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CONNECTIVITY OF REGULAR GRAPHS AND THE EXISTENCE OF 1-FACTORS

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The notions and denotations not defined heɪe can be found in [3]. A *graph* G (always finite and loopless) will be denoted by (V, H) where V and H are the sets of its points and lines, respectively. If $M \subseteq V$, then $G(M)$ denotes the *induced subgraph of* G on the points of M. By a *u-v path* we mean a path from the point u to the point v. An *r-regular* graph is a regular graph of degree r. $\lambda(G)$ and $\varkappa(G)$ denote the *line-connectivity* and the *point-connectivity* of G, respectively. G is *k-line-connected* if $\lambda(G) \geqslant k$ (where k is a non-negative integer).

The problem of the existence of factors is very old. For example, J. Petersen [5] has shown that *every bridgeless cubic graph has a 1-factor.*

Further, T. Schönberger (see [4], p. 192) in 1934 has proved that *every bridgeless cubic graph has a 1-factor not containing two arbitrarily prescribed lines.*

F. Baebler [1] has observed that there is some relationship between the connectivity and the factorisation of graphs, namely:

If G is a $(2m + 1)$-regular graph with $\lambda(G) \geqslant 2n$ (where m and n are positive integers), then G has a $2n$-factor; particularly, if $\lambda(G) \geqslant 2m$, then G has a $2m$ -factor and therefore also a 1-factor.

C. Berge ([2], Chapter 18, Theorems 6 and 7) in the two following results has generalized Petersen's result and Baebler's one (as for the existence of a 1-factor):

Every r-regular graph G (where $r > 0$) with an even number of points and with $\lambda(G) \geqslant r - 1$ has a 1-factor.

Every $(2m + 1)$-regular graph G with $\lambda(G) \geqslant 2m$ has a 1-factor containing an arbitrarily prescribed line.

A. Kotzig (oral communication) has conjectured the following generalization of the abovementioned results of Schönberger and Berge:

Every r-regular graph G with an even number of points and with $\lambda(G) \geqslant$

$\geqslant r - 1$ (which holds e. g. if $\varkappa(G) \geqslant r - 1$) *has a* 1*-factor not containing* $r - 1$ *arbitrarily prescribed lines.*

In this paper this conjecture is proved and moreover it is shown that for $\varkappa(G) < r - 1$ the assertion does not hold in general.

Lemma 1. (Tutte [6].) *A graph* $G = (V, H)$ *has a* 1*-factor if and only if* $|V|$ *is even and there is no set* S *of points such that the number of odd components of the induced subgraph* $G\ (V - S)$ *exceeds* $|S|$. (By an odd component of G we mean that with an odd number of points.)

Theorem 1. *Let* $G = (V, H)$ *be an* $(r-1)$*-line-connected regular graph of degree* $r > 0$, *with an even* $|V|$ *and let* $H' \subseteq H$ *be an arbitrary set of* $r - 1$ *lines. Then the graph* $G' = (V, H - H')$ *has a* 1*-factor.*

Proof. Let us suppose the graph G' has no 1-factor. Then by Lemma 1 there is a set $S \subseteq V$ such that the induced subgraph $G'(V - S)$ has n odd components $G_1', G_2', \ldots, G_n'$, where $n > |S|$. Let V_i denotes the point set of the graph G_i' for $i = 1, 2, \ldots, n$. The number of all lines of G incoming to V_i from S, or from $V - S - V_i$ will be denoted by s_i, or by t_i, respectively. Since G is $(r-1)$-line-connected and for every $i = 1, 2, \ldots, n$ we have $|V_i| \equiv 1 \pmod 2$, thus denoting by σ_i the sum of degrees of points in $G(V_i)$, we have $0 \equiv \sigma_i = r|V_i| - (s_i + t_i) \leqslant r|V_i| - (r - 1) = r(|V_i| - 1) + 1 \equiv 1 \pmod 2$. It follows that $r - 1 < s_i + t_i$, or

$$r \leqslant s_i + t_i. \tag{1}$$

From S exactly $\sum_{i=1}^{n} s_i$ lines income to $\bigcup_{i=1}^{n} V_i$. Since G is r-regular, thus

$$\sum_{i=1}^{n} s_i \leqslant r|S|. \tag{2}$$

Further, the condition $|H'| = r - 1$ gives

$$\sum_{i=1}^{n} t_i \leqslant 2(r - 1). \tag{3}$$

Using (1), (2) and (3) we obtain

$$r(|S| + 2 - n) - 2 \geqslant 0. \tag{4}$$

Since $|S| + n \equiv |V| \equiv 0 \pmod 2$, thus from the inequality $|S| < n$ it follows that $|S| + 2 \leqslant n$, which combined with (4) gives a contradiction. This completes the proof.

Theorem 1 is best possible in the sense that no less connectivity will suffice as it can be seen from the following result.

Theorem 2. *Let k and r be integers with $0 \leqslant k \leqslant r - 2$. Then there is an r-regular graph $G = (V, H)$ with $|V| \equiv 0 \pmod 2$, with no 1-factor and such that*

(a) $\varkappa(G) = k$;

(b) $\lambda(G) = \begin{cases} k + 1, & \text{if } r \text{ is even and } k \text{ is odd;} \\ k, & \text{otherwise.} \end{cases}$

Note that if $r \equiv 0$, $k \equiv 1 \pmod 2$, then obviously no r-regular graph G with $\lambda(G) = k$ can exist.

We will find it convenient to use the following lemmas in the proof of Theorem 2.

Lemma 2. *Fof any graph G, $\varkappa(G) \leqslant \lambda(G)$.*
(For the proof see e. g. [3], p. 43.)

Lemma 3. *Let $G_1 = (V, H)$ be a graph. Let $\{v_1, v_2, \ldots, v_m\} \subseteq V$ with $m \geqslant \varkappa(G_1)$ and let $v \notin V$. Then for the graph $G = (V \cup \{v\}, H \cup \{v_1 v, v_2 v, \ldots, v_m v\})$ we have $\varkappa(G) \geqslant \varkappa(G_1)$.*

Proof. If $\varkappa(G_1) = 0$, then the assertion is clear and therefore let $\varkappa(G_1) \geqslant 1$. Let the deleting of some $\varkappa(G_1) - 1$ points from G give a graph G'. We will show that G' is connected. Let x, y be any two points of G'. If $x, y \in V$, then there is an $x - y$ path in G' by our assumption. If $x \in V$, $y = v$, then there is at least one point $u \in V$ adjacent with v (since less than m points have been deleted). By the preceding considerations, in G' there is an $x - u$ path and hence also an $x - v$ path. The lemma is proved.

Lemma 4. *Let any integer $c \geqslant 0$ be given and let $G_i = (V_i, H_i)$, $i = 1, 2$, be two point-disjoint graphs with $\varkappa(G_i) \geqslant c$. Let $\{u_1, u_2, \ldots, u_m\} \subseteq V_1$ and $\{v_1, v_2, \ldots, v_m\} \subseteq V_2$ be two point sets with $m \geqslant c$. Then for the graph $G = (V_1 \cup V_2, H_1 \cup H_2 \cup \{u_1 v_1, u_2 v_2, \ldots, u_m v_m\})$ we have $\varkappa(G) \geqslant c$.*

Proof. In the case of $c = 0$ the lemma is trivial and therefore let $c \geqslant 1$. Let the deleting of some $c - 1$ points of G result in a graph G'. To show that G' is connected, let any two points x, y of G' be taken. If $x, y \in V_1$ or $x, y \in V_2$, then there is an $x - y$ path by the assumption. Now, let $x \in V_1$ and $y \in V_2$. Since $m > c - 1$, thus at least one line $u_i v_i$ exists for some i. But by the just proved, there are $x - u_i$ and $v_i - y$ paths in G' and hence there is an $x - y$ path, too. The lemma is proved.

Lemma 5. *Let any integers m. n, c with $m, n \geqslant c \geqslant 0$ be given. Let $U = \{u_1, u_2, \ldots, u_m\}$ be a set of m points and let $G_1 = (V_1, H_1)$, $G_2 = (V_2, H_2)$, $\ldots$, $G_n = (V_n, H_n)$ be graphs with $\varkappa(G_i) \geqslant c$, $V_i \cap V_j = V_i \cap U = \emptyset$ for $i, j = 1, 2, \ldots, n$; $i \neq j$.*

Given sets $U_i = \{u_{i1}, u_{i2}, \ldots, u_{im}\} \subseteq V_i$ *for* $i = 1, 2, \ldots, n$, *then the graph* $G = (U \cup V_1 \cup V_2 \cup \ldots \cup V_n,\ H_1 \cup H_2 \cup \ldots \cup H_n \cup \{u_j u_{ij} | i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\})$ *has* $\varkappa(G) \geqslant c$.

Proof. We have a trivial case if $c = 0$, therefore let $c \geqslant 1$. Let a set of $c - 1$ points be deleted from G giving a graph G'. To prove that G' is connected, let two points x, y of G' be considered. If $x, y \in V_i$ for some i, then there is an $x - y$ path by the assumption about G_i. If $x \in V_i$ and $y \in V_j$, $i \neq j$, then there is at least one path $u_{ik} u_k u_{jk}$ for some k (since $m > c - 1$, such point — disjoint paths from V_i to V_j in the graph G exist). But then using the just proved we have an $x - u_{ik}$ path and an $u_{jk} - y$ one and hence also an $x - y$ path. Let $x \in V_i$ and $y = u_k \in U$. The degree of u_k in G is greater than $c - 1$, hence in G' there is a line $u_k u_{jk}$ for some j. Since the existence of some $x - u_{jk}$ path follows by the preceding thus there is an $x - y$ path. Finally let $x = u_i \in U$ and $y = u_j \in U$, $i \neq j$. Since $n > c - 1$ and $\varkappa(G_s) \geqslant 1$ for all s thus there is a $u_{ki} - u_{kj}$ path P_k (in $G_k \cap G'$) for some k. The path $u_i P_k u_j$ is an $x - y$ path in G'. This completes the proof.

Lemma 6. *Let any integers* m, n, c *with* $m, n \geqslant c \geqslant 0$ *be given. Then the complete bigraph* $K_{m,n}$ *has* $\varkappa(K_{m,n}) \geqslant c$.

The proof is easy and can be made analogously to the preceding proof.

The following two lemmas can be found in [3], (p. 89).

Lemma 7. *The complete graph* $K_{2n+\cdot}$ *is a sum of* n *spanning cycles.*

Lemma 8. *The complete graph* K_{2n} *is* 1*-factorable.*

Proof of Theorem 2. If $k = 0$, we can take for G any graph with two components, both isomorphic to the same connected r-regular graph with no 1-factor. Therefore we can suppose that $k \geqslant 1$ so that $r \geqslant 3$.

We shall give examples of the required graphs. A few cases will be considered.

(1) $r \equiv k \pmod 2$. Let us consider the graph $G = (V, H)$ sketched in Fig. 1. In this graph $V = \{u_1, u_2, \ldots, u_k\} \cup \bigcup_{i=1}^{r} (A_i \cup B_i \cup C_i)$, where $A_i = \{a_{i1}, a_{i2}, \ldots, a_{ik}\}$, $B_i = \{b_{i1}, b_{i2}, \ldots, b_{i,r-1}\}$, $C_i = \{c_{i1}, c_{i2}, \ldots, c_{i,r-k}\}$ and $H = \bigcup_{i=1}^{r} H_i$ where $H_i = \{u_j a_{ij} | j = 1, 2, \ldots, k\} \cup \{xy | x \in B_i,\ y \in A_i \cup C_i\} \cup \{c_{i1}c_{i2}, c_{i3}c_{i4}, \ldots, c_{i,r-k-1}c_{i,r-k}\}$.

Thus it can be seen that the subgraphs $G_1, G_2, \ldots, G_r$ sketched in Fig. 1. are mutually isomorphic and therefore we have drawn out only G_1. G is obviouslyobviously r-regular. Further, each subgraph G_i has $2r - 1 \equiv 1 \pmod 2$ points. Thus the number of points in G is equal to $r'2r - 1) + k = 2r^2 - (r - k) \equiv 0 \pmod 2$. According to Lemma 1 G has no 1-factor because

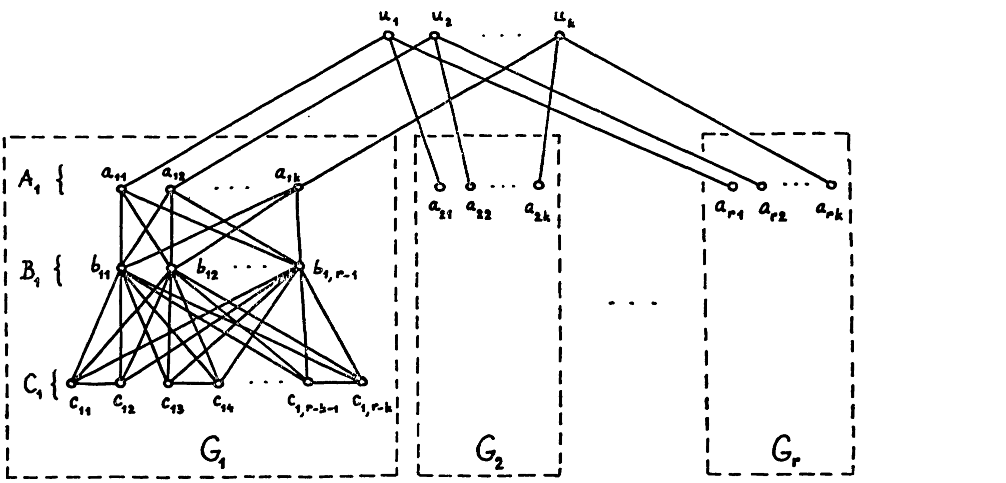

Fig. 1

if we put $S = \{u_1, u_2, \ldots, u_k\}$, then $G_1, G_2, \ldots, G_r$ will be the odd components of $G(V - S)$ and $k < r$. To show that $\varkappa(G) = \lambda(G) = k$, we start from the induced subgraph $G(A_i \cup B_i)$ which is in fact the complete bigraph $K_{k,r-1}$. Using Lemma 6 we have $\varkappa(G(A_i \cup B_i)) \geqslant k$. Now, if the points of C_i are successively added, then we result in the graph G_i and by Lemma 3 $\varkappa(G_i) \geqslant k$. If Lemma 5 is used, we see that $\varkappa(G) \geqslant k$. Since deleting the edges $u_1a_{11}, u_2a_{12}, \ldots, u_ka_{1k}$ results in a disconnected graph, thus $\lambda(G) \leqslant k$. Now, according to Lemma 2 the required equality $\lambda(G) = \varkappa(G) = k$ follows.

Note that in the following two cases ((2) and (3)) the graph of Fig. 2 will be used and therefore we denote for the next: $U = \{u_1, u_2, \ldots, u_{k+1}\}$, $A_i =$

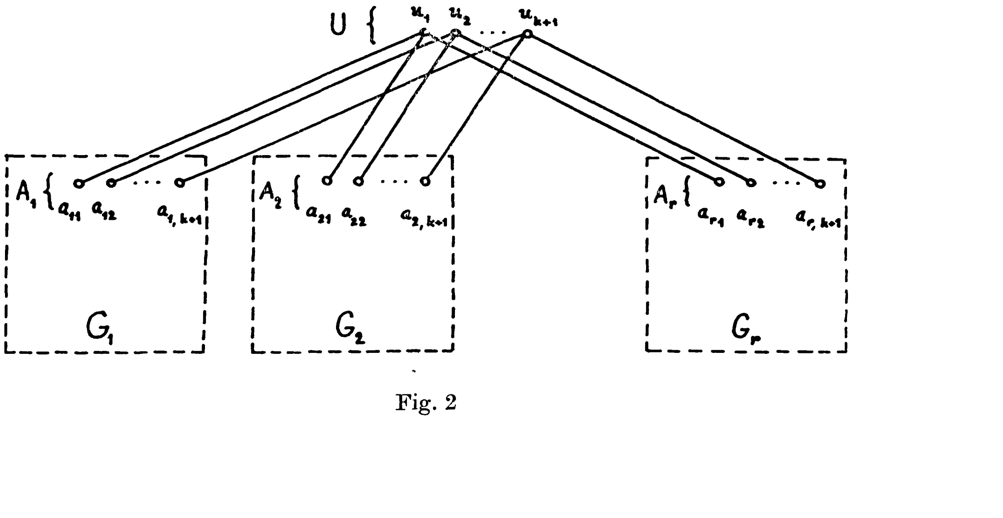

Fig. 2

$= \{a_{i1}, a_{i2}, \ldots, a_{i,k+1}\}$ and $L_i = \{u_1 a_{i1}, \ u_2 a_{i2}, \ldots, u_{k+1} a_{i,k+1}\}, \quad i = 1, 2, \ldots, r$.

(2) $r \quad 1, k \equiv 0 \pmod 2$.

(a) If $2k + 1 \leqslant r$, then denote by G the graph sketched in Fig. 2, with each G_i equal to the graph of Fig. 3. Thus $G = (V, H)$ $V = U \cup \bigcup_{i=1}^{r} (A_i \cup \cup B_i \cup C_i \cup D_i \cup E_i)$, where $B_i = \{b_{i1}, b_{i2}, \ldots, b_{i,r-1}\}$, $C_i = \{c_{i1}, c_{i2}, \ldots$

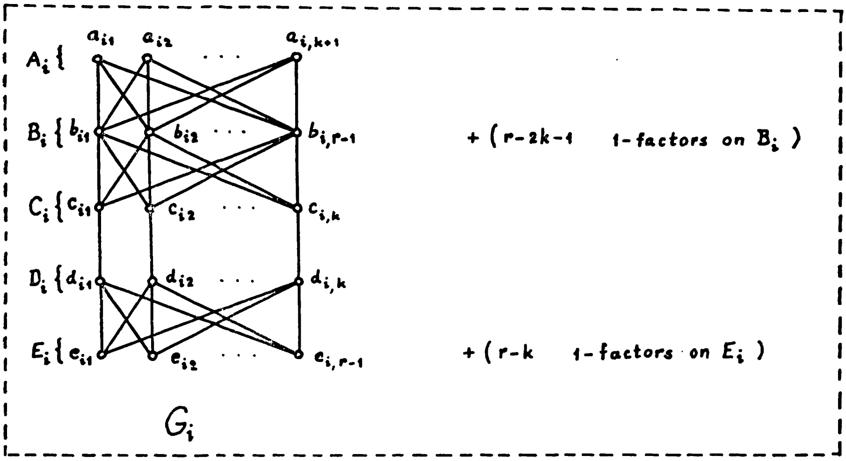

Fig. 3

$c_{i,k}\}$, $D_i = \{d_{i1}, d_{i2}, \ldots, d_{ik}\}$, $E_i = \{e_{i1}, e_{i2}, \ldots, e_{i,r-1}\}$ and with $H = \bigcup_{i=1}^{r} (L_i \cup \cup H_i)$, where $H_i = \{xy | x \in B_i, y \in A_i \cup C_i\} \cup \{c_{i1}d_{i1}, c_{i2}d_{i2}, \ldots, c_{ik}d_{ik}\} \cup \cup \{xy | x \in D_i, y \in E_i\} \cup H_i^1 \cup H_i^2$. Here H_i^1 consists of the lines of $r - 2k - 1$ 1-factors on B_i (as denoted in Fig. 3). (By Lemma 8 the complete graph with $r - 1$ vertices of B_i can be decomposed into $r - 2$ 1-factors. Now, we take $r - 2k - 1$ of these 1-factors and we delete the other ones. Thus the induced subgraph $G(B_i)$ is a sum of its $r - 2k - 1$ 1-factors.) Similarly, H_i^2 consists of $r - k$ 1-factors on E_i. It can be easily verified that G is r-regular with $|V| \quad 0 \pmod 2$. Using Lemma 1, where we put $S = U$, the graph G appears with no 1-factor. To show $\varkappa(G_i) \geqslant k$ it is sufficient to consider H_i without $H_i^1 \cup H_i^2$. Then the induced subraphs on $A_i \cup B_i \cup C_i$ and $D_i \cup E_i$ are $K_{2k+1,r-1}$, or $K_{k,r-1}$, respectively. If the Lemmas 6 and 4 are used, then we have $\varkappa(G_i) \geqslant \geqslant k$ and by Lemma 5 also $\varkappa(G) \geqslant k$.

If the k lines from C_1 to D_1 are deleted, then we have a disconnected graph and $\lambda(G) \leqslant k$. Thus $\lambda(G) = \varkappa(G) = k$ follows.

(b) If $2k+1 > r$, then we take the graph $G = (V, H)$ of Fig 2 with G_i from Fig. 4. We have: $V = U \cup \bigcup_{i=1}^{r} (A_i \cup B_i \cup C_i \cup D_i)$, where $B_i = \{b_{i1}, b_{i2}, \ldots, b_{ik}\}$, $C_i = \{c_{i1}, c_{i2}, \ldots, c_{ik}\}$, $D_i = \{d_{i1}, d_{i2}, \ldots, d_{i,r-1}\}$ and $H = \bigcup_{i=1}^{r} (L_i \cup H_i)$, where $H_i = \{xy | x \in A_i,\ y \in B_i\} \cup \{c_{i1}d_{i1}, c_{i2}d_{i2}, \ldots, c_{ik}d_{ik}\} \cup \{xy | x \in C_i, y \in D_i\} \cup H_i^1 \cup H_i^2 \cup H_i^3$. Here, the set H_i^1 consists of $(r-k-1)/2$ spanning cycles on A_i which can be taken from the $k/2$ spanning cycles of K_{k+1} (considered on A_i) as Lemma 7 provides. The sets H_i^2 and H_i^3 consist of $r-k-2$ 1-factors on B_i, or $r-k$ 1-factors on D_i, respectively (see Lemma 8). Now, it can be seen that G is r-regular with $|V| \equiv 0 \pmod 2$.

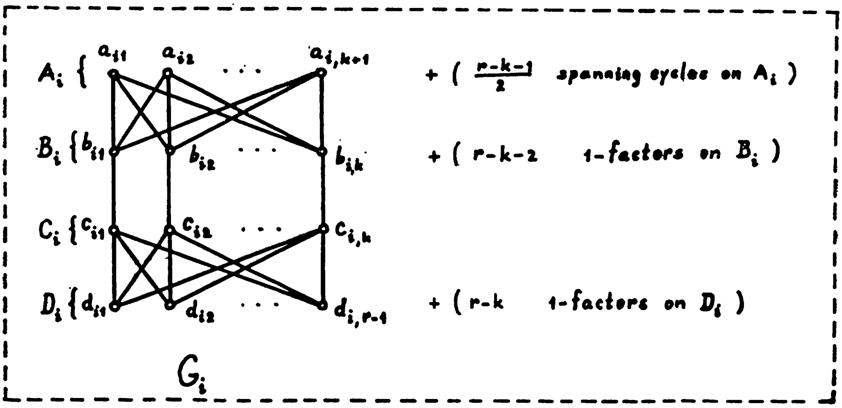

Fig. 4

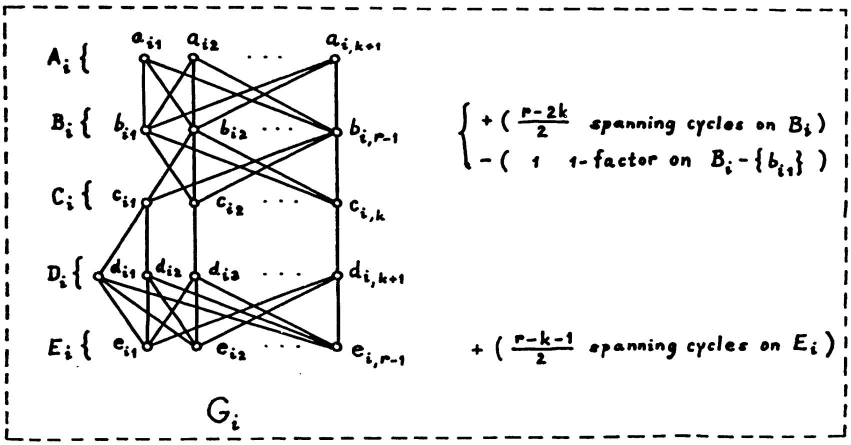

Fig. 5

Analogously as before it can be shown that G has no 1-factor and $\lambda(G) = \varkappa(G) = k$.

(3) $r \equiv 0$, $k \equiv 1 \pmod 2$.

(a) If $2k + 1 \leqslant r$, then we take for $G = (V, H)$ the graph of Fig 2, where for G_i the graph of Fig. 5. have been substituted.

Thus $V = U \cup \bigcup_{i=1}^{r} (A_i \cup B_i \cup C_i \cup D_i \cup E_i)$, where $B_i = \{b_{i1}, b_{i2}, \ldots, b_{i,r-1}\}$, $C_i = \{c_{i1}, c_{i2}, \ldots, c_{ik}\}$, $D_i = \{d_{i1}, d_{i2}, \ldots, d_{i,k+1}\}$, $E_i = \{e_{i1}, e_{i2}, \ldots, e_{i,r-1}\}$ and $H = \bigcup_{i=1}^{r} (L_i \cup H_i)$, where $H_i = (\{xy | x \in A_i \cup C_i, y \in B_i\} - \{b_{i1}c_{i1}\}) \cup \cup \{c_{i1}d_{i1}, c_{i1}d_{i2}, c_{i2}d_{i3}, c_{i3}d_{i4}, \ldots, c_{ik}d_{i,k+1}\} \cup \{xy | x \in D_i, y \in E_i\} \cup H_i^1 \cup H_i^2$. Here, H_i^1 consists of the lines of $(r-2k)/2$ spanning cycles on B_i (see Lemma 7) without the lines of one 1-factor of one of these cycles on $B_i - \{b_{i1}\}$. (This can be done; since $r - 2k > 0$, thus at least one spanning cycle has been added, then the 1-factor as a subgraph of the spanning cycle can be formed. H_i^2 consists of the lines of $(r-k-1)/2$ spanning cycles on E_i (see Lemma 7). Analogously as before we can verify that G is r-regular with no 1-factor and with $|V| \equiv 0 \pmod 2$. Also analogously using Lemmas 6, 3, 4 and 5 we find out that $\varkappa(G) \geqslant k$. However, the graph $G(V - C_i)$ is disconnected for any i. This yields $k = |C_i| \geqslant \varkappa(G)$ and hence $\varkappa(G) = k$. By Lemma 2 we have $\lambda(G) \geqslant k$. But a regular graph of an even degree r cannot have an odd $\lambda(G) = k$. Therefore $\lambda(G) \geqslant k + 1$. As removing $k + 1$ edges $u_1a_{11}, u_2a_{12}, \ldots, u_{k+1}a_{1,k+1}$ disconnects G, we have $\lambda(G) = k + 1$.

(b) If $2k + 1 > r$, then we take the graph $G = (V, H)$ of Fig. 2 again, where for each G_i the graph of Fig. 6. has been substituted. Here $V = U \cup \bigcup_{i=1}^{r} (A_i \cup B_i \cup C_i \cup D_i)$, where $B_i = \{b_{i1}, b_{i2}, \ldots, b_{ik}\}$, $C_i = \{c_{i1}, c_{i2}, \ldots,$

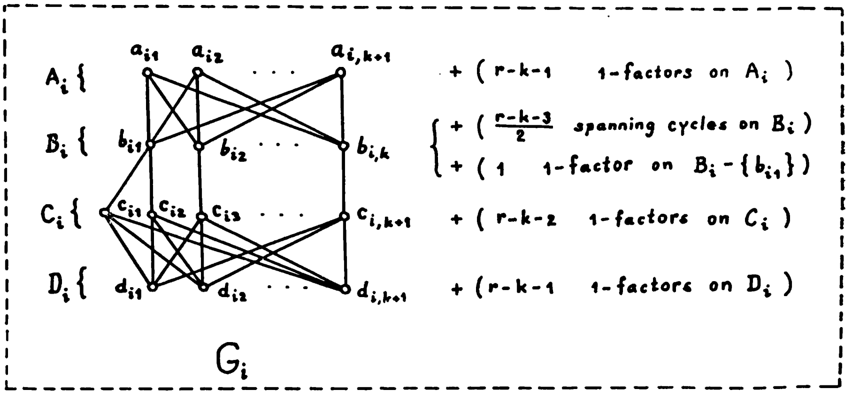

Fig. 6

$c_{i,k+1}\}$, $D_{i1}=\{d_{i1}, d_{i2}, \ldots, d_{i,k+1}\}$ and $H = \bigcup_{i=1}^{r} (L_i \cup H_i)$, where $H_i = \{xy | x \in A_i,$ $y \in B_i\} \cup \{b_{i1}c_{i1}, b_{i1}c_{i2}, b_{i2}c_{i3}, \ldots, b_{ik}c_{i,k+1}\} \cup \{xy | x \in C_i,\ y \in D_i\} \cup H_i^1 \cup$ $\cup H_i^2 \cup H_i^3 \cup H_i^4$. Here H_i^1, H_i^3, or H_i^4 correspond to the adding of $r - k - 1$ 1-factors on A_i, $r - k - 2$ 1-factors on C_i, or $r - k - 1$ 1-factors on D_i, respectively (see Lemma 8). According to Lemma 7 if the complete graph K_k is considered on B_i, then it can be decomposed into $(k-1)/2$ spanning cycles. Now, we take into H_i^2 $(r-k-3)/2$ from these cycles and then another spanning cycle is considered from which its 1-factor on $B_i - \{b_{i1}\}$ is taken into H_i^2 (this can be done since $(k - 1) - (r - k - 3) = (2k + 1) - r + 1 > 1$). Analogously as before we can find out again: G is r-regular with no 1-factor, with $|V| \equiv 0 \pmod 2$, with $\varkappa(G) = k$ and $\lambda(G) = k + 1$.

Now, we have considered all cases and Theorem 2 is proved.

Remark. We note that our results hold also in the case when multigraphs or pseudographs are admitted. Especially, Theorem 1 then follows from the validity of Tutte's theorem (Lemma 1) (since our proof is based on it) also for pseudographs.

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