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ONE-SIDED BASES OF SEMIGROUPS

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T. Tamura in [5] introduced the notion of a right (left) base of a semigroup and by means of this notion some properties of semigroups are investigated. In the present paper we shall describe the structure of semigroups containing one-sided bases. We shall consider only right bases, since for left bases analogous statements hold.

Definition 1. ([5]). *We say that a subset A of a semigroup S is a right base of S if $SA \cup A = S$, but there exists no proper subset $B \subset A$, for which $SB \cup \cup B = S$.*

Now we introduce a quasi-ordering into S, namely

$$a \leqslant b \text{ means } a \cup Sa \subseteq b \cup Sb.$$

Lemma 1. ([5]). *Let A be a right base of S. If $a, b \in A$ and $a \in Sb$, then $a = b$.*

Lemma 2. ([5]). *A non-empty subset A of S is a right base of S if and only if A satisfies the following conditions:*

(1) *for any $x \in S$ there exists $a \in A$ such that $x \leqslant a$.*

(2) *for any two distinct elements $a, b \in A$ neither $a \leqslant b$, nor $b \leqslant a$.*

The set of all elements of S generating the same principal left ideal as a fixed element of S is called an *L-class* of S (see [2]). The principal left ideal $a \cup Sa$ will be denoted by $(a)_L$.

Simple examples of semigroups show that a right base A of S need not be a subsemigroup and therefore not a left ideal, either.

Further we show some conditions when a right base of S is a left ideal, and also a subsemigroup of S.

Remark 1. We can show easily that a right base A of a semigroup S is a left ideal of S if and only if $A = S$.

A semigroup S is called *right singular* if for every two elements $x, y \in S$ we have $xy = y$.

Theorem 1. *A right base A of a semigroup S is a subsemigroup of S if and only if A is a right singular semigroup.*

Proof. (a) Let a right base A of S be a subsemigroup of S. It is necessary to show that for any $a, b \in A$, $ab = b$. According to the assumption $ab \in A$. Therefore, $ab = c$ for some $c \in A$, whence it follows that $c \in Sb$. Lemma 1 implies that $c = b$, therefore for arbitrary $a, b \in A$, $ab = b$.

(b) The converse statement is evident.

Corollary. *If a right base A of a semigroup S is a subsemigroup of S, then S contains at least one idempotent.*

By the following example of a semigroup we can ascertain that if a right base of S is a subsemigroup and therefore a right singular subsemigroup, the whole semigroup need not be such one.

Example 1. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table

	a	b	c	d
a	a	a	a	a
b	a	b	a	d
c	c	c	c	c
d	a	b	a	d

$A = \{b, d\}$ is a right base and a subsemigroup, but the whole semigroup is not right singular.

The notion of a *maximal proper left ideal* is used in the same sense as in [3].

We say that a semigroup S contains a left ideal L^*, if L^* is such a maximal proper left ideal, in which every proper left ideal of S is included (see [4]).

Lemma 3. *Let A be a right base of a semigroup S. Let $a_0 \in A$ be any element of A. If $(a_0)_L = (b)_L$ for some $b \in S, b \neq a_0$, then b is an element of a right base of S, distinct from A.*

Proof. Let $B = [A - \{a_0\}] \cup \{b\}$. It is clear that $A \neq B$. We show that B is also a right base of S. It is sufficient to show that B satisfies conditions (1), (2) of Lemma 2. Let x be an arbitrary element of S. Then, since A is a right base of S, there exists $a \in A$ such that $x \leqslant a$. Now, there are only two possibilities. 1. $a \neq a_0$, 2. $a = a_0$. If $a \neq a_0$, then $a \in B$. If $a = a_0$, then $a \bar{\in} B$, but $(a_0)_L = (b)_L$, therefore if $x \leqslant a$ then $x \cup Sx \subseteq a \cup Sa = b \cup Sb$, whence it follows that $x \leqslant b$ and $b \in B$. It means that B satisfies condition (1) of Lemma 2. Now let $b_1, b_2 \in B$ be arbitrary elements, but distinct. If both elements are distinct from b, then $b_1 \in A$, $b_2 \in A$ and since A is a right base of S, then neither $b_1 \leqslant b_2$, nor $b_2 \leqslant b_1$. Let for instance $b_1 = b$. If $b_1 \leqslant b_2$ then $a_0 \leqslant b_2$, where $a_0 \in A$, $b_2 \in A$. But A is a right base of S, therefore this is not possible. Similarly we can show that the relation $b_2 \leqslant b_1$ cannot be fulfilled. But it means that B also satisfies condition (2) of Lemma 2, therefore, B is a right base of S, distinct from A.

Theorem 2. *Let $\mathscr{A}$ be the union of all right bases of a semigroup S. If $L = S - \mathscr{A}$ is non-empty, then L is a left ideal of S.*

Proof. To prove our statement, we must show that if $x \in S$, $a \in L = S - \mathscr{A}$, then $xa \in L$. Let us assume that $xa \overline{\in} L$. Then $b = xa \in \mathscr{A}$, and thus b belongs at least to one right base A of S, and we have that $b \in Sa$, therefore $Sb \subseteq Sa$, $b \cup Sb \subseteq a \cup Sa$. We show that $(b)_L \neq (a)_L$. If $(b)_L = (a)_L$, then, since $b \in \mathscr{A}$, according to Lemma 3 $a \in \mathscr{A}$, and it is a contradiction to the assumption, because $a \in S - \mathscr{A}$. It means that $(b)_L \leqslant (a)_L$, $(b)_L \neq (a)_L$. And since A is a right base, then to the element a there exists an element $b_1 \in A$ such that $a \leqslant b_1$. Thus, we have $b \leqslant a \leqslant b_1$, therefore $b \leqslant b_1$, but it is a contradiction to condition (2) of Lemma 2, because b, $b_1 \in A$. Hence, $xa \in S - \mathscr{A}$.

The following example of a semigroup shows that $L = S - \mathscr{A}$ need not be a maximal left ideal of S.

Example 2. Let $S = \{a, b, c, d\}$ be a semigroup with the multiplication table:

	a	b	c	d
a	a	b	a	a
b	a	b	a	a
c	a	b	c	c
d	a	b	d	d

All right bases of S are: $A_1 = \{b, c\}$ and $A_2 = \{b, d\}$. $S - \mathscr{A} = \{a\}$ is a left ideal of S, but it is not a maximal proper one.

In the following we shall find conditions when $L = S - \mathscr{A}$ is a maximal proper left ideal of S.

Theorem 3. *Let $\mathscr{A} \neq \emptyset$. $S - \mathscr{A}$ is a maximal proper left ideal of a semigroup S if and only if $\mathscr{A} \neq S$ and $\mathscr{A} \subseteq a \cup Sa$ for all $a \in \mathscr{A}$.*

Proof. (a) Let $L = S - \mathscr{A}$ be a maximal proper left ideal of a semigroup S. Therefore $\mathscr{A} \neq S$. Let $a \in \mathscr{A}$. If $\mathscr{A} \subseteq a \cup Sa$ does not hold, then $(S - \mathscr{A}) \cup \cup (a)_L$ as a union of two left ideals is a left ideal of S, but a proper one. Then $S - \mathscr{A}$ is not a maximal left proper ideal, which is a contradiction to the assumption.

(b) Let $\mathscr{A} \subseteq a \cup Sa$ for all $a \in \mathscr{A}$, and $\mathscr{A} \neq S$. We have to prove that $S - \mathscr{A}$ is a maximal proper left ideal of S. According to Theorem 2, $S - \mathscr{A}$ is a (proper) left ideal of S. Let $S - \mathscr{A} \subseteq L'$, where L' is a left ideal of S and $S - \mathscr{A} \neq L'$. Then $L' \cap \mathscr{A} \neq \emptyset$. Let $a \in L' \cap \mathscr{A}$, so $a \in L'$. It follows that $Sa \subseteq SL' \subseteq L'$, $a \cup Sa \subseteq L'$. Whence, and according to the assumption, we obtain $\mathscr{A} \subseteq a \cup Sa \subseteq L'$. Consequently, $\mathscr{A} \subseteq L'$, $S - \mathscr{A} \subseteq L'$, therefore $S = L'$.

It is clear that S may contain maximal proper left ideals distinct from

$S - \mathscr{A}$. The question arises: when will $S - \mathscr{A}$ be such a maximal proper left ideal of S that every proper left ideal of S will be included in it?

We say that an element $a \in S$ is *left invertible* if $Sa = S$.

Lemma 4. ([1]). *Let a semigroup S contain at least one left invertible element. Then S contains the ideal L^* and the complement of this ideal is the set of all left invertible elements.*

Theorem 4. *Let $\emptyset \neq \mathscr{A} \neq S$. $S - \mathscr{A} = L^*$ if and only if every right base of S is one-element and one from the following conditions holds:*

(1) *Every right base of S is formed by a left invertible element.*

(2) *A semigroup S contains only one right base $A = \{a\}$ and we have: $a \cup Sa = S$, but $a \bar{\in} Sa$.*

Proof. (a) Let be $S - \mathscr{A} = L^*$. Thus $S - \mathscr{A}$ is a maximal proper left ideal. Theorem 3 implies that for any $a \in \mathscr{A}$ the following holds: $\mathscr{A} \subseteq a \cup Sa$. But, moreover, every proper left ideal of S is included in $S - \mathscr{A}$. Now we shall show that $S - \mathscr{A} \subseteq a \cup Sa$ for any $a \in \mathscr{A}$ as well. Till now we have: $\mathscr{A} \subseteq a \cup Sa$ and $S - \mathscr{A} = L^*$. There are only two possibilities: either $a \cup Sa$ is a proper left ideal of S and then $a \cup Sa \subseteq S - \mathscr{A}$, or $a \cup Sa = S$. The first possibility cannot hold, because at least $a \bar{\in} S - \mathscr{A}$. Therefore the other possibility must hold, so $a \cup Sa = S$, for any $a \in \mathscr{A}$. Thus $\{a\}$ is a right base of S. And as a is an arbitrary element of $\mathscr{A}$, then all right bases are one-element. Therefore, only the following three cases are possible.

(1) $a \cup Sa = S$, $a \in Sa$ for any element $a \in \mathscr{A}$. (It means that every element $a \in \mathscr{A}$ is left invertible.)

(2) $a \cup Sa = S$, $a \bar{\in} Sa$ for any element $a \in \mathscr{A}$.

(3) $a \cup Sa = S$, $a \in Sa$ for some element $a \in \mathscr{A}$, but $b \cup Sb = S$, $b \bar{\in} Sb$ for another element $b \in \mathscr{A}$, $b \neq a$.

We shall show that if $S - \mathscr{A}$ is a maximal proper left ideal of S, then case (3) cannot occur and in case (2) the semigroup S contains only one such base.

Let us assume that in case (2) a semigroup S contains at least two right bases, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$ such that $a_1 \cup Sa_1 = S$, $a_1 \bar{\in} Sa_1$, $a_2 \cup Sa_2 = S$, $a_2 \bar{\in} Sa_2$. Then $S - \mathscr{A} \subseteq Sa_1 \subset S$, where $S - \mathscr{A} \neq Sa_1$, because $a_2 \in Sa_1$. But it means that $S - \mathscr{A}$ is not a maximal proper left ideal, and this is a contradiction. If case (3) occurs, then again $S - \mathscr{A} \subseteq Sb \subset S$, where $S - \mathscr{A} \neq Sb$, because $a \in Sb$. It means that $S - \mathscr{A}$ is not a maximal proper left ideal, which is again a contradiction to the assumption.

(b) Let us assume that all right bases of S are one-element and that one condition from (1), (2) is satisfied. If (1) holds, then the statement follows from Lemma 4. If (2) holds, then $S - \{a\} = S - \mathscr{A} = L$ is a left ideal. It is evident that it is a maximal proper left one. We show that every proper left ideal of S is included in L. Let L_1 be a left ideal of S which is not included in L. Then, evidently $a \in L_1$, therefore $Sa \subseteq SL_1 \subseteq L_1$. But, since $a \in L_1$, then $S = a \cup Sa \subseteq L_1$, therefore $L_1 = S$. It means that $L = L^*$.

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