \leq 5$,

$$q(G) \leq 7s + 5 = 3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right].$$

Case D. $d = 6s + 5$. Then $t = 2s + 3$, $u = 6$, $q(I) \leq 8$,

$$q(G) \leq 7s + 8 = 3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right].$$

Q. E. D.

Remark. A comparison of (1) and (7) shows that (for graphs fulfilling the assumptions of Theorem) the estimate (7) is never worse than (1). Moreover, if $d \in \{6, 9, 10\}$, or if $d \geq 12$, then (7) is better than (1).

From another known estimates of the chromatic index for the considered class of graphs there can be applied that by Shannon [6]:

$$q(G) \leq \left[\frac{3}{2} d\right], \tag{8}$$

and if $d \geq 4$, the estimate by Vizing [7]:

$$q(G) \leq \left[\frac{3}{2} d\right] - 1. \tag{9}$$

(For infinite graphs, the proofs of (8) and (9) are given in [2], Theorem 3.) However, (7) is better than (8) for $d \geq 8$ and better than (9) for $d \geq 12$.

Corollary. *Let G be a graph of an even degree d such that* (i), (ii) *and* (iii) *hold. Then we have:*

$$q(G) \leq \left[\frac{7d + 4}{6}\right] \tag{10}$$

and this estimate is best possible.

Proof. For an even d we have

$$3\left[\frac{d+1}{2}\right] - \left[\frac{d}{3}\right] = \left[\frac{7d + 4}{6}\right].$$

Thus we need only to show that (10) is sharp. We shall construct for any even d a graph G satisfying (i), (ii) and (iii) and with chromatic index

$$q(G) = \left[\frac{7d + 4}{6}\right]. \tag{11}$$

Namely, let G be a graph obtained from the graph of a circuit of length 7 by replacing each edge by $\frac{d}{2}$ multiple edges. Evidently, (i), (ii) and (iii) hold. Moreover, using results of [1] (Chapter 12, Theorem 5), [2] (Lemma 3), or [5] (Theorem 14.1.4), it is easy to check (11) to be true.

Conjecture. *The estimate* (10) *holds also in case of an odd degree.*

Remark. It can be easily shown by examples of 7-vertex graphs that if Conjecture is true, then it is sharp.

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