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ON THE CLASSIFICATION AND TOUGHNESS OF GENERALIZED PERMUTATION STAR-GRAPHS

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Abstract. We use an algebraic method to classify the generalized permutation star-graphs, and we use the classification to determine the toughness of all generalized permutation star-graphs.

1. INTRODUCTION

The graphs which we consider here are finite, undirected, loopless and simple. Let $X = (V_1, E_1)$ be a graph where the vertex-set $V_1 = V_1(X) = \{v_{11}, v_{12} \ldots, v_{1n}\}$ and $E_1 = E(X)$ is its edge-set, and σ be a permutation on V_1. A permutation X-graph (X, σ) is a graph with $2n$ vertices, $V(X,\sigma) = V_1 \cup V_2$ where $V_i = \{v_{i1}, v_{i2}, \ldots v_{in}\}$ for $i = 1, 2$ and $V_1 \cap V_2 = \varphi$, and $E(X,\sigma) = E_1 \cup E_2 \cup E_{12}$ where $E_1 = E(X)$, $E_2 = \{[v_{2t}, v_{2s}];\ [v_{1t}, v_{1s}] \in E_1\}$ and $E_{12} = \{[v_{1t}, v_{2s}];\ \sigma(v_{1t}) = v_{1s}\}$.

Example 1. Let C_5 be a 5-cycle with $V(C_5) = \{v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$ and

$$\sigma = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ v_1 & v_4 & v_2 & v_5 & v_3 \end{pmatrix}.$$

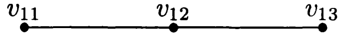
For simplicity, we shall write σ as (1)(2453). Permutation C_5-graph (C_5, σ) is the Petersen graph.

Permutation graphs were first considered by Chartrand and Harary in [3]. Dörfler, in [5] and [6], obtained some interesting results on automorphisms and isomorphisms of permutation graphs. Here, we shall consider a generalization of permutation graphs.

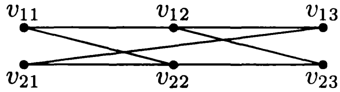
Let m be an integer $\geqslant 2$. $X = (V_1, E_1)$ and σ be a permutation on V_1. A generalized permutation X^m—graph, denoted by (X^m, σ), is a graph with mn vertices,

$V(X^m, \sigma) = V_1 \cup V_2 \cup \ldots \cup V_m$ where $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$ for $i = 1, 2, \ldots, m$, and $V_i \cap V_j = \varphi$ for $i \neq j$, and $E(X^m, \sigma) = (E_1 \cup E_2 \cup \ldots \cup E_m) \cup (E_{1,2} \cup E_{2,3} \cup \ldots \cup E_{m-1,m})$ where $E_1 = E(X)$, $E_i = \{[v_{it}, v_{is}];\ [v_{1t}, v_{1s}] \in E_1\}$ for $i = 2, 3, \ldots, m$, and $E_{j(j+1)} = \{[v_{jt}, v_{(j+1)s}];\ \tau(v_{1t}) = v_{1s}\}$ where $\tau = \sigma$ for j an odd integer and $1 \leqslant j \leqslant m-1$, and $\tau = \sigma^{-1}$ (the inverse of σ) for j an even integer and $1 \leqslant j \leqslant m-1$.

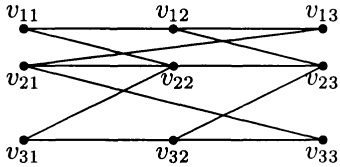
Example 2. Let X be the following graph with 3 vertices:



and $\sigma = (123)$. The permutation graph (X^2, σ) is the following graph with 6 vertices:



The generalized permutation graph (X^3, σ) is the following graph with 9 vertices:



where $\sigma^{-1} = (132)$ is used. The adjacency matrix $A = A(X)$ of X with the ordering v_{11}, v_{12}, v_{13} and the permutation matrix P_σ corresponding to σ are respectively:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We order the vertices of (X^2, σ) as $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}$. Then the adjacency matrix $A(X^2, \sigma)$ is the following 6×6 matrix consisting of four 3×3 block matrices

$$\begin{pmatrix} A_1 & P_\sigma \\ P_\sigma^t & A_2 \end{pmatrix}$$

where $A_1 = A_2 = A$ and $P_\sigma^t (= P_\sigma^{-1})$ is the transpose of P_σ. We also order the vertices of (X^3, σ) as $v_{11}, v_{12}, v_{13}, v_{21}, v_{22}, v_{23}, v_{31}, v_{32}, v_{33}$. Then the adjacency matrix $A(X^3, \sigma)$ is the following 9×9 matrix consisting of seven 3×3 nonzero block matrices and two 3×3 zero block matrices

$$\begin{pmatrix} A_1 & P_\sigma & \\ P_\sigma^t & A_2 & P_\sigma^t \\ & P_\sigma & A_3 \end{pmatrix}$$

where $A_1 = A_2 = A_3 = A, P_\sigma^t (= P_\sigma^{-1})$ is the transpose of P_σ and each of the blank entries is a 3×3 block matrix with all entries being zero.

Here our purposes are:

1. Use an algebraic method to obtain some results on the isomorphisms and automorphisms of generalized permutation graphs. Some of our results are generalizations of those in [5] and [6]. Our algebraic method depends on the Lemma A, on p. 480 in [1] which states: Let X and Y be graphs, σ be a one-to-one map of $V(X)$ onto $V(Y)$, and P_σ be the permutation matrix corresponding to σ. Then σ is an isomorphism of X onto Y if and only if

$$A(X)P_\sigma = P_\sigma A(Y). \tag{1}$$

On p. 489 in [1], Corollary A.1 states: Let X be a graph, σ be a permutation of $V(X)$, and P_σ be the permutation matrix corresponding to σ. Then σ is an automorphism of X if and only if

$$A(X)P_\sigma = P_\sigma A(X). \tag{2}$$

2. We shall use our results on isomorphisms and automorphisms to classify generalized permutation star-graphs. . . star-graph with $n+1$ vertices, $n \geqslant 1$, is a complete bipartite graph $K(1,n)$ with $n+1$ vertices having one vertex of degree n and each of the other n vertices of degree 1. In the Example 2 above, X is a star-graph $K(1,2)$.

3. We shall use our classification to determine the toughness of all generalized permutation star-graphs, i.e., to determine the toughness of $\big((K(1,n))^m, \sigma\big)$ for every positive integer n, every integer $m \geqslant 2$ and every permutation σ in the symmetric group S_{n+1} on $n+1$ vertices. The toughnesss of a graph $X, t(X)$, is defined as

$$t(X) = \min\left\{\frac{|S|}{\omega(X-S)}\right\}$$

where the minimum is taken over all disconnecting sets S of $V(X)$, $|S|$ is the cardinality of S, and $\omega(X-S)$ is the number of components of the induced graph $X-S$. (See [4].)

Lemma 1. *Let m be an integer $\geqslant 2$, X be a graph with n vertices, $G(X)$ be its group of automophisms, σ and μ be permutations on $V(X)$, and (X^m, σ) and (X^m, μ) be generalized permutation graphs. If there exists an α in $G(X)$ such that $\alpha^{-1}\sigma\alpha = \mu$, then (X^m, σ) and (X^m, μ) are isomorphic.*

Proof. Let $\alpha' = (\alpha, \alpha, \ldots, \alpha)$ be a map from $V(X^m, \sigma) = V_1 \cup V_2 \cup \ldots \cup V_m$ to $V(X^m, \sigma)$ defined by $\alpha'(V_1) = \alpha(V_1)$ and $\alpha'(v_{jt}) = v_{js}$ if and only if $\alpha(v_{1t}) = v_{1s}$ for $t = 1, 2, \ldots, n$, and $j = 2, 3, \ldots, m$. Then α' is a permutation of $V(X^m, \sigma)$. We order the vertices in $V(X^m, \sigma)$ lexicographically, i.e., in the following order:

$$v_{11}, v_{12}, \ldots, v_{1n}, v_{21}, v_{22}, \ldots, v_{2n}, \ldots, v_{m1}, v_{m2}, \ldots, v_{mn}.$$

Thus, the corresponding permutation matrix is

$$P_{\alpha'} = \begin{pmatrix} P_\alpha & & & \\ & P_\alpha & & \\ & & \ddots & \\ & & & P_\alpha \end{pmatrix} = (\text{diag.}(P_\alpha, P_\alpha, \ldots, P_\alpha))$$

where P_α is the permutation matrix corresponding to α, and the adjacency matrix of (X^m, σ) is

$$A(X^m, \sigma) = \begin{pmatrix} A_1 & P_\sigma & & & & \\ P_\sigma^t & A_2 & P_\sigma^t & & & \\ & P_\sigma & A_3 & & & \\ & & & \ddots & & \\ & & & & A_{m-1} & P_\sigma^{\pm\, t} \\ & & & & P_\sigma^{\mp\, t} & A_m \end{pmatrix}$$

where $A_1 = A_2 = A_3 = \ldots = A_m = A, P_\sigma$ is the permutation matrix corresponding to σ and $P_\sigma^{\pm\, t} = P_\sigma^t$ if m is an odd integer, and $P_\sigma^{\pm\, t} = P_\sigma^{-t} = P_\sigma$ if m is an even integer.

Since $\alpha \in G(X)$, by using (2), $\alpha^{-1}\sigma\alpha = \mu$, and the isomorphism of the symmetric group S_n on n vertices and the group of $n \times n$ permutation matrices, we have

$$P_{\alpha'}^{-1} A(X^m, \sigma) P_{\alpha'} = (\text{diag. } (P_\alpha^{-1}, P_\alpha^{-1}, \ldots, P_\alpha^{-1})) A(X^m, \sigma)(\text{diag. } (P_\alpha, P_\alpha, \ldots, P_\alpha))$$

$$= \begin{pmatrix} P_\alpha^{-1}A_1P_\alpha & P_\alpha^{-1}P_\sigma P_\alpha & & & \\ P_\alpha^{-1}P_\sigma^tP_\alpha & P_\alpha^{-1}A_2P_\alpha & P_\alpha^{-1}P_\sigma^tP_\alpha & & \\ & & \ddots & & \\ & & & P_\alpha^{-1}A_{m-1}P_\alpha & P_\alpha^{-1}P_\sigma^{\pm\ t}P_\alpha \\ & & & P_\alpha^{-1}P_\sigma^{\mp\ t}P_\alpha & P_\alpha^{-1}A_mP_\alpha \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & P_\mu & & & \\ P_\mu^t & A_2 & P_\mu^t & & \\ & & \ddots & & \\ & & & A_{m-1} & P_\mu^{\pm\ t} \\ & & & P_\mu^{\mp\ t} & A_m \end{pmatrix} = A(X^m, \mu).$$

By using (1), (X^m, σ) and (X^m, μ) are isomorphic. □

Corollary 1.1. *Let $\alpha \in G(X)$. Then $\alpha' = \overbrace{(\alpha, \alpha, \dots, \alpha)}^{m}$ belongs to the group of autormorphisms, $G(X^m, \sigma)$, of (X^m, σ) if and only if $\sigma\alpha = \alpha\sigma$.*

Proof. If $\sigma\alpha = \alpha\sigma$, then by Lemma 1 and (2), $\alpha' \in G(X^m, \sigma)$. Conversely, if $\alpha' \in G(X^m, \sigma)$, then, by (2), we have

$$A(X^m, \sigma) = (\mathrm{diag}(P_\alpha^{-1}, P_\alpha^{-1}, \dots, P_\alpha^{-1}))A(X^m, \sigma)(\mathrm{diag}(P_\alpha, P_\alpha, \dots, P_\alpha)),$$

i.e.,

$$\begin{pmatrix} A_1 & P_\sigma & & & \\ P_\sigma^t & A_2 & P_\sigma^t & & \\ & & \ddots & & \\ & & & A_{m-1} & P_\sigma^{\pm\ t} \\ & & & P_\sigma^{\mp\ t} & A_m \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & P_\alpha^{-1}P_\sigma P_\alpha & & & \\ P_\alpha^{-1}P_\sigma^tP_\alpha & A_2 & P_\alpha^{-1}P_\sigma P_\alpha & & \\ & & \ddots & & \\ & & & A_{m-1} & P_\alpha^{-1}P_\sigma^{\pm\ t}P_\alpha \\ & & & P_\alpha^{-1}P_\sigma^{\pm\ t}P_\alpha & A_m \end{pmatrix}$$

Thus, $P_\sigma = P_\alpha^{-1}P_\sigma P_\alpha$ and $\alpha\sigma = \sigma\alpha$. □

Remark. In our Corollary 1.1, if X and σ are given, how do we find $\alpha \in G(X)$ such that $\alpha' = (\alpha, \alpha) \in G(X, \sigma)$, i.e., which α in $G(X)$ such that $\alpha\sigma = \sigma\alpha$? The answer is that we have to find the centralizer ring, $R(\langle\sigma\rangle)$, of the cyclic group, $\langle\sigma\rangle$, generated by σ. Then take the intersection of $G(X)$ and $R(\langle\sigma\rangle)$. In general, there are not "many" such permutations α, although the intersection is not empty. In [1] and [2], there is an algorithm to find $R(H)$ for any given permutation group H. $R(H)$ is also a finite dimensional vector space over a field. The algorithm is to find a basis for the vector space. For instance, consider the Petersen graph $(X, (1)(2453))$ where X is the 5-cycle with $V(X) = \{1,2,3,4,5\}$. Then $G(X)$ is the dihedral group generated by (12345) and (1)(25)(34), and $R(\langle(1)(2453)\rangle)$ is

$$\left\{ \begin{pmatrix} a_{11} & a_{12} & a_{12} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{32} & a_{25} \\ a_{21} & a_{32} & a_{22} & a_{25} & a_{23} \\ a_{21} & a_{23} & a_{25} & a_{22} & a_{32} \\ a_{21} & a_{25} & a_{32} & a_{23} & a_{22} \end{pmatrix} ;\ a_{ij} \in \{0,1\} \right\}.$$

Consequently, $G(X) \cap R(\langle(1)(2453)\rangle)$ consists of the identity and (1)(25)(34) permutations. We know that the group of automorphisms of the Petersen graph is isomorphic to S_5 on 10 points. (See [7]).

Lemma 2. *Let X be a graph with n vertices, $G(X)$ be the group of automorphisms of X, and S_n be the symmetric group on n vertices.*

(a) *If σ and μ are in the same right coset of $G(X)$ in S_n, then the generalized permutation graphs (X^m, σ) and (X^m, μ) are isomorphic for any integer $m \geqslant 2$.*

(b) *If σ and μ are in the same left coset of $G(X)$ in S_n, then the generalized permutation graphs (X^m, σ) and (X^m, μ) are isomorphic for any integer $m \geqslant 2$.*

Proof. (a) Since σ and μ belong to the same right coset of $G(X)$ in S_n, there exists a $\beta \in G(X)$ such that $\sigma = \beta\mu$. Let ε be the identity permutation on $G(X)$, and

$$\beta' = \begin{cases} (\beta, \varepsilon, \beta, \varepsilon, \ldots, \beta, \varepsilon), & \text{if } m \text{ is even}, \\ (\beta, \varepsilon, \beta, \varepsilon, \ldots, \beta), & \text{if } m \text{ is odd}, \end{cases}$$

be a map from $V(X^m, \sigma) = V_1 \cup V_2 \cup \ldots \cup V_m$ to $V(X^m, \sigma)$ defined by $\beta'(V_1) = \beta(V_1), \beta'(v_{jt}) = \varepsilon(v_{jt}) = v_{jt}$ for $t = 1, 2, \ldots, n$ and j being even and $2 \leqslant j \leqslant m$, and $\beta'(v_{it}) = v_{is}$ if and only if $\beta(v_{1t}) = v_{1s}$ for $t = 1, 2, \ldots, n$ and j being odd and $2 < j \leqslant m$. Then β' is a permutation of $V(X^m, \sigma)$. Let $P_\varepsilon = I_n$ be the $n \times n$ identity matrix. Since $\sigma = \beta\mu, P_\beta^{-1}P_\sigma = P_\mu$, and

$$\begin{aligned}
&P_{\beta'}^{-1}A(X^m,\sigma)P_{\beta'}\\
&= (\operatorname{diag}(P_\beta^{-1}, I_n, P_\beta^{-1}, I_n, \ldots))A(X^m,\sigma)(\operatorname{diag}(P_\beta, I_n, P_\beta, I_n, \ldots))\\
&= \begin{pmatrix} P_\beta^{-1}A_1P_\beta & P_\beta^{-1}P_\sigma & & & \\ P_\sigma^t P_\beta & A_2 & P_\sigma^t P_\beta & & \\ & P_\beta^{-1}P_\sigma & P_\beta^{-1}A_3P_\beta & P_\beta^{-1}P_\sigma & \\ & & & \ddots & \end{pmatrix} = \begin{pmatrix} A_1 & P_\mu & & & \\ P_\mu^t & A_2 & P_\mu^t & & \\ & P_\mu & A_3 & P_\mu & \\ & & & \ddots & \end{pmatrix}\\
&= A(X^m,\mu)
\end{aligned}$$

where (2) is used. By (1), (X^m,σ) and (X^m,μ) are isomorphic.

(b) Similar to (a), there exists a $\gamma \in G(X)$ such that $\sigma = \mu\gamma$. Let

$$\gamma' = \begin{cases} (\varepsilon, \gamma, \varepsilon, \gamma, \ldots, \varepsilon, \gamma), & \text{if } m \text{ is even},\\ (\varepsilon, \gamma, \varepsilon, \gamma, \ldots, \gamma, \varepsilon), & \text{if } m \text{ is odd}, \end{cases}$$

be a map from $V(X^m,\sigma) = V_1 \cup V_2 \cup \ldots \cup V_m$ to $V(X^m,\sigma)$ defined by $\gamma'(v_{jt}) = \varepsilon(v_{jt}) = v_{jt}$ for $t = 1,2,\ldots,n$ and j being odd and $1 \leqslant j \leqslant m$, and $\gamma'(v_{it}) = v_{is}$ if and only if $\gamma(v_{it}) = v_{is}$ for $t = 1,2,\ldots,n$, and i being even and $1 < i \leqslant m$. Then γ' and $(\gamma')^{-1}$ are permutations of $V(X^m,\sigma)$. Since $\sigma = \mu\gamma, P_\sigma P_\gamma^{-1} = P_\mu$, and, similar to (a), we have

$$(P_{\gamma'}^{-1})^{-1}A(X^m,\sigma)P_{\gamma'}^{-1} = A(X^m,\mu).$$

By (1), (X^m,σ) and (X^m,μ) are isomorphic.

For $m = 2$, our Lemma 2 is the same as Theorem 9 and Theorem 9′ in [5]. □

Theorem 1. *Let m be an integer $\geqslant 2$, X be a graph with n vertices, $G(X)$ be its group of automorphisms, S_n be the symmetric group on n vertices, and $N(X^m)$ be the number of nonisomorphic classes of generalized permutation X^m−graphs. Then*

$$1 \leqslant N(X^m) \leqslant \frac{|S_n|}{|G(X)|},$$

i.e., $N(X^m)$ is bounded by the index of $G(X)$ in S_n for any integer $m \geqslant 2$.

The proof follows from Lemma 2. □

We note that if X is the complete graph or the null graph N_n, then $G(X)$ is S_n and $N(X^m) = 1$ for any integer $m \geqslant 2$, i.e., $(X^m,\sigma) \backsimeq (X^m,\varepsilon)$ for any $\sigma \in S_n$ and any integer $m \geqslant 2$.

Theorem 2. *The number of nonisomorphic classes of generalized permutation star-graphs with $n+1$ vertices is 2 for each integer $n \geqslant 2$, i.e., $N((K(1,n))^m) = 2$ for each integer $n \geqslant 2$ and for each integer $m \geqslant 2$.*

(We note that $N((K(1,1))^m) = N((K_2)^m) = 1$ for any integer $m \geqslant 2$.)

Proof. For $n \geqslant 2$, let $X = K(1,n)$ be a star-graph with $V(K(1,n)) = \{v_{11}, v_{12}, \dots, v_{1\ n+1}\}$ where the degree of v_{11} is n, and the degree of v_{1i} is 1 for $i = 2,3,\dots,n+1$. Clearly, $G(K(1,n))$ is $\{\sigma \in S_{n+1}\,;\, \sigma(v_{11}) = v_{11}\}$ of order $n!$, and it is isomorphic to S_n. The number of right cosets of $G(K(1,n))$ in S_{n+1} is $n+1$.

We claim that these $n+1$ right cosets of $G(K(1,n))$ in S_{n+1} can be represented as

$$G(K(1,n)), G(K(1,n))(12), G(K(1,n))(13), \dots, G(K(1,n))(1(n+1)),$$

i.e., they are pairwise disjoint, and $S_{n+1} = G(K(1,n)) \bigcup_{i=2}^{n+1} (G(K(1,n))(1i))$. Suppose that for $i \neq j$, $\sigma \in G(K(1,n))(1i) \cap G(K(1,n))(1j)$. Then there exist α and β in $G(K(1,n))$ such that $\sigma = \alpha(1i)$ and $\sigma = \beta(1j)$. If $\alpha(i) = k$ and $\beta(j) = q$, then $\sigma = (1ik\dots)$ and $\sigma = (1jq\dots)$. Since $i \neq j$, this is a contradiction, and $G(K(1,n))(1i) \cap G(K(1,n))(1j) = \varphi$ for $i,j = 2,3,\dots,n+1$, and $i \neq j$. Since each coset contains $n!$ permutations in S_{n+1},

$$S_{n+1} = G(K(1,n)) \cup \bigcup_{i=2}^{n+1} (G(K(1,n))(1i)).$$

It follows from Lemma 2 (a) that for any two permutations σ_1, σ_2 in the same right coset, the generalized permutation graphs $((K(1,n))^m, \sigma_1)$ and $((K(1,n))^m, \sigma_2)$ are isomorphic.

We claim that for any permutation $(1i)$, $i = 3,4,\dots,n+1$, the generalized permutation star-graphs $((K(1,n))^m, (1i))$ and $((K(1,n))^m, (12))$ are isomorphic. Since $(23\dots(n+1)) \in G(X)$ and

$$((23\dots(n+1))^{i-2})^{-1}(12)(23\dots(n+1))^{i-2} = (1i),$$

by Lemma 1, $((K(1,n))^m, (1i))$ and $((K(1,n))^m, (12))$ are isomorphic for $i = 3,4,\dots,n+1$.

We show that for the permutation (12) and the identity permutation ε in S_{n+1}, the generalized permutation star-graphs $((K(1,n))^m, \varepsilon)$ and $((K(1,n))^m, (12))$ are not isomorphic.

Every cycle in $((K(1,n))^m, \varepsilon)$ is of even length. But in $((K(1,n))^m, (12))$, the cycle $v_{11} - v_{22} - v_{21} - v_{23} - v_{13} - v_{11}$ is of length 5. Thus, $((K(1,n))^m, \varepsilon)$ and $((K(1,n)^m, (12))$ are not isomorphic, and the number of nonisomorphic classes of generalized permutation star-graphs with $n+1$ vertices is 2 for each integer $n \geqslant 2$ and for each integer $m \geqslant 2$. $\square$

3. The toughness

We shall determine the toughness of$((K(1,n))^m,\sigma)$ for every positive integer n, every integer $m \geqslant 2$ and every permutation σ in the symmetric group S_{n+1} on $n+1$ vertices. By using our classification, we only need to consider the toughness of $((K(1,n))^m,\varepsilon)$ and the toughness of $((K(1,n))^m,(12))$ for every positive integer n and every integer $m \geqslant 2$.

Theorem 3. *Let m and n be integers such that $m \geqslant 2$ and $n \geqslant 1$, $X = K(1,n)$ be a star-graph with $n+1$ vertices, and (X^m,σ) be a generalized permutation star-graph. Then*

$$t(X^m,\varepsilon) = \begin{cases} 1, & n=1 \text{ and } m \geqslant 2, & \text{(i)} \\ 1, & n=2,\ m \text{ even and } m \geqslant 2, & \text{(ii)} \\ \dfrac{3m-1}{3m+1}, & n=2,\ m \text{ odd and } m > 2, & \text{(iii)} \\ \dfrac{[\frac{m-1}{2}]n+[\frac{m+2}{2}]}{[\frac{m+1}{2}]n+[\frac{m-1}{2}]}, & 3 \leqslant n \leqslant m+1 \text{ and } m \geqslant 2, & \text{(iv)} \\ \dfrac{m}{n}, & n \geqslant m+2 \text{ and } m \geqslant 2, & \text{(v)} \end{cases}$$

where $[\frac{N}{2}]$ is the largest integer $\leqslant \frac{N}{2}$, and

$$t(X^m,(12)) = \frac{m}{(n-1)+m}, \quad n \geqslant 1 \text{ and } m \geqslant 2. \qquad \text{(vi)}$$

In order to prove Theorem 3, we need the following lemmas.

Lemma 3.

$$t(X^m,\varepsilon) \leqslant \frac{[\frac{m-1}{2}]n+[\frac{m+2}{2}]}{[\frac{m+1}{2}]n+[\frac{m-1}{2}]} < 1 \quad \text{for } n \geqslant 3 \text{ and } m \geqslant 2.$$

Proof. Let $S = S_1 \cup S_2 \cup \ldots \cup S_m$ be the disconnecting set of (X^m, ε) with

$$S_i = \{v_{i1}\} \text{ for } i \text{ being odd and } 1 \leqslant i \leqslant m-2,$$
$$S_k = \{v_{kj}\,;\, j = 2,3,\ldots,n+1\} \text{ for } k \text{ being even and } 1 < k \leqslant m-2,$$
$$S_{m-1} = \begin{cases} \{v_{(m-1)1}\}, & \text{if } m \text{ is even, and} \\ \{v_{(m-1)j}\,;\, j = 2,3,\ldots,n+1\}, & \text{if } m \text{ is odd,} \end{cases}$$

and

$$S_m = \{v_{m1}\}.$$

If m is even, then the components of the induced graph $(X^m, \varepsilon) - S$ are: $\{v_{1j}\}$ for $j = 2,3,\ldots,n+1, \{v_{21}\}, \{v_{3j}\}$ for $j = 2,3,\ldots,n+1, \{v_{41}\}, \ldots \{[v_{(m-1)j}, v_{mj}]\}$ for $j = 2,3,\ldots,n+1$.

If m is odd, then the components of the induced graph $(X^m, \varepsilon) - S$ are: $\{v_{1j}\}$ for $j = 2,3,\ldots,n+1, \{v_{21}\}, \{v_{3j}\}$ for $j = 2,3,\ldots,n+1, \{v_{41}\}, \ldots, \{v_{(m-1)1}\}, \{v_{mj}\}$ for $j = 2,3,\ldots,n+1$.

Thus, we have $|S| = [\frac{m-1}{2}]n + [\frac{m+2}{2}], \omega((X^m, \varepsilon) - S) = [\frac{m+1}{2}]n + [\frac{m-1}{2}]$, and

$$t(X^m, \varepsilon) \leqslant \frac{|S|}{\omega((X^m, \varepsilon) - S)} = \frac{[\frac{m-1}{2}]n + [\frac{m+2}{2}]}{[\frac{m+1}{2}]n + [\frac{m-1}{2}]} \quad \text{for } n \geqslant 3 \text{ and } m \geqslant 2.$$

We claim that $\dfrac{|S|}{\omega((X^m, \varepsilon) - S)} < 1$. If m is even and $n \geqslant 3$, then

$$t(X^m, \varepsilon) \leqslant \frac{|S|}{\omega((X^m, \varepsilon) - S)} = \frac{(\frac{m-2}{2})n + (\frac{m+2}{2})}{(\frac{m}{2})n + (\frac{m-2}{2})} = \frac{nm - 2n + m + 2}{nm + m - 2} < 1.$$

If m is odd, then

$$t(X^m, \varepsilon) \leqslant \frac{|S|}{\omega((X^m, \varepsilon) - S)} = \frac{(\frac{m-1}{2})n + (\frac{m+1}{2})}{(\frac{m+1}{2})n + (\frac{m-1}{2})} = \frac{nm - n + m + 1}{nm + n + m - 1} < 1.$$

We note that Lemma 3 also holds for $n = 2$, m odd and $m > 2$. □

Lemma 4.

$$t(X^m, (12)) \leqslant \frac{m}{(n-1) + m} < 1 \quad \textit{for } n \geqslant 2 \textit{ and } m \geqslant 2.$$

Proof. Let $S = S_1 \cup S_2, \cup \ldots \cup S_m$ be the disconnecting set with $S_i = \{v_{i1}\}$ for $i = 1, 2, \ldots, m$. Then the components of the induced graph $(X^m, (12)) - S$ are $\{v_{i2}\}$ for $i = 1, 2, \ldots, m$ and the chains

$$v_{1j} - v_{2j} - \ldots - v_{mj}, \quad \text{for } j = 3, 4, \ldots, n+1.$$

Thus, $|S| = m$, and $\omega((X^m, (12)) - S) = (n-1) + m$, and

$$t(X^m, (12)) \leqslant \frac{|S|}{\omega((X^m, (12)) - S)} = \frac{m}{(n-1)+m} < 1 \quad \text{for } n \geqslant 2 \text{ and } m \geqslant 2.$$

□

Let $F(X^m, \sigma) = \{S \subseteq V(X^m, \sigma);\ \frac{|S|}{\omega((X^m, \varepsilon) - S)} = t(X^m, \sigma)\}$, and $S = \bigcup_{i=1}^{m} S_i$ where $S_i = S \cap V(X_i)$ for $i = 1, 2, \ldots, m$.

Lemma 5. *If $S \in F(X^m, \varepsilon)$, then $S_i \neq \varphi$ for $i = 1, 2, \ldots, m$.*

Proof. Case 1. $S_i = \varphi$ and $S_{i+1} \neq \varphi, 1 \leqslant i \leqslant m-1$.

Case 1.1. $v_{(i+1)1} \notin S_{i+1}$. We claim that none of $v_{(i+1)j} \in S_{i+1}$ for $j = 2, 3, \ldots, n+1$. Suppose the contrary, i.e., $v_{(i+1)j} \in S_{i+1}$ for some $j \in \{2, 3, \ldots, n+1\}$. Let $S'_{i+1} = S_{i+1} \setminus \{v_{(i+1)j}\}$, and $S' = S_1 \cup \ldots \cup S_i \cup S'_{i+1} \cup S_{i+2} \cup \ldots \cup S_m$. Then $|S'| = |S| - 1$. If there is a component C of the induced graph $(X^m, \varepsilon) - S$ such that $v_{(i+2)j} \in C$ and $v_{i1} \notin C$ where $i + 2 \leqslant m$, then we have

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) - 1.$$

(The case of $i + 2 > m$ belongs to the case of having no such component.)

If there is no such component C, then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) > \omega((X^m, \varepsilon) - S) - 1.$$

Thus, in any case, we have

$$\frac{|S'|}{\omega((X^m, \varepsilon) - S')} \leqslant \frac{|S| - 1}{\omega((X^m, \varepsilon) - S) - 1} < \frac{|S|}{\omega((X^m, \varepsilon) - S)}$$

where Lemma 3 is used, i.e., $t(X^m, \varepsilon) = \frac{|S|}{\omega((X^m, \varepsilon) - S)} < 1$ is used. That is a contradiction to $S \in F(X^m, \varepsilon)$.

Case 1.2. $v_{(i+1)1} \in S_{i+1}$.. We claim that none of $v_{(i+1)j} \in S_{i+1}$ for $j = 2, 3, \ldots, n+1$. Suppose the contrary, i.e., $v_{(i+1)j} \in S_{i+1}$ for some $j \in \{2, 3, \ldots, n+1\}$. By using the same reasoning as in the Case 1.1, we have a contradiction. Consequently, $S_{i+1} = \{v_{(i+1)1}\}$. Let $S''_{i+1} = S_{i+1} \setminus \{v_{(i+1)1}\}$ and $S'' = S_1 \cup \ldots \cup S_i \cup S''_{i+1} \cup S_{i+2} \ldots \cup S_m$. Then $\omega((X^m, \varepsilon) - S'') \geqslant \omega((X^m, \varepsilon) - S) - 1$, and

$$\frac{|S''|}{\omega((X^m, \varepsilon) - S'')} \leqslant \frac{|S| - 1}{\omega((X^m, \varepsilon) - S) - 1} < \frac{|S|}{\omega((X^m, \varepsilon) - S)}.$$

That is a contradiction to $S \in F(X^m, \varepsilon)$.

By the Case 1.1 and the Case 1.2, we know that $S_{i+1} = \varphi$, i.e., the case $S_i = \varphi$ and $S_{i+1} \neq \varphi$, $1 \leqslant i \leqslant m - 1$, does not exist.

Case 2. $S_i = \varphi$ and $S_{i-1} \neq \varphi$ for $2 \leqslant i \leqslant m$.

Case 2.1. $v_{(i-1)1} \notin S_{i-1}$. Similar to the proof of the Case 1.1, we know that none of $v_{(i-1)j} \in S_{i-1}$ for $j = 2, 3, \ldots, n + 1$.

Case 2.2. $v_{(i-1)1} \in S_{i-1}$. Similar to the proof of the Case 1.2, we know that it is impossible, i.e., $S_{i-1} = \varphi$.

By the Case 2.1 and the Case 2.2, we know that $S_{i-1} = \varphi$, i.e., the case $S_i = \varphi$ and $S_{i-1} \neq \varphi$ for $2 \leqslant i \leqslant m$ does not exist.

Since (X^m, ε) is connected and $S \in F(X^m, \varepsilon), S \neq \varphi$. Say $S_k \neq \varphi$ for some k such that $1 \leqslant k \leqslant m$. Repeatedly using the Case 1, we have $S_{k-1} \neq \varphi$, $S_{k-2} \neq \varphi$, $\ldots$, $S_1 \neq \varphi$. Repeatedly using the Case 2, we have $S_{k+1} \neq \varphi, S_{k+2} \neq \varphi, \ldots, S_m \neq \varphi$. Hence, if $S \in F(X^m, \varepsilon)$, then $S_i \neq \varphi$ for $i = 1, 2, \ldots, m$. □

Lemma 6. *If $S \in F(X^m, (12))$, then $S_i \neq \varphi$ for $i = 1, 2, \ldots, m$.*

Proof. Case 1. $S_i = \varphi$ and $S_{i+1} \neq \varphi$, $1 \leqslant i \leqslant m - 1$.

Case 1.1. $v_{(i+1)1} \notin S_{i+1}$. The proof for the case that none of $v_{(i+1)j} \in S_{i+1}$ for $j = 3, 4, \ldots, n + 1$ is the same as the Case 1.1 in Lemma 5. We claim that $v_{(i+1)2} \notin S_{i+1}$. Suppose the contrary, i.e., $v_{(i+1)2} \in S_{i+1}$. Let $S'_{i+1} = S_{i+1} \setminus \{v_{(i+1)2}\}$, and $S' = S_1 \cup \ldots \cup S_i \cup S'_{i+1} \cup S_{i+2} \cup \ldots \cup S_m$. Then $|S'| = |S| - 1$. If there is a component C of the induced graph $(X^m, (12)) - S$ such that $v_{(i+2)1} \in C$ and $v_{i1} \notin C$ where $i + 2 \leqslant m$ (The case of $i + 1 > m$ belongs to the case of having no such component.), then we have

$$\omega((X^m, (12)) - S') = \omega((X^m, (12)) - S) - 1.$$

If there is no such component C, then

$$\omega((X^m, (12)) - S') = \omega((X^m, (12)) - S) > \omega((X^m, (12)) - S) - 1.$$

Thus, in any case, we have

$$\frac{|S'|}{\omega((X^m, (12)) - S')} \leqslant \frac{|S| - 1}{\omega((X^m, (12)) - S) - 1} < \frac{|S|}{\omega((X^m, (12)) - S)}$$

where Lemma 4 is used, i.e., $t(X^m, (12)) = \frac{|S|}{\omega((X^m,(12))-S)} < 1$ is used. That is a contradiction to $S \in F(X^m, (12))$.

Case 1.2. $v_{(i+1)1} \in S_{i+1}$. By using the same reasoning as in the Case 1.1, we know that none of $v_{(i+1)j} \in S_{i+1}$ for $j = 2, 3, \dots, n+1$. Thus, $S_{i+1} = \{v_{(i+1)1}\}$. Using the same reasoning as the Case 1.2 in Lemma 5, we have $S_{i+1} = \varphi$. By the Case 1.1 and the Case 1.2, we know $S_{i+1} = \varphi$, i.e., the case $S_i = \varphi$ and $S_{i+1} \neq \varphi$ for $1 \leqslant i \leqslant m-1$ does not exist.

Case 2. $S_i = \varphi$ and $S_{i-1} \neq \varphi$ for $2 \leqslant i \leqslant m$.

Case 2.1. $v_{(i-1)1} \notin S_{i-1}$. Similar to the proof of the Case 1.1, we know that none of $v_{(i-1)j} \in S_{i-1}$ for $i = 2, 3, \dots, n+1$.

Case 2.2. $v_{(i-1)1} \in S_{i-1}$. Similar to the proof of the Case 1.2, we know that it is impossible, i.e., $S_{i-1} = \varphi$.

By the Case 2.1 and the Case 2.2, we know that $S_{i-1} = \varphi$, i.e., the case $S_i = \varphi$ and $S_{i-1} \neq \varphi$ for $2 \leqslant i \leqslant m$ does not exist.

Similar to Lemma 5, repeatedly using the Case 1 and the Case 2, we have $S_i \neq \varphi$ for $i = 1, 2, \dots, m$. □

Lemma 7. *Let $X \in F(X^m, \varepsilon)$. If $v_{i1} \in S_i$, for $i = 1, 2, \dots, m$, then $v_{ij} \notin S_i$ for $j = 2, 3, \dots, n+1$.*

Proof. Suppose the contrary, i.e, $v_{ij} \in S_i$ for some j such that $2 \leqslant j \leqslant n+1$. Then let $S_i' = S_i \setminus \{v_{ij}\}$ and $S' = S_1 \cup \ldots \cup S_{i-1} \cup S_i' \cup S_{i+1} \cup \ldots \cup S_m$. Thus, $|S'| = |S| - 1$. If there is a component C of the induced graph $(X^m, \varepsilon) - S$ such that one of $v_{(i-1)j}$ and $v_{(i+1)j}$ belongs to C and the other does not (The case of $i = 1$ or $i = m$ belongs to the case of having no such component.), then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) - 1.$$

If there is no such component C, then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) > \omega((X^m, \varepsilon) - S) - 1.$$

Thus, in any case, we have

$$\frac{|S'|}{\omega((X^m, \varepsilon) - S')} \leqslant \frac{|S| - 1}{\omega((X^m, \varepsilon) - S) - 1} < \frac{|S|}{\omega((X^m, \varepsilon) - S)}$$

where the Lemma 3 is used, i.e., $t(X^m, \varepsilon) = \dfrac{|S|}{\omega((X^m, \varepsilon) - S)} < 1$ is used. That is a contradiction to $S \in F(X^m, \varepsilon)$. □

Lemma 8. *Let $S \in F(X^m,(12))$.*
(a) *If $v_{i1} \in S_i$, for $i = 1, 2, 3, \dots, m$, then $v_{ij} \notin S_i$ for $j = 3, 4, \dots, n+1$.*
(b) *If $v_{i1} \in S_i$, then $v_{i2} \notin S_i$ for $i = 1, 2, \dots, m$.*

Proof. (a) We replace $2 \leqslant j \leqslant m$, (X^m, ε), and Lemma 3 in the proof of Lemma 7 by $2 < j \leqslant m$, $(X^m, (12))$, and Lemma 4 respectively.

(b) We replace v_{ij}, $2 \leqslant j \leqslant m$, $(X^m, \varepsilon), v_{(i-1)j}$, $v_{(i+1)j}$, and Lemma 3 in the proof of the Lemma 5 by v_{i2}, $j = 2$, $(X^m, (12))$, $v_{(i-1)1}$, $v_{(i+1)1}$ and Lemma 4 respectively. □

Lemma 9. *Let $S \in F(X^m, \varepsilon)$. Then $v_{11} \in S_1$ and $v_{m1} \in S_m$.*

Proof. Suppose that $v_{11} \notin S_1$. By Lemma 5, $S_1 \neq \varphi$. If $v_{1j} \in S_1$ for some j such that $2 \leqslant j \leqslant n+1$, then let $S_1' = S_1/\{v_{1j}\}$ and $S' = S_1' \cup S_2 \cup \ldots \cup S_m$. Thus, $|S'| = |S| - 1$. If there is a component C of the induced graph $(X^m, \varepsilon) - S$ which contains only one of v_{11} and v_{2j}, then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) - 1.$$

If there is no such a component C, then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) > \omega((X^m, \varepsilon) - S) - 1.$$

Thus, $\frac{|S'|}{\omega((X^m,\varepsilon)-S')} \leqslant \frac{|S|-1}{\omega((X^m,\varepsilon)-S)-1} < \frac{|S|}{\omega((X^m,\varepsilon)-S)}$ where Lemma 1 is used, i.e., $t(X^m, \varepsilon) = \frac{|S|}{\omega((X^m,\varepsilon)-S)} < 1$ is used. That is a contradiction to $S \in F(X^m, \varepsilon)$, and $v_{11} \in S_1$. Similarly, $v_{m1} \in S_m$. □

Lemma 10. *Let $S \in F(X^m, (12))$. Then $v_{11} \in S_1$ and $v_{m1} \in S_m$.*

Proof. Suppose that $v_{11} \notin S_1$. By Lemma 6, $S_1 \neq \varphi$. If $v_{1j} \in S_1$ for some j such that $2 \leqslant j \leqslant n+1$, then let $S_1' = S_1/\{v_{1j}\}$ and $S = S_1' \cup S_2 \cup \ldots \cup S_m$. thus, $|S'| = |S| - 1$. If there is a component C of the induced graph $(X^m, (12)) - S$ which contains only one of v_{11} and v_{2j} for $2 \leqslant j \leqslant n+1$ or contains only one of v_{11} and v_{22}, then

$$\omega((X^m, (12)) - S') = \omega((X^m, (12)) - S) - 1.$$

If there is no such a component C, then

$$\omega((X^m, (12)) - S') = \omega(X^m, (12)) - S) > \omega((X^m, (12)) - S) - 1.$$

Thus, $\frac{|S'|}{\omega((X^m,(12))-S')} \leqslant \frac{|S|-1}{\omega((X^m,(12))-S)-1} < \frac{|S|}{\omega((X^m,(12))-S)}$ where Lemma 4 is used, i.e., $t(X^m, (12)) = \frac{|S|}{\omega((X^m,(12))-S)} < 1$ is used. That is a contradiction to $S \in F(X^m, (12))$ and $v_{11} \in S_1$. Similarly, $v_{m1} \in S_m$. □

Lemma 11. *There does not exist any S in $F(X^m, \varepsilon)$ with the property that $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$ where $1 \leqslant i \leqslant m-1$.*

Proof. Suppose the contrary, i.e., there existed a $S \in F(X^m, \varepsilon)$ with the property that $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$ where $2 \leqslant i \leqslant m-1$. Since $S_i \neq \varphi$ by Lemma 5, there would be a $v_{ij} \in S_i$ for some j such that $2 \leqslant j \leqslant n+1$.

Let $S_i' = S_i \backslash \{v_{ij}\}$ and $S' = S_1 \cup \ldots \cup S_{i-1} \cup S_i' \cup S_{i+1} \cup \ldots \cup S_m$. Then $|S'| = |S| - 1$. If $v_{(i+1)j}$ is in the induced graph $(X^m, \varepsilon) - S$, then $v_{(i+1)j}, v_{(i+1)1}$ and v_{i1} are in the same component, since $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$. If there is a component C in the induced graph $(X^m, \varepsilon) - S$ which contains only one of $v_{(i-1)j}$ and v_{i1}, then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) - 1.$$

(The case of $i = 2$ belongs to the following case.) If there is no such a component C, then

$$\omega((X^m, \varepsilon) - S') = \omega((X^m, \varepsilon) - S) > \omega((X^m, \varepsilon) - S) - 1.$$

Thus,

$$\frac{|S'|}{\omega((X^m, \varepsilon) - S')} \leqslant \frac{|S| - 1}{\omega((X^m, \varepsilon) - S) - 1} < \frac{|S|}{\omega((X^m, \varepsilon) - S)} < 1$$

where Lemma 3 is used, i.e., $t(X^m, \varepsilon) = \dfrac{|S|}{\omega((X^m, \varepsilon) - S)} < 1$ is used. That is a contradiction to $S \in F(X^m, \varepsilon)$. Hence with $v_{11} \in S_1$ (Lemma 7), there does not exist any $S \in F(X^m, \varepsilon)$ with the property that $v_{i1} \in S_i$ and $v_{(i+1)1} \notin S_{i+1}$ where $1 \leqslant i \leqslant m-1$. □

Lemma 12. *There does not exist any S in $F(X^m, (12))$ with the the property that $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$ where $1 \leqslant i \leqslant m-1$.*

Proof. Suppose the contrary, i.e., there existed a $S \in F(X^m, (12))$ with the property that $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$ where $2 \leqslant i \leqslant m-1$. Since $S_i \neq \varphi$ by Lemma 6, there would be a $v_{ij} \in S_i$ for some j such that $2 \leqslant j \leqslant n+1$. There are two cases:

Case 1. $j = 2$, i.e., $v_{i2} \in S_i$. We may assume that i is the smallest positive integer with the proprety $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$. Since by Lemma 10, $v_{11} \in S_1$ and $v_{m1} \in S_m$, we have $1 < i < m$. That means that for $1 < i < m$, there are S_{i-1}, S_i, S_{i+1} in S such that $v_{(i-1)1} \in S_{i-1}, v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$. Let $S_i' = S_i \setminus \{v_{i2}\}$ and $S' = S_1 \cup \ldots \cup S_{i-1} \cup S_i' \cup S_{i+1} \cup \ldots \cup S_m$. Then $|S'| = |S| - 1$.

If there is a component C in the induced graph $(X^m,(12)) - S$ which contains only one of the vertices v_{i1} and $v_{(i+1)1}$, then

$$\omega((X^m,(12)) - S') = \omega((X^m,(12)) - S) - 1.$$

If there is no such component C, then

$$\omega((X^m,(12)) - S') = \omega((X^m,(12)) - S) > \omega((X^m,(12)) - S) - 1.$$

Thus,

$$\frac{|S'|}{\omega((X^m,(12)) - S')} \leqslant \frac{|S| - 1}{\omega((X^m,(12)) - S) - 1} < \frac{|S|}{\omega((X^m,(12)) - S)} < 1$$

where Lemma 4 is used, i.e., $t(X^m,(12)) = \dfrac{|S|}{\omega((X^m,(12)) - S)} < 1$ is used. That is a contradiction to $S \in F(X^m,(12))$ with the property that $v_{i1} \notin S_i$ and $v_{(i+1)1} \notin S_{i+1}$ for $1 \leqslant i \leqslant m-1$.

Case 2. $j > 2$, i.e., $v_{ij} \in S_i$ for some j such that $2 < j \leqslant n+1$. The proof is similar to the one in Lemma 11. □

Lemma 13. *Let $S \in F(X^m,\varepsilon)$ and $[\frac{N}{2}]$ be the largest integer $\leqslant \frac{N}{2}$. Then*

$$\frac{|S|}{\omega((X^m,\varepsilon) - S)} \geqslant \frac{[\frac{m-1}{2}]n + [\frac{m+2}{2}]}{[\frac{m+1}{2}]n + [\frac{m-1}{2}]}$$

for $3 \leqslant n \leqslant m+1$ and $m \geqslant 2$, and $\frac{|S|}{\omega((X^m,\varepsilon)-S)} \geqslant \frac{m}{n}$ for $n \geqslant m+2$ and $m \geqslant 2$.

P r o o f. By Lemma 5, we know that $S_i \neq \varphi$ for $i = 1,2,\ldots,m$. By Lemma 9, $v_{11} \in S_1$ and $v_{m1} \in S_m$. By Lemma 7, $S_1 = \{v_{11}\}$ and $S_m = \{v_{m1}\}$. Thus, let $S_{i_1}, S_{i_2}, \ldots, S_{i_j}$ be the ones with $v_{i_p1} \notin S_{i_p}$ for $p = 1,2,\ldots,j$, and $1 < i_1 < i_2 < \ldots < i_j < m$, and $|S_{i_p}| = k_p$ for $p = 1,2,\ldots,j$. Then we have

$$(3) \qquad |S| = \left(\sum_{p=1}^{j} |S_{i_p}|\right) + (m-j) = \left(\sum_{p=1}^{j} k_p\right) + (m-j).$$

By Lemma 11, we know that $(i_p + 1) < i_{p+1}$ for $p = 1,2,\ldots,j-1$. Consider the induced graph from X_1 to X_{i_1}, denoted by $[X_1, X_{i_1}]$, of $X_1 \cup X_2 \cup \ldots \cup X_{i_1-1} \cup X_{i_1}$. If $v_{(i_1)q} \in S_{i_1}$ for $2 \leqslant q \leqslant n+1$, then the chain $v_{1q} - v_{2q} - \ldots - v_{(i_1-1)q}$ is a component in $[X_1, X_{i_1},]$.

If $v_{(i_1)r} \notin S_{i_1}$ for $2 \leqslant r \leqslant n+1$, then the chain $v_{1r} - v_{2r} - \ldots - v_{(i_1)r}$ is in the component which contains $v_{(i_1)1}$ in the induced graph $[X_1, X_{i_1}]$. Hence, the number of components in $[X_i, X_{i_1}]$ is $|S_{i_1}| + 1 = k_1 + 1$.

Consider the induced graph from X_1 to X_{i_2}, $[X_1, X_{i_1}, X_{i_2}]$, of $X_1 \cup X_2 \cup \ldots \cup X_{i_1} \cup X_{i_1+1} \cup \ldots \cup X_{i_2}$. If $v_{(i_1)q} \in S_{i_1}$ and $v_{(i_2)q} \in S_{i_2}$ for $2 \leqslant q \leqslant n+1$, then the chain $v_{(i_1+1)q} - v_{(i_1+2)q} - \ldots - v_{(i_2-1)q}$ is a component in $[X_1, X_{i_1}, X_{i_2}]$. Let k_{12} be the number of such components in $[X_1, X_{i_1}, X_{i_2}]$. If $v_{(i_1)q} \notin S_{i_1}$ and $v_{(i_2)q} \in S_{i_2}$ for $2 \leqslant q \leqslant n+1$, then the chain $v_{(i_1)q} - v_{(i_1+1)q} \cdots - v_{(i_2-1)q}$ is in the component which contains $v_{(i_1)1}$. If $v_{(i_1)q} \in S_{i_1}$ and $v_{(i_2)q} \notin S_{i_2}$ for $2 \leqslant q \leqslant n+1$, then the chain $v_{(i_1+1)q} - v_{(i_1+2)q} - \ldots - v_{(i_2)q}$ is in the component which contains $v_{(i_2)1}$. If $v_{(i_1)q} \notin S_{i_1}$ and $v_{(i_2)q} \notin S_{i_2}$ for $2 \leqslant q \leqslant n+1$, then the chain $v_{(i_1)q} - v_{(i_1+1)q} - \ldots - v_{(i_2)q}$ is in the component which contains $v_{(i_1)1}$ and $v_{(i_2)1}$. Thus, the total number of components in $[X_1, X_{i_1}, X_{i_2}]$ is $\leqslant (k_1+1) + (k_{12}+1)$. Similarly, the total number of components in $[X_1, X_{i_1}, X_{i_2}, X_{i_3}]$, is $\leqslant (k_1+1) + (k_{12}+1) + (k_{23}+1) \ldots$. The total number of components in $[X_1, X_{i_1}, X_{i_2}, \ldots, X_{i_j}]$ is $\leqslant (k_1+1) + (k_{12}+1) + (k_{23}+1) + \ldots + (k_{(j-1)j}+1)$. Clearly, $k_{r(r+1)} \leqslant k_r$ and $k_{r(r+1)} \leqslant k_{r+1}$ for $r = 1, 2, \ldots, j-1$. Since $S_m = \{v_{m1}\}$, the total number of components in $[X_1, X_{i_1}, X_{i_2}, \ldots, X_{i_j}, X_m] = X^m - S$ is $\leqslant (k_1+1) + (k_{12}+1) + (k_{23}+1) + \ldots + (k_{(j-1)j}+1) + k_j$, i.e.,

$$\omega((X^m, \varepsilon) - S) \leqslant k_1 + \left(\sum_{r=1}^{j-1} k_{r(r+1)}\right) + k_j + j. \tag{4}$$

By using (3) and (4), we have

$$\frac{|S|}{\omega((X^m, \varepsilon) - S)} \geqslant \frac{\left(\sum_{q=1}^{j} k_q\right) + (m-j)}{k_1 + \left(\sum_{r=1}^{j-1} k_{r(r+1)}\right) + k_j + j}. \tag{5}$$

We claim that

$$\frac{\left(\sum_{q=1}^{j} k_q\right) + (m-j)}{k_1 + \left(\sum_{r=1}^{j-1} k_{r(r+1)}\right) + k_j + j} \geqslant \frac{jn + (m-j)}{(j+1)n + j}. \tag{6}$$

By using $k_1 \leqslant n$, $k_j \leqslant n$, $k_{r(r+1)} \leqslant k_r \leqslant n$ and $k_{r(r+1)} \leqslant k_{r+1} \leqslant n$ for $r = 1, 2, \ldots, j-1$, we have

$$\left[\left(\sum_{p=1}^{j} k_p\right) + (m-j)\right][(j+1)n + j] - \left[k_1 + \left(\sum_{r=1}^{j-1} k_{r(r+1)}\right) + k_j + j\right][jn + (m-j)]$$
$$= \left[\left(\sum_{p=1}^{j} k_p\right)(j+1) - \left(k_1 j + \left(\sum_{r=1}^{j-1} k_{r(r+1)}\right) j + k_j j\right)\right] n$$
$$+ (m-j)(j+1)n + (m-j)j + \left(\sum_{p=1}^{j} k_p\right) j$$
$$- \left[k_1(m-j) + \left(\sum_{r=1}^{j-1} k_{r(r+1)}\right)(m-j) + (k_j + j)(m-j) + j^2 n\right]$$
$$\geqslant 0 + (m-j)(j+1)n$$
$$- \left(j^2 n + (m-2j)k_1 + (m-2j)\left(\sum_{r=1}^{j-1} k_{r(r+1)}\right) + (m-j)k_j\right)$$
$$\geqslant 0 + (m-j)(j+1)n - (j^2 + j(m-2j) + (m-j))n$$
$$\geqslant 0 + (m-j)(j+1)n - (m-j)(j+1)n = 0.$$

Hence,

$$\frac{\left(\sum\limits_{p=1}^{j} k_p\right) + (m-j)}{k_1 + \left(\sum\limits_{r=1}^{j-1} k_{r(r+1)}\right) + k_j + j} \geqslant \frac{jn + (m-j)}{(j+1)n + j}.$$

We claim that, for $3 \leqslant n \leqslant m+1$,

$$\frac{jn + m - j}{(j+1)n + j} \geqslant \frac{[\frac{m-1}{2}]n + [\frac{m+2}{2}]}{[\frac{m+1}{2}]n + [\frac{m-1}{2}]}. \tag{7}$$

Let $f(j) = \dfrac{jn + m - j}{(j+1)n + j}$. We show that $f(j)$ is decreasing for all integers $j \geqslant 0$, i.e., $f(j+1) > f(j)$ for all integers $j \geqslant 0$. By using $n \leqslant m+1$, we have

$$[(j+1)n + m - (j+1)][(j+1)n + j] - [(j+2)n + (j+1)][jn + m - j]$$
$$= n^2 - (m+1)n - m < 0$$

for all integers $j \geqslant 0$.

Since $0 \leqslant j \leqslant [\frac{m-1}{2}]$, $f(j) \geqslant \frac{[\frac{n-1}{2}]n+[\frac{m+2}{2}]}{[\frac{m+1}{2}]n+[\frac{m-1}{2}]}$, i.e., the inequality (7) holds, and by (5), (6) and (7), we have

$$\frac{|S|}{\omega((X^m,\varepsilon)-S)} \geqslant \frac{[\frac{n-1}{2}]n+[\frac{m+2}{2}]}{[\frac{m+1}{2}]n+[\frac{m-1}{2}]} \quad \text{for } 3 \leqslant n \leqslant m+1 \text{ and } m \leqslant 2.$$

We claim that, for $n \geqslant m+2$

$$\frac{jn+(m-j)}{(j+1)n+j} \geqslant \frac{m}{n} \tag{8}$$

hold for all integers $j \geqslant 0$.

Clearly, if $j=0$, then (8) is an equality. Since $n \geqslant m+2$, we have

$$m \leqslant n-2 = (n+1) - \frac{3(n+1)}{(n+1)} = \frac{(n+1)^2-3(n+1)}{(n+1)} = \frac{n^2-n-2}{n+1} < \frac{n^2-n}{n+1},$$

i.e.,

$$n^2-n > m(n+1) \quad \text{or } n^2-n-mn-m > 0.$$

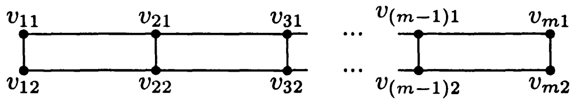
Since $(jn+(m-j))n-((j+1)n+j)m = j(n^2-n-mn-m) > 0$ for integers $j > 0$, the inequality (8) holds ior all integers $j \geqslant 0$. By (5), (6) and (8), we have

$$\frac{|S|}{\omega((X^m,\varepsilon)-S)} \geqslant \frac{m}{n} \quad \text{for } n \geqslant m+2 \text{ and } m \geqslant 2.$$

□

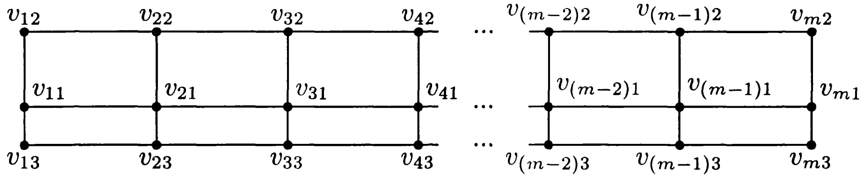
The proof of Theorem 3 goes as follows:

(i) For $n=1$ and $m \geqslant 2$, we have $X = K(1,1)$, and (X^m,ε) is the following graph:



(X^m,ε) is a Hamiltonian graph. In [4], a result states that the toughness of a Hamiltonian graph is $\geqslant 1$. Let $S = \{v_{11}, v_{22}, v_{31}, \dots, v_{m1}\}$ if m is odd, and $S = \{v_{11}, v_{22}, \dots, v_{m2}\}$ if m is even. Then $|S| = \omega((X^m,\varepsilon)-S) = \frac{1}{2}|V(X^m,\varepsilon)|$, and $\frac{|S|}{\omega((X^m,\varepsilon)-S)} = 1$. Hence, $t(X^m,\varepsilon) = 1$.

(ii) For $n = 2$, m even and $m \geqslant 2$, we have $X = K(1,2)$, and (X^m, ε) is the following graph:



Since m is even and $m \geqslant 2$, (X^m, ε) has a Hamiltonian cycle: $v_{12} - v_{22} - v_{32} - \ldots - v_{(m-2)2} - v_{(m-1)2} - v_{m2} - v_{m1} - v_{m3} - v_{(m-1)3} - v_{(m-1)1} - v_{(m-2)1} - v_{(m-2)3} - \ldots v_{41} - v_{43} - v_{33} - v_{31} - v_{21} - v_{23} - v_{13} - v_{11}$. Thus, by the result in [4], $t(X^m, \varepsilon) \geqslant 1$. Let $S = \{v_{11}, v_{22}, v_{23}, v_{31}, v_{42}, v_{43}, \ldots, v_{(m-1)1}, v_{m2}, v_{m3}\}$. Then $|S| = \omega((X^m, \varepsilon) - S) = \frac{1}{2}\left|V(X^m, \varepsilon)\right|$, and $\frac{|S|}{\omega((X^m,\varepsilon)-S)} = 1$. Hence, $t(X^m, \varepsilon) = 1$.

We shall prove (iv) first before we prove (iii).

(iv) For $n \geqslant 3$ and $m \geqslant 2$, we have $X = K(1, n)$. By Lemma 3, we have

$$t(X^m, \varepsilon) \leqslant \frac{[\frac{m-1}{2}]n + [\frac{m+2}{2}]}{[\frac{m+1}{2}]n + [\frac{m-1}{2}]}.$$

By Lemma 13, we have, for $3 \leqslant n \leqslant m+1$ and $m \geqslant 2$,

$$t(X^m, \varepsilon) \geqslant \frac{[\frac{m-1}{2}]n + [\frac{m+2}{2}]}{[\frac{m+1}{2}]n + [\frac{m-1}{2}]}.$$

Hence, (iv) holds.

(iii) For $n = 2$, $m =$ odd and $m > 2$, we have $X = K(1,2)$. The note at the end of Lemma 3 states that Lemma 3 also holds for $n = 2, m$ being odd and $m > 2$. Thus Lemmas 5, 7, 9, 11 and 13 also hold for this case, and

$$t(X^m, \varepsilon) = \frac{[\frac{m-1}{2}]2 + [\frac{m+2}{2}]}{[\frac{m+1}{2}]2 + [\frac{m-1}{2}]} \quad \text{where } m \text{ is odd and } m > 2,$$

i.e.,

$$t(X^m, \varepsilon) = \frac{(\frac{m-1}{2})2 + (\frac{m+1}{2})}{(\frac{m+1}{2})2 + \frac{m-1}{2}} = \frac{2m - 2 + m + 1}{2m + 2 + m - 1} = \frac{3m-1}{3m+1}.$$

(v) Let $S = \{v_{11}, v_{21}, \ldots, v_{m1}\}$ be a disconnecting set in (X^m, ε). Then $|S| = m$, and $\omega((X^m, \varepsilon) - S) = n$, i.e., for each $j = 2, 3, \ldots, n+1$, the chain $v_{1j} - v_{2j} - v_{3j} - \ldots - v_{mj}$ is a component in the induced graph $(X^m, \varepsilon) - S$, and there are n of them. Thus,

$$t(X^m, \varepsilon) \leqslant \frac{|S|}{\omega((X^m, \varepsilon) - S)} = \frac{m}{n}.$$

By Lemma 13, for $n \geqslant m+2$ and $m \geqslant 2$, we have $t(X^m, \varepsilon) \geqslant \frac{m}{n}$. Hence,

$$t(X^m, \varepsilon) = \frac{m}{n} \quad \text{for } n \geqslant m+2 \text{ and } m \geqslant 2.$$

(vi) Let $n \geqslant 1$, $m \geqslant 2$ and $X = K(1, n)$. We want to show that

$$t(X^m, (12)) = \frac{m}{(n-1)+m}.$$

Case 1. $n = 1$ and $m \geqslant 2$. With $X = K(1,1)$, $(X^m, (12))$ and (X^m, ε) are clearly isomorphic. Thus, $t(X^m, (12)) = t(X^m, \varepsilon) = 1 = \frac{m}{(1-1)+m}$, and $t(X^m, (12)) = \frac{m}{(n-1)+m}$ holds for $n = 1$ and $m \geqslant 2$.

Case 2. $n \geqslant 2$ and $m \geqslant 2$. Let $S \in F(X^m, (12))$. Then by Lemma 4, we know that

$$\frac{|S|}{\omega((X^m, (12)) - S)} \leqslant \frac{m}{(n-1)+m} < 1.$$

We claim that there exists a $S' \in F(X^m, (12))$ such that

$$S'_i = \{v_{i1}\} \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Let $S \in F(X^m, (12))$ such that $S_i \neq \{v_{i1}\}$ for some i. By Lemma 8, 10, 12, $1 < i < m$, $v_{i1} \notin S_i$, $v_{(i-1)1} \in S_{i-1}$, and $v_{(i+1)1} \in S_{i+1}$. Since $S_i \neq \varphi$, there exists a vertex $v_{ij} \in S_i$ such that $j = \min\{t \geqslant 2; v_{it} \in S_i\}$. Let $S'_i = (S_i \setminus \{v_{ij}\}) \cup \{v_{i1}\}$ and $S' = S_1 \cup S_2 \cup \ldots \cup S_{i-1} \cup S'_i \cup \ldots \cup S_m$. Then $\{v_{i2}\}$ is a component of $(X^m, (12)) - S'$. Thus,

$$\omega((X^m, (12)) - S') \geqslant \omega((X^m, (12)) - S)$$

i.e., $S' \in F(X^m, (12))$. By Lemma 8, $S'_i = \{v_{i1}\}$

Repeatedly using the above method on $1 < i < m$, we have that $S' \in F(X^m, (12))$ such that $S'_i = \{v_{i1}\}$ for $i = 1, 2, 3, \ldots, m$.

Hence, by Lemma 2,

$$t(X^m, (12)) = \frac{m}{(n-1)+m} \quad \text{for} \quad n \geqslant 2 \quad \text{and} \quad m \geqslant 2.$$

□

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