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OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HIGHER ORDER DAMPED NONLINEAR DIFFERENCE EQUATIONS

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Abstract. The asymptotic and oscillatory behavior of solutions of mth order damped nonlinear difference equation of the form

$$\Delta(a_n\Delta^{m-1}y_n) + p_n\Delta^{m-1}y_n + q_n f(y_{\sigma(n+m-1)}) = 0$$

where m is even, is studied. Examples are included to illustrate the results.

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1. Introduction

The problem of determining oscillation criteria for difference equations has been the subject of intensive investigations in the last few years, see for example [2–3, 6–18] and the references cited therein. We refer particularly to [2, 3, 12, 13, 16–18] in which oscillation theorems for higher order nonlinear difference equations are presented. Following this trend, in this paper we are concerned with the oscillatory and asymptotic behavior of the m^{th} order nonlinear damped difference equation of the form

$$\text{(E)} \qquad \Delta(a_n\Delta^{m-1}y_n) + p_n\Delta^{m-1}y_n + q_n f(y_{\sigma(n+m-1)}) = 0, \quad n \in \mathbb{N}$$

where m is even, $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ and $\Delta^i y_n = \Delta(\Delta^{i-1}y_n)$, $1 \leqslant i \leqslant m$. The real sequences $\{a_n\}$, $\{p_n\}$, $\{q_n\}$, $\{\sigma(n)\}$ and the function f satisfy the following hypotheses:

(H_1) $a_n > 0$ with $\Delta a_n > 0$ and $\{p_n\}$ and $\{q_n\}$ are given infinite sequences such that $p_n \geqslant 0$ and $q_n > 0$ for all $n \geqslant n_0 \in \mathbb{N}$;

(H_2) $\{\sigma(n)\}$ is a given monotonic increasing sequence of integers such that $\sigma(n) \to \infty$ as $n \to \infty$;

(H_3) $f\colon \mathbb{R} \to \mathbb{R} = (-\infty, \infty)$ is continuous and nondecreasing such that $uf(u) > 0$ for $u \neq 0$.

By a solution of equation (E) we mean a real sequence $\{y_n\}$ satisfying equation (E) for all $n \in \mathbb{N}$. A solution of equation (E) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory.

Our purpose in this paper is to obtain sufficient conditions for all solutions of equation (E) to be oscillatory. Thandapani and Sundaram [14] have recently considered a special case of equation (E)

$$(E_1) \qquad \Delta^m y_n + q_n f(y_{n-\sigma_n}) = 0, \quad n \geqslant n_0$$

where $\{q_n\}$ is an eventually positive sequence. Our results include, as special cases, known oscillation theorems not only for equation (E_1), but also for several other particular difference equations considered in [1]. Further, our results generalize those in [10, 11]. Finally, we remark that the motivation of this paper comes from [4, 5].

2. Main results

In the sequel, we need the following two lemmas of which the first can be found in [18] and the second in [1].

Lemma 1. *Let $\{y_n\}$ be a sequence of real numbers defined in $\mathbb{N}$. Let $\{y_n\}$ and $\{\Delta^m y_n\}$ be of constant sign with $\Delta^m y_n$ being not identically zero on any subset of the form $\{n_1, n_1+1, \ldots\}$ of $\mathbb{N}$. If*

$$y_n \Delta^m y_n \leqslant 0,$$

then

(i) *there is a natural member $n_2 \geqslant n_1$ such that the sequence $\{\Delta^j y_n\}$, $j = 1, 2, \ldots m-1$ is of constant sign on $\{n_2, n_2+1, \ldots\}$.*

(ii) *there exists a number $l \in \{0, 1, 2, \ldots m-1\}$ with $(-1)^{m-l-1} = 1$ such that*

$$y_n \Delta^j y_n > 0 \quad \text{for} \quad j = 0, 1, 2, \ldots l,\ n \geqslant n_2,$$
$$(-1)^{j-l} y_n \Delta^j y_n > 0 \quad \text{for} \quad j = l+1, \ldots m-1,\ n \geqslant n_2.$$

Lemma 2. *Let $y_n > 0$ and $\Delta^{m-1} y_n > 0$ be defined for $n \geqslant n_0 \in \mathbb{N}$ with $\Delta^m y_n \leqslant 0$ for all $n \geqslant n_0$, and not eventually identically equal to zero. Then there exists an integer $n_1 \geqslant n_0$ such that*

$$y_n \geqslant \frac{(n-n_1)^{(m-1)}}{(m-1)!} \Delta^{m-1} y_{2^{m-l-1} n}$$

for $n \geqslant n_1$, where l is defined in Lemma 1 and $(n-n_1)^{(m-1)}$ is the usual factorial notation.

Remark 1. Observe that under the hypotheses of Lemma 1, if $\{y_n\}$ is increasing, then

$$y_{n+m-1} \geqslant \frac{1}{(m-1)!} \left(\frac{n}{2^{m-1}}\right)^{(m-1)} \Delta^{m-1} y_n$$

for $n \geqslant 2^{m-1} n_1$.

Theorem 1. *Suppose that*

$$a_n - p_n > 0 \quad \textit{for} \quad n \geqslant n_0 \in \mathbb{N}, \tag{1}$$

and

$$\sum_{n=n_0}^{\infty} \frac{q_n}{a_{n+1}} = \infty. \tag{2}$$

Further assume that there exists a positive real sequence $\{\beta_n\}$ such that

$$\Delta\beta_n \leqslant 0, \quad \Delta(p_n \beta_{n+1}) \leqslant 0 \quad \textit{and} \quad \Delta(a_n \Delta\beta_n) \geqslant 0 \tag{3}$$

for all $n \geqslant n_0 \in \mathbb{N}$. If

$$\sum_{n=n_0}^{\infty} q_n \beta_{n+1} = \infty \tag{4}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n \beta_n} \left(\sum_{s=n_0}^{n-1} q_s \beta_{s+1} \right) = \infty$$

hold then every solution of equation (E) *is either oscillatory or tends to zero monotonically as $n \to \infty$.*

P r o o f. Let $\{y_n\}$ be a nonoscillatory solution of equation (E) which is eventually of constant sign. Without loss of generality we may assume that $y_n > 0$ and $y_{\sigma(n)} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$, since the proof for the case $y_n < 0$ for $n \geqslant n_0$ is similar. Now we consider the following cases for the behavior of $\{\Delta^{m-1} y_n\}$.

Case 1. Suppose $\{\Delta^{m-1}y_n\}$ is oscillatory. Then there exists an integer $n_1 \geqslant n_0$ such that

$$\Delta^{m-1}y_{n_1} < 0 \quad \text{or} \quad \Delta^{m-1}y_{n_1} = 0.$$

First we consider $\Delta^{m-1}y_{n_1} < 0$. Now equation (E) implies

$$\begin{aligned}\Delta(a_{n_1}\Delta^{m-1}y_{n_1}) &= -p_{n_1}\Delta^{m-1}y_{n_1} - q_{n_1}f(y_{\sigma(n_1+m-1)}) \\ &< -p_{n_1}\Delta^{m-1}y_{n_1}\end{aligned}$$

since $-q_{n_1}f(y_{\sigma(n_1+m-1)}) < 0$. Hence

$$a_{n_1+1}\Delta^{m-1}y_{n_1+1} - a_{n_1}\Delta^{m-1}y_{n_1} < -p_{n_1}\Delta^{m-1}y_{n_1}$$

or

$$a_{n_1+1}\Delta^{m-1}y_{n_1+1} < (a_{n_1} - p_{n_1})\Delta^{m-1}y_{n_1} < 0.$$

Thus we get

$$\Delta^{m-1}y_{n_1+1} < 0.$$

By induction we obtain

$$\Delta^{m-1}y_n < 0 \qquad \text{for all} \quad n \geqslant n_1.$$

Next, consider $\Delta^{m-1}y_{n_1} = 0$. Then equation (E) implies $\Delta^{m-1}y_{n_1+1} < 0$ and we obtain as above $\Delta^{m-1}y_n < 0$ for all $n > n_1$. Hence in both cases, we obtain $\Delta^{m-1}y_n < 0$ for all $n > n_1$ which however contradicts the assumption that $\{\Delta^{m-1}y_n\}$ oscillates. Thus $\{\Delta^{m-1}y_n\}$ is eventually of fixed sign.

Case 2. $\Delta^{m-1}y_n > 0$ for all $n \geqslant n_1$ for some integer $n_1 \geqslant n_0 \in \mathbb{N}$. Using (H_1)–(H_3), it follows from (E) that

$$a_{n+1}\Delta^m y_n + \Delta a_n\Delta^{m-1}y_n = -p_n\Delta^{m-1}y_n - q_nf(y_{\sigma(n+m-1)}) < 0 \quad \text{for} \quad n \geqslant n_1.$$

Hence we have

$$\Delta^m y_n \leqslant 0 \quad \text{for} \quad n \geqslant n_1.$$

Now from Lemma 1, we have (here l is odd and $1 \leqslant l \leqslant m-1$)

$$\Delta y_n > 0 \quad \text{and} \quad \Delta y_{\sigma(n)} > 0 \quad \text{for} \quad n \geqslant n_1.$$

Define

$$z_n = \frac{\beta_n v_n}{f(y_{\sigma(n+m-2)})} \quad \text{where} \quad v_n = a_n\Delta^{m-1}y_n.$$

Note that $z_n > 0$. Then for $n \geqslant n_1$ we have

$$\Delta z_n = -q_n\beta_{n+1} - \frac{p_n\beta_{n+1}\Delta^{m-1}y_n}{f(y_{\sigma(n+m-1)})} + \frac{v_n\Delta\beta_n}{f(y_{\sigma(n+m-1)})} - \frac{v_n\beta_n\Delta f(y_{\sigma(n+m-2)})}{f(y_{\sigma(n+m-2)})f(y_{\sigma(n+m-1)})}. \tag{6}$$

From the hypotheses and condition (3) we obtain

$$\Delta z_n \leqslant -q_n\beta_{n+1} \quad \text{for} \quad n \geqslant n_1.$$

Summing the above inequality from n_1 to n, we have

$$\sum_{s=n_1}^{n} q_s\beta_{s+1} \leqslant z_{n_1} - z_{n+1} \leqslant z_{n_1} < \infty,$$

which contradicts (4).

Case 3. $\Delta^{m-1}y_n < 0$ for $n \geqslant n_1$ for some integer $n_1 \geqslant n_0 \in \mathbb{N}$. For $m \geqslant 4$ we have from Lemma 1 either

$$\Delta y_n > 0, \quad \Delta^2 y_n > 0 \tag{7}$$

or

$$\Delta y_n < 0, \quad \Delta^2 y_n > 0 \tag{8}$$

for $n \geqslant n_1$. Suppose (7) holds. Let $L = \lim\limits_{n\to\infty} y_{\sigma(n+m-1)}$.

Then, since $\sigma(n+m-1) \to \infty$ and y_n and Δy_n are increasing for large n, we have $L = \infty$. Since f is nondecreasing, there exists an integer $n_2 \geqslant n_1$ such that $f(y_{\sigma(n+m-1)}) \geqslant A$, $n \geqslant n_2$ for some $A > 0$. Now, from equation (1), we get

$$a_{n+1}\Delta^{m-1}y_{n+1} \leqslant (a_n - p_n)\Delta^{m-1}y_n - Aq_n, \qquad n \geqslant n_2,$$

which in view of (1) leads to

$$\Delta^{m-1}y_{n+1} \leqslant -\frac{Aq_n}{a_{n+1}}, \quad n \geqslant n_2. \tag{9}$$

Summing (9) from n_2 to $n-1$, we obtain

$$\Delta^{m-2}y_{n+1} \leqslant \Delta^{m-2}y_{n_2+1} - A\sum_{s=n_2}^{n-1}\frac{q_s}{a_{s+1}}. \tag{10}$$

By (3), the right hand side of (10) tends to $-\infty$ as $n \to \infty$. Thus, there exists an integer $n_3 \geqslant n_2$ such that

$$\Delta^{m-2} y_{n+1} < 0 \quad \text{for} \quad n \geqslant n_3,$$

which implies that $y_n \to -\infty$ as $n \to \infty$ (see, Lemma (1.7.10 [1])). This contradicts the assumption that $\{y_n\}$ is eventually positive. Thus condition (8) is fulfilled. For $m = 2$, we automatically have $\Delta y_n < 0$ for $n \geqslant n_1$. Hence we have for $m \geqslant 2$, $\Delta y_n < 0$ for all $n \geqslant n_1$. Since $y_n > 0$ for $n \geqslant n_0 \in \mathbb{N}$, it follows that

$$\lim_{n\to\infty} y_n = b, \qquad b \geqslant 0.$$

We claim that $b = 0$. To prove it, assume $b > 0$. Define

$$u_n = \beta_n a_n \Delta^{m-1} y_n \quad \text{for} \quad n \geqslant n_1.$$

We then obtain for $n \geqslant n_1$

$$\Delta u_n = -\beta_{n+1} p_n \Delta^{m-1} y_n - \beta_{n+1} q_n f(y_{\sigma(n+m-1)}) + a_n \Delta^{m-1} y_n \Delta\beta_n.$$

Hence for all $n \geqslant n_1$ we have

$$\begin{aligned}
u_n &= u_{n_1} - \sum_{s=n_1}^{n-1} \beta_{s+1} p_s \Delta^{m-1} y_s - \sum_{s=n_1}^{n-1} \beta_{s+1} q_s f(y_{\sigma(s+m-1)}) \\
&\quad + \sum_{s=n_1}^{n-1} a_s \Delta\beta_s \Delta^{m-1} y_s \\
&= u_{n_1} - \sum_{s=n_1}^{n-1} \beta_{s+1} p_s \Delta^{m-1} y_s - f(y_{\sigma(n+m-1)}) \sum_{s=n_1}^{n-1} q_s \beta_{s+1} \\
&\quad + \sum_{s=n_1}^{n-1} \Delta f(y_{\sigma(s+m-1)}) \sum_{t=n_1}^{s} q_t \beta_{t+1} + \sum_{s=n_1}^{n-1} a_s \Delta\beta_s \Delta^{m-1} y_s.
\end{aligned} \tag{11}$$

Since $\{y_n\}$ is positive decreasing and f is nondecreasing, we have $\Delta f(y_{\sigma(n+m-1)}) \leqslant 0$ for all $n \geqslant n_1$. Then we have from (11)

$$u_n \leqslant u_{n_1} - \sum_{s=n_1}^{n-1} \beta_{s+1} p_s \Delta^{m-1} y_s - f(y_{\sigma(n+m-1)}) \sum_{s=n_1}^{n-1} q_s \beta_{s+1} + \sum_{s=n_1}^{n-1} a_s \Delta\beta_s \Delta^{m-1} y_s.$$

Now using condition (3), summation by parts and the fact that $\Delta^{m-2} y_n < 0$ we obtain

$$u_n \leqslant u_{n_1} + p_{n_1} \beta_{n_1+1} \Delta^{m-2} y_{n_1} - f(b) \sum_{s=n_1}^{n-1} q_s \beta_{s+1} - a_{n_1} \Delta^{m-2} y_{n_1}.$$

So, for every $n \geqslant n_1$, we have

$$u_n \leqslant M - f(b) \sum_{s=n_1}^{n-1} q_s \beta_{s+1}$$

where $M = u_{n_1} + p_{n_1}\beta_{n_1+1}\Delta^{m-2}y_{n_1} - a_{n_1}\Delta\beta_{n_1}\Delta^{m-2}y_{n_1}$. By assumption (4), there exists an integer $n_2 \geqslant n_1$ such that

$$u_n \leqslant -\frac{f(b)}{2} \sum_{s=n_2}^{n-1} q_s \beta_{s+1} \quad \text{for} \quad n \geqslant n_2.$$

Thus

$$\sum_{s=n_2}^{n-1} \Delta^{m-1} y_s \leqslant -\frac{f(b)}{2} \sum_{s=n_2}^{n-1} \frac{1}{a_s \beta_s} \sum_{t=n_2}^{s-1} q_t \beta_{t+1}.$$

This, in view of condition (5), leads to

$$\Delta^{m-2} y_n \to -\infty \quad \text{as} \quad n \to \infty,$$

which in turn implies $y_n \to -\infty$ as $n \to \infty$ (see Lemma 1.7.10 [1]). This contradicts the assumption that $y_n > 0$ for all $n \geqslant n_0 \in \mathbb{N}$. This completes the proof of the theorem. □

Remark 2. When $m = 2$ and $\sigma(n) = n$, Theorem 1 reduces to Theorem 1 given in [10].

Example 1. The difference equation

$$(\mathrm{E}_2) \quad \Delta(n(n+1)(n+2)(n+3)\Delta^3 y_n) + n(n+1)\Delta^3 y_n + \frac{6(n+3)^4}{n+2} y_{n+3}^5 = 0, \quad n \geqslant 1$$

satisfies all conditions of Theorem 1 when $\beta_n = \frac{1}{n(n+1)(n+2)}$ and hence every solution of equation (E_2) is either oscillatory or tends to zero monotonically as $n \to \infty$. In fact, equation (E_2) admits a solution $\{y_n\} = \{1/n\} \to 0$ monotonically as $n \to \infty$.

Theorem 2. *Let conditions* (1), (3)–(5) *hold. Then every bounded (all) solution(s) of equation* (E) *is (are) oscillatory when* $m \geqslant 4$ ($m \geqslant 2$).

P r o o f. The proof is similar to that of Theorem 1 except Case 3. In this case if we take into account the boundedness of solutions of equation (E) for $m \geqslant 4$, then condition (8) holds for $m \geqslant 2$. The rest of the proof is similar to that of Case 3 and hence the details are omitted.

Next, we study the oscillatory behavior of equation (E) using the following lemma, which is a discrete analogue of Lemma 3 of Grace and Lalli [4]. □

Lemma 3. *Suppose that condition* (1) *holds. If*

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n}\left[\prod_{s=n_0}^{n-1}\left(1-\frac{p_s}{a_s}\right)\right]=\infty \tag{12}$$

and if $\{y_n\}$ *is a nonoscillatory solution of equation* (E), *then there is an integer* $n_1 \geqslant n_0 \in \mathbb{N}$ *such that*

$$y_n \Delta^{m-1} y_n > 0 \quad \textit{for all} \quad n \geqslant n_1.$$

P r o o f. Let $\{y_n\}$ be a nonoscillatory solution of equation (E) which must then eventually be of constant sign. Without loss of generality, we can assume that $y_n > 0$, and $y_{\sigma(n)} > 0$ for all $n \geqslant n_0 \in \mathbb{N}$, since the proof for the case $y_n < 0$, $n \geqslant n_0$ is similar. As in the proof of Theorem 1 (Case 3), $\{\Delta^{m-1} y_n\}$ cannot oscillate. Therefore $\{\Delta^{m-1} y_n\}$ is eventually of fixed sign. Let $\Delta^{m-1} y_n < 0$ for all $n \geqslant n_1 \geqslant n_0$; then if $w_n = -a_n \Delta^{m-1} y_n$, $n \geqslant n_1$, we get from equation (E)

$$\Delta w_n + \frac{p_n}{a_n} w_n \geqslant 0.$$

From the above equation, we obtain

$$w_n \geqslant w_{n_1} \prod_{s=n_1}^{n-1}\left(1-\frac{p_s}{a_s}\right)$$

or

$$\Delta^{m-1} y_n \leqslant -\frac{w_n}{a_n} \prod_{s=n_1}^{n-1}\left(1-\frac{p_s}{a_s}\right), \quad n \geqslant n_1.$$

Summing the above inequality from n_1 to $n-1$ and using the condition (12) yields

$$\Delta^{m-2} y_n \to -\infty \quad \text{as} \quad n \to \infty,$$

which implies that $y_n \to -\infty$ as $n \to \infty$, a contradiction. This complete the proof. □

Theorem 3. *Let conditions* (1), (3), (4) *and* (12) *be satisfied. Then every solution of equation* (E) *is oscillatory.*

P r o o f. The proof is similar to that of Theorem 1 and Lemma 3 and hence the details are omitted. □

Example 2. The difference equation

$$\Delta(n\Delta^3 y_n) + \frac{1}{n}\Delta^3 y_n + \frac{27(3n^2+n-2)2^{2n+5}}{n}\, y_{n+3}^3 = 0, \quad n \geqslant 2 \tag{E$_3$}$$

satisfies all conditions of Theorem 3 if we choose $\beta_n = 1$. Hence all solutions of (E$_3$) are oscillatory. In fact, one such solution of (E$_3$) is $\{y_n\} = \{(-1)^n/2^n\}$.

Remark 3. In Theorems 1-3 we do not require that the function f be superlinear or sublinear. Furthermore, the statement of Theorems 1–3 holds when the argument $\sigma(n)$ is of ordinary, retarded, advanced or mixed type.

In the following theorem we study the oscillatory behavior of equation (E) subject to the condition

$$\frac{f(u)}{u} \geqslant M > 0 \quad \text{for} \quad u \neq 0. \tag{13}$$

Theorem 4. *Let $\sigma(n) \leqslant n$. Suppose that conditions* (1), (12) *and* (13) *hold. Assume that there exists a positive nondecreasing real sequence $\{\beta_n\}$ such that condition* (4) *and*

$$\sum_{n=n_0}^{\infty} \frac{a_n \Delta\beta_n}{(\sigma(n+m-1)/2^{m-1})^{(m-1)}} < \infty \tag{14}$$

hold. Then every solution of equation (E) *is oscillatory.*

P r o o f. Let $\{y_n\}$ be a nonoscillatory solution of equation (E) which must then eventually be of constant sign. In view of Lemma 3, there is no loss in generality in assuming that there is an integer $n_1 \geqslant n_0 \in \mathbb{N}$ such that

$$y_n > 0, \quad y_{\sigma(n)} > 0 \quad \text{and} \quad \Delta^{m-1} y_n > 0 \quad \text{for all} \quad n \geqslant n_1.$$

Using the function z_n defined in the proof of Theorem 1 (Case 2), we obtain (6). This, in view of the hypothesis of the theorem, implies

$$\Delta z_n \leqslant -q_n\beta_{n+1} + \frac{v_n \Delta\beta_n}{f(y_{\sigma(n+m-1)})}, \quad n \geqslant n_1$$

or

$$\Delta z_n \leqslant -q_n\beta_{n+1} + \frac{a_n \Delta\beta_n \Delta^{m-1} y_n}{f(y_{\sigma(n+m-1)})}. \tag{15}$$

By Lemma 2, there exists an integer $n_2 \geqslant n_1$ such that

$$y_{\sigma(n+m-1)} \geqslant \frac{1}{(m-1)!}(\sigma(n+m-1)/2^{m-1})^{(m-1)}\Delta^{m-1}y_n,$$
$$n \geqslant 2^{m-1}n_2 = n_3 \text{ (say)}.$$

Using the above inequality in (15), one gets

$$\Delta z_n \leqslant -q_n\beta_{n+1} + \frac{(m-1)!a_n\Delta\beta_n}{(\sigma(n+m-1)/2^{m-1})^{(m-1)}}\frac{y_{\sigma(n+m-1)}}{f(y_{\sigma(n+m-1)})}.$$

Now using condition (13) in the last inequality, we obtain

$$\Delta z_n \leqslant -q_n\beta_{n+1} + \frac{(m-1)!}{M}\frac{a_n\Delta\beta_n}{(\sigma(n+m-1)/2^{m-1})^{(m-1)}}, \quad n \geqslant n_3,$$
$$\sum_{s=n_3}^{n} q_s\beta_{s+1} < z_{n_3} - z_{n+1} + \frac{(m-1)!}{M}\sum_{s=n_3}^{n}\frac{a_s\Delta\beta_s}{(\sigma(n+m-1)/2^{m-1})^{(m-1)}}.$$

By using condition (14) and in view of $z_n > 0$, $n \geqslant n_3$, we have

$$\sum_{n=n_3}^{\infty} q_n\beta_{n+1} < \infty,$$

which contradicts condition (4). This completes the proof of the theorem.

Example 3. The difference equation

$$(\mathrm{E}_4) \quad \Delta(n\Delta^3 y_n) + \frac{1}{n}\Delta^3 y_n + 27\frac{3n^2+n-2}{n}\frac{2^{2n+5}}{1+2^{2n+6}}(y_{n+3} + y_{n+3}^3) = 0, \quad n \geqslant 1$$

satisfies all conditions of Theorem 4 if we choose $\beta_n = 1$. Hence all solutions of (E_4) are oscillatory. In fact, one such solution of (E_4) is $\{y_n\} = \{(-1)^n/2^n\}$.

Finally, we study the oscillatory behavior of equation (E) subject to the condition

$$(15) \qquad -f(-uv) \geqslant f(uv) \geqslant Kf(u)f(v)$$

on $\mathbb{R} - \{0\}$, where K is a positive constant. □

Theorem 5. *Let $\sigma(n) \leqslant n$. Suppose that conditions* (1), (13), (15) *hold. Assume*

$$(16) \qquad \int_{-\alpha}^{0}\frac{\mathrm{d}u}{f(u)} > -\infty \quad \text{and} \quad \int_{0}^{\alpha}\frac{\mathrm{d}u}{f(u)} < \infty, \quad \text{for all} \quad \alpha > 0,$$

and

$$(17) \qquad \sum_{n=n_0}^{\infty} A_n q_n = \infty,$$

where

$$A_n = f\Big[\frac{1}{a_n}\Big(\frac{\sigma(n+m-1)}{2^{m-1}}\Big)^{(m-1)}\Big].$$

Then every solution of equation (E) *is oscillatory.*

Proof. Let $\{y_n\}$ be a nonoscillatory solution of equation (E). As in Theorem 4, there exists an integer $n_1 \geqslant n_0 \in \mathbb{N}$ such that

$$y_n > 0, y_{\sigma(n)} > 0 \quad \Delta^{m-1} y_n > 0 \quad \text{for all} \quad n \geqslant n_1,$$

which in turn by Lemma 1 that

$$\Delta y_n > 0 \quad \text{and} \quad \Delta y_{\sigma(n)} > 0 \quad \text{for} \quad n \geqslant n_1.$$

Since $p_n \geqslant 0$, equation (E) yields

$$(18) \qquad \Delta(a_n \Delta^{m-1} y_n) + q_n f(y_{\sigma(n+m-1)}) \leqslant 0, \quad n \geqslant n_1.$$

By Lemma 2, there exists an integer $n_2 \geqslant n_1$ such that

$$(19) \qquad y_{\sigma(n+m-1)} \geqslant \frac{1}{(m-1)!}\Big(\frac{\sigma(n+m-1)}{2^{m-1}}\Big)^{(m-1)} \frac{1}{a_n}[a_n \Delta^{m-1} y_n], \quad n \geqslant 2^{m-1} n_2 = n_3.$$

Using condition (15) and the nondecreasing nature of f, we have from (19)

$$(20) \qquad f(y_{\sigma(n+m-1)}) > K^2 f\Big(\frac{1}{(m-1)!}\Big) A_n f(a_n \Delta^{m-1} y_n), \quad n \geqslant n_3.$$

Then, using (2) in (18) we have

$$(21) \qquad \frac{\Delta(a_n \Delta^{m-1} y_n)}{f(a_n \Delta^{m-1} y_n)} + \mu\, A_n q_n \leqslant 0, \qquad n \geqslant n_3,$$

where $\mu = k^2 f(\frac{1}{(m-1)!})$.

Observe that, for $a_n \Delta^{m-1} y_n \geqslant u \geqslant a_{n+1} \Delta^{m-1} y_{n+1}$, we have

$$\frac{1}{f(u)} \geqslant \frac{1}{f(a_n \Delta^{m-1} y_n)}$$

and consequently

$$-\int_{a_{n+1}\Delta^{m-1}y_{n+1}}^{a_n\Delta^{m-1}y_n}\frac{\mathrm{d}u}{f(u)}\leqslant\frac{\Delta(a_n\Delta^{m-1}y_n)}{f(a_n\Delta^{m-1}y_n)}.$$

Using the last inequality in (21) and summing the resulting inequality from n_3 to n, we obtain

$$\sum_{s=n_3}^{n}A_nq_n<\frac{1}{\mu}\int_{a_{n+1}\Delta^{m-1}y_{n+1}}^{a_{n_3}\Delta^{m-1}y_{n_3}}\frac{\mathrm{d}u}{f(u)},$$

which is by (16) an immediate contradiction to (17). Hence the proof of the theorem is complete.

Remark 4. Theorem 5 generalizes Theorem 4 given in [11].

Example 4. The difference equation

$$(\mathrm{E}_5)\qquad \Delta(n\Delta^3y_n)+\frac{1}{n}\Delta^3y_n+q_ny_{n+3}^{1/3}=0,\qquad n\geqslant n_2$$

where $q_n=\frac{27(4^{3/4})(3n^2+2n-1)}{2n}$, satisfies all conditions of Theorem 5. Hence every solution of equation (E_5) is oscillatory. One such solution is $\{y_n\}=\{(-1)^n2^n\}$. $\square$

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