Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 50 (2000), No. 1, 103–112

Persistent URL: http://dml.cz/dmlcz/127554

Czechoslovak Mathematical Journal, 50 (125) (2000), 103–112

A REMARK ON THE CENTERED n-DIMENSIONAL HARDY-LITTLEWOOD MAXIMAL FUNCTION

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(Received May 13, 1997)

Abstract. We study the behaviour of the n-dimensional centered Hardy-Littlewood maximal operator associated to the family of cubes with sides parallel to the axes, improving the previously known lower bounds for the best constants c_n that appear in the weak type $(1,1)$ inequalities.

Keywords: Hardy-Littlewood maximal function

MSC 2000: 42A99

Introduction. Let $\mathcal{C}$ be the family of cubes with sides parallel to the axes, and let $\mathcal{C}_0$ be the collection of sets in $\mathcal{C}$ that are centered at zero. Given a locally integrable real valued function f on $\mathbb{R}^n$, the centered Hardy-Littlewood maximal operator associated to $\mathcal{C}_0$ is defined as

$$Mf(x) := \sup_{C\in\mathcal{C}_0} \frac{1}{|C|}\int_{x+C} |f|,$$

where $|C|$ denotes the Lebesgue measure of C. An analogous definition can be given for more general bounded, convex, symmetric subsets of $\mathbb{R}^n$, but here we shall only be concerned with cubes.

While several authors have sought to find better upper bounds for the best constants in the weak type $(1,1)$ inequalities satisfied by M, there is very little work done with respect to lower bounds. Denote by c_n the best constant appearing in the weak type $(1,1)$ inequality satisfied by the operator M. It is shown in [MS], Theorem 6, that $c_n \geqslant \bigl(\frac{1+2^{1/n}}{2}\bigr)^n$, and conjectured in [M], Note I.7, p. 22, that in fact $\bigl(\frac{1+2^{1/n}}{2}\bigr)^n$ is the best constant for every n, i.e., if $f\in L^1(\mathbb{R}^n)$ and $\alpha>0$, then $\alpha|\{Mf>\alpha\}| \leqslant \bigl(\frac{1+2^{1/n}}{2}\bigr)^n \|f\|_1$.

The one-dimensional version of this conjecture ($c_1 = \frac{3}{2}$) originally appeared in [BH] (see Problem 7.74 c, proposed by A. Carbery), and a counterexample can be found in [A]. Here we show that the general conjecture fails for every $n \geqslant 2$, and also asymptotically, i.e., $\liminf c_n > \lim\left(\frac{1+2^{1/n}}{2}\right)^n = \sqrt{2}$. The method we use is the same as in [MS] and [A], that is, we study the behaviour of the operator M when it acts over finite sums of Dirac deltas, a technique which leads to arguments of a completely elementary nature. It is easy to justify such procedure in order to obtain lower bounds for the best constants. Interestingly enough, Dirac deltas also suffice to give upper bounds. (This fact is the basis of the discretization method, developed by M. de Guzmán—see [Gu], Theorem 4.1.1—and refined by M. Trinidad Menárguez and F. Soria—cf. Theorem 1 of [MS].) In this note, however, we will study lower bounds only.

I am indebted to Prof. F. Soria for several useful conversations regarding this subject, and to Prof. A. Bravo Zarza for her help during the preparation of this paper.

Results. Recall from the introduction that $\mathcal{C}$ denotes the family of cubes in $\mathbb{R}^n$ with sides parallel to the axes, and $\mathcal{C}_0$ the collection of cubes centered at zero. Given a finite sum $\mu = \sum_1^k \delta_{x_i}$ of Dirac deltas, where the x_i's need not be all different, let $\sharp(x + C)$ stand for the number of point masses from μ contained in $x + C$, i.e., $\sharp(x+C) := \operatorname{card}\{x_{i_j} : x_{i_j} \in x+C\}$. The discrete centered Hardy-Littlewood maximal function is then defined as

$$M\mu(x) := \sup_{C\in\mathcal{C}_0} \frac{\sharp(x + C)}{|C|}.$$

The n-dimensional maximal operator M is said to satisfy a *weak type* $(1, 1)$ inequality if there exists a constant c such that for every $f \in L^1(\mathbb{R}^n)$ and every $\alpha > 0$, we have $\alpha|\{Mf > \alpha\}| \leqslant c\|f\|_1$.

From the viewpoint of weak type inequalities it makes no difference, first, whether $Mf \geqslant \alpha$ is used instead of the strict inequality $Mf > \alpha$, and second, whether open cubes or closed cubes are used in the definition of the Hardy-Littlewood maximal operator. For us it will be more convenient, when dealing with the discrete case, to utilize the nonstrict inequality and closed cubes.

The result we present next is well known. We include the proof here for completeness.

Lemma 1.1. *Let c be a fixed constant. If there exists a finite sum $\mu = \sum_1^k \delta_{x_i}$ of Dirac deltas in $\mathbb{R}^n$ such that for some $\alpha > 0$, $\alpha|\{M\mu \geqslant \alpha\}| > ck$, then there exists a function $f \in L^1(\mathbb{R}^n)$ and a $\beta > 0$ with $\beta|\{Mf \geqslant \beta\}| > c\|f\|_1$.*

Proof. Suppose $\mu = \sum_1^k \delta_{x_i}$ and $\alpha > 0$ are such that the inequality $\alpha|\{M\mu \geqslant \alpha\}| > ck$ holds. We prove that for every $\varepsilon > 0$ there exists an $f \in L^1(\mathbb{R}^n)$ with $\|f\|_1 = k$ and $\{M\mu \geqslant \alpha\} \subset \{Mf > (1+\varepsilon)^{-1}\alpha\}$. Let $\ell(C)$ denote the sidelength of the cube C. For each $x \in \{M\mu \geqslant \alpha\}$, select a cube C_x centered at x with

$$\frac{\sharp(C_x)}{|C_x|} \geqslant \alpha.$$

We make the additional assumption that $\ell(C_x) \geqslant \alpha^{-1/n}$; this is always possible to do since each C_x must contain at least one point mass. Now α is strictly positive, so $d := \sup\{\ell(C_x)\colon x \in \{M\mu \geqslant \alpha\}\} < \infty$. Given any $\varepsilon > 0$, choose $\delta(\varepsilon) > 0$ such that for every $y \in [\alpha^{-1/n}, d]$, $(y+\delta)^n/y^n < 1+\varepsilon$. Let E_i be the (closed) cube centered at x_i of sidelength δ and define $f := \delta^{-n}\sum_1^k \chi_{E_i}$. Replacing each C_x by the cube C'_x centered at x and of sidelength $\ell(C'_x) = \ell(C_x) + \delta$, we see that $\{M\mu \geqslant \alpha\} \subset \{Mf > (1+\varepsilon)^{-1}\alpha\}$. Pick $\varepsilon > 0$ so small that $\alpha|\{M\mu \geqslant \alpha\}| > (1+\varepsilon)ck$. Since $\|f\|_1 = k$, we get

$$\alpha|\{Mf \geqslant \alpha(1+\varepsilon)^{-1}\}| \geqslant \alpha|\{M\mu \geqslant \alpha\}| > (1+\varepsilon)c\|f\|_1,$$

and setting $\beta = \alpha(1+\varepsilon)^{-1}$, the conclusion follows. □

In one dimension, it is sufficient to consider configurations of point masses for which the set $\{M\mu \geqslant \alpha\}$ is an interval: If it has more than one component, shifting the masses contained in the leftmost component towards the right, by the minimum amount that will make the first disconnection disappear, reduces the number of components by one. Then repeat it till there is only one component left. This procedure never leads to any losses in measure, though it may lead to no gain either, as the following example shows: Set $\mu := \sum_1^4 k_i\delta_{x_i}$, where $x_1 = 0$, $k_1 = 6$, $x_2 = 5.5$, $k_2 = 1$, $x_3 = 7$, $k_3 = 1$, $x_4 = 10$, $k_4 = 3$. It is not difficult to check that $[0,10] \setminus \{M(\sum_1^4 k_i\delta_{x_i}) \geqslant 1\} = (4, 4.5)$. If we leave the last three positions fixed, then the minimum amount one has to displace x_1 to make the disconnection disappear is $\frac{1}{2}$ (to the right), so $\left|\left\{M\left(\sum_1^4 k_i\delta_{x_i}\right) \geqslant 1\right\}\right| = \left|\left\{M\left(6\delta_{1/2} + \sum_2^4 k_i\delta_{x_i}\right) \geqslant 1\right\}\right|$.

For $n \geqslant 2$, however, it is unclear which topological structure the set $\{M\mu \geqslant \alpha\}$ should have in order to maximize $\alpha|\{M\mu \geqslant \alpha\}|$. In [M] and [MS] the following measures μ on $\mathbb{R}^n$ are considered: Given a large cube C, the measure μ supported on C is obtained by placing a Dirac delta at each point of the integer lattice contained in C. We prove that for the plane and this particular type of measures, it is best to choose α so that the set $\{M\mu \geqslant \alpha\}$ is not simply connected. Having to consider sets with nontrivial fundamental groups makes of course harder to compute the measure of $\{M\mu \geqslant \alpha\}$, pointing out to new difficulties when trying to improve estimates by this method. These considerations should be of particular relevance when, via numerical methods, one tries to obtain experimental evidence on the behaviour of c_n. Numerical searches in the one dimensional case have been carried out in [DrGaSt]. Incidentally, the proof of the next proposition also yields that $c_2 > 1.47 > \left(\frac{1+2^{1/2}}{2}\right)^2$. However, we will see later on that a better estimate for c_2 can be obtained in a simpler way.

As in [MS], Theorem 6, we define in $\mathbb{R}^n$ the measure $\mu_T := \sum_{i=1}^{(T+1)^n} \delta_{x_i}$, where T is a large natural number and $x_i \in \mathbb{R}^n$ ranges over all points with integer coordinates between 0 and T. What matters here is that we are placing point masses on a square lattice. The fact that we are using integer coordinates is due to mere convenience, which can be achieved simply by rescaling. Given a positive constant $c \neq 1$, the measure μ_{cT} is defined as before, save that contiguous masses are now placed at the distance c.

Proposition 1.2. *For $n = 2$, $\alpha > 0$ and μ_T as above, in order to maximize $\alpha|\{M\mu_T \geqslant \alpha\}|$, α must be chosen so that $\{M\mu_T \geqslant \alpha\}$ is not simply connected.*

P r o o f. We break up the proof into five steps. The first shows that in order to maximize $\alpha|\{M\mu_T \geqslant \alpha\}|$, α must lie in the interval $(0, \frac{9}{4}]$, which in particular entails that the set $\{M\mu_T \geqslant \alpha\} \cap [0,T]^2$ is connected. The second step proves that if $\alpha \leqslant \frac{9}{4}$ and $\{M\mu_T \geqslant \alpha\}$ is simply connected, then $\alpha < 1.462$. It is shown in the third step that taking $\alpha = \frac{8}{5}$ and T sufficiently large, we obtain $\alpha|\{M\mu_T \geqslant \alpha\}| > 1.47(T+1)^2$. The fourth step tells us that if $\alpha \geqslant 1$, then $\alpha|\{M\mu_T \geqslant \alpha\}| \leqslant \alpha(T+4)^2$, and the fifth, that $\alpha|\{M\mu_T \geqslant \alpha\}|$ is not maximized when $\alpha < 1$. These steps yield the result: We only need to consider values of α in $[1, \frac{9}{4}]$, and then the previous estimates imply that if T is sufficiently large and $\{M\mu_T \geqslant \alpha\}$ is simply connected, we have

$$\alpha|\{M\mu_T \geqslant \alpha\}| \leqslant \alpha(T+4)^2 < 1.462(T+4)^2 < 1.47(T+1)^2 < \frac{8}{5}\Big|\Big\{M\mu_T \geqslant \frac{8}{5}\Big\}\Big|.$$

Step 1. *The set $\{M\mu_T \geqslant \alpha\} \cap [0,T]^2$ is connected if and only if $\alpha \leqslant \frac{9}{4}$. Furthermore, if $\alpha > \frac{9}{4}$, then $\alpha|\{M\mu_T \geqslant \alpha\}|$ is not maximized.*

P r o o f. Since the point masses have all integer coordinates, in order to evaluate $M\mu_T(v)$ we only need to consider (closed) squares centered at v, such that for one of their sides, either the projection onto the x-axis, or the projection onto the y-axis, is an integer. Clearly, $\{M\mu_T \geqslant \alpha\} \cap [0,T]^2$ is connected if and only if for

$0 \leqslant n, m \leqslant T-1$, the diagonals of the squares $[n, n+1] \times [m, m+1]$ are contained in $\{M\mu_T \geqslant \alpha\}$. So in order to check whether $\{M\mu_T \geqslant \alpha\} \cap [0,T]^2$ is connected, it suffices to study $M\mu_T$ on these diagonals. If $\alpha > \frac{9}{4}$, then no point on the horizontal lines $y = m + \frac{1}{3}$ belongs to $\{M\mu_T \geqslant \alpha\}$, for if $v = (n + \frac{1}{3}, m + \frac{1}{3})$, then $M\mu_T(v)$ is bounded above by one of the functions

$$h_1(n) := \frac{(2n+1)^2}{(2n+\frac{2}{3})^2}$$

or

$$h_2(n) := \frac{(n+1)^2}{(n+\frac{2}{3})^2}.$$

(The functions h_i correspond to what one would obtain if there were one Dirac delta at each point of the integer lattice $\mathbb{Z}^2$.) These functions are maximized at $n = 0$, so $M\mu_T(v) \leqslant \frac{9}{4}$. On the other hand, it is easy to check, using centered squares of sidelength at most $\frac{4}{3}$, that $M\mu_T \geqslant \frac{9}{4}$ on the diagonals of the squares $[n, n+1] \times [m, m+1]$, $0 \leqslant n, m \leqslant T-1$. Thus, $\{M\mu_T \geqslant \alpha\} \cap [0,T]^2$ is connected if and only if $\alpha \leqslant \frac{9}{4}$.

Notice next that we can choose α so that $\{M\mu_T \geqslant \alpha\} \cap [0,T]^2$ is connected. Otherwise we shrink the original integer lattice by the minimum amount $k < 1$ needed for $\{M\mu_{kT} \geqslant \alpha\} \cap [0,kT]^2$ to consist of only one component. Clearly, nothing is lost by doing so. But $kv \in \{M\mu_{kT} \geqslant \alpha\}$ if and only if $v \in \{M\mu_T \geqslant k^2\alpha\}$, whence $\alpha|\{M\mu_{kT} \geqslant \alpha\}| = k^2\alpha|\{M\mu_T \geqslant k^2\alpha\}|$. Thus, from the viewpoint of maximizing $\alpha|\{M\mu_T \geqslant \alpha\}|$, it makes no difference whether we shrink the lattice or lower α to $k^2\alpha$. □

From now on we always assume that $\alpha \leqslant \frac{9}{4}$.

Step 2. *If $\{M\mu_T \geqslant \alpha\}$ is simply connected, then $\alpha < 1.462$.*

Proof. Since $\alpha \leqslant \frac{9}{4}$, we know from Step 1 that the diagonals of the squares $[n, n+1] \times [m, m+1]$ are contained in $\{M\mu_T \geqslant \alpha\}$ for $0 \leqslant n, m \leqslant T-1$. So to prove $\{M\mu_T \geqslant \alpha\}$ is not simply connected if $\alpha \geqslant 1.462$, it suffices to show that for some integers $i, j \in [1, T-2]$, there is an $x \in [i, i+1] \times [j, j+1] \setminus \{M\mu_T \geqslant \alpha\}$. To see this, we study the value of $M\mu_T$ at each of the points $(n_1 + 0.415, n_2 + 0.1)$, where n_1 and n_2 are integers with $0 \leqslant n_1, n_2 \leqslant T$. It is enough to consider squares centered at $(n_1 + 0.415, n_2 + 0.1)$ such that one of their sides has integer projection (as in the previous step). Depending on which of the sides has integer projection, we obtain four functions of $n \in \mathbb{N}$: If it is the lower (horizontal) side, then

$$f_1(n) \leqslant h_1(n) := \frac{(2n+1)2n}{(2n+0.2)^2}.$$

If it is the upper side, then

$$f_2(n) \leqslant h_2(n) := \frac{(2n+2)^2}{(2n+1.8)^2}.$$

If it is the left side, then

$$f_3(n) \leqslant h_3(n) := \frac{(2n+1)^2}{(2n+0.83)^2}.$$

And if it is the right side, then

$$f_4(n) \leqslant h_2(n) := \frac{(2n+2)(2n+1)}{(2n+1.17)^2}.$$

Now h_1 attains its maximum at $n = 1$, and the other functions h_i, at $n = 0$. So $M\mu_T(n_1 + 0.415, n_2 + 0.1) = \max\{f_i(n)\} \leqslant \max\{h_1(1), h_2(0), h_3(0), h_4(0)\} < 1.462$. Taking $x = (1.415, 1.1)$, the conclusion follows. $\square$

Step 3. *If T is sufficiently large, then $\frac{8}{5}|\{M\mu_T \geqslant \frac{8}{5}\}| > 1.47(T+1)^2$.*

Proof. Set $\alpha = \frac{8}{5}$. Let a, b, and c be such that $\frac{8}{5} = (2a)^{-2} = 2(2b)^{-2} = 4(2c)^{-2}$. Set

$$A = [0,a]^2 \cup ([0,a] \times [1-a,1]) \cup ([1-a,1] \times [0,a]) \cup [1-a,1]^2,$$
$$B = ([1-b,b] \times [0,1]) \cup ([0,1] \times [1-b,b]), \quad \text{and} \quad C = [1-c,c]^2.$$

If $x \in A \cup B \cup C$, by choosing a square centered at x of sidelength $2a, 2b$, or $2c$ respectively (for definiteness, if $x \in A \cap C$, then take a square of sidelength $2a$), we see that $M\mu_T(x) \geqslant \frac{8}{5}$. Therefore

$$|\{M\mu_T \geqslant \tfrac{8}{5}\} \cap [0,1]^2| \geqslant 1 - 8(1-a-b)(1-c) > 0.92,$$

whence $\frac{8}{5}|\{M\mu_T \geqslant \frac{8}{5}\} \cap [0,1]^2| > 1.47$. By translation we also get that for every pair $0 \leqslant n_1, n_2 < T$, we have $\frac{8}{5}|\{M\mu_T \geqslant \frac{8}{5}\} \cap [n_1, n_1+1] \times [n_2, n_2+1]| > 1.47$. Now the number of unit squares with integer coordinates contained in $[0,T]^2$ is T^2, while the number of integer lattice points is $(T+1)^2$. Since $\lim_{T\to\infty} T^2/(T+1)^2 = 1$, for T large enough we have $\frac{8}{5}|\{M\mu_T \geqslant \frac{8}{5}\}| > 1.47(T+1)^2$. $\square$

Step 4. *If $\alpha \geqslant 1$, then $\alpha|\{M\mu_T \geqslant \alpha\}| \leqslant \alpha(T+4)^2$.*

Proof. By symmetry, it is enough to consider the behaviour of $M\mu_T$ at points (x,y) such that $x < 0$ and for $0 \leqslant i < T$, either $y = i$ or $y = i + \frac{1}{2}$. In every case we have that

$$M\mu_T((x,y)) \leqslant \max_{0\leqslant n\leqslant T} H(n), \qquad \text{where} \qquad H(n) := \frac{(2(|x|+n)+1)(n+1)}{(2(|x|+n))^2}.$$

If $|x| > 2$, then

$$M\mu_T((x,y)) \leqslant \max_{0\leqslant n\leqslant T} \frac{|x|+n+1}{2(|x|+n)} < \frac{1}{2} + \frac{1}{4}.$$

So for $\alpha \geqslant 1$, $\{M\mu_T \geqslant \alpha\} \subset [-2, T+2]^2$ and the conclusion follows. □

Step 5. *If $\alpha < 1$, then $\alpha|\{M\mu_T \geqslant \alpha\}|$ is not maximized.*

Proof. As we did in Step 4, we consider points (x,y) such that $x < 0$ and for $0 \leqslant i < T$, either $y = i$ or $y = i + \frac{1}{2}$, so $M\mu_T((x,y)) \leqslant \max_{0\leqslant n\leqslant T} H(n)$. We study the behaviour of $H(n)$ for $d := |x|$ in different ranges, and we set $\alpha := \sup\{M\mu_T(v)\}$, where the sup is taken over all v lying on the boundary of the square $[-d, T+d]^2$. It then follows that $\{M\mu_T \geqslant \alpha\} \subset [-d, T+d]^2$, so

$$\frac{1}{(T+1)^2}\alpha|\{M\mu_T \geqslant \alpha\}| < \frac{1}{(T+1)^2}\alpha(T+2d)^2.$$

First, we assume that $|x| > 2$ (else, the bound $\alpha|\{M\mu_T \geqslant \alpha\}| < \alpha(T+4)^2$ from Step 4 holds). For $2 < d \leqslant \frac{T}{10}$ we use the estimate $\max_{0\leqslant n\leqslant T} H(n) \leqslant \frac{3}{4}$. Then

$$\frac{1}{(T+1)^2}\alpha|\{M\mu_T \geqslant \alpha\}| < \frac{3}{4}\frac{(T+2d)^2}{(T+1)^2} \leqslant \frac{27}{25}.$$

For $d \in (\frac{T}{10}, \frac{T}{2}]$ we distinguish two cases: If d is such that the maximum is attained when $n \geqslant \frac{T}{2}$, then nothing is lost if we change μ_T to the measure ν obtained by concentrating all the Dirac deltas at the point $(\frac{T}{2}, \frac{T}{2})$. But in this case,

$$\frac{1}{(T+1)^2}\alpha|\{M\mu_T \geqslant \alpha\}| \leqslant \frac{1}{(T+1)^2}\alpha|\{M\nu \geqslant \alpha\}| = 1$$

for every $\alpha > 0$. And if the maximum is attained when $n < \frac{T}{2}$ then

$$\max_{0\leqslant n<\frac{T}{2}} H(n) \leqslant \max_{0\leqslant n\leqslant\frac{T}{2}} \frac{d+n+1}{2(d+n)} \max_{0\leqslant n\leqslant\frac{T}{2}} \frac{n+1}{d+n} \leqslant \frac{1}{2}\Big(1+\frac{1}{d}\Big)\frac{T+2}{2d+T},$$

so

$$\begin{aligned}\frac{1}{(T+1)^2}\alpha|\{M\mu_T \geqslant \alpha\}| &\leqslant \frac{1}{(T+1)^2}\frac{1}{2}\Big(1+\frac{1}{d}\Big)\frac{T+2}{2d+T}(T+2d)^2 \\ &\leqslant \frac{1}{(T+1)^2}\frac{1}{2}\Big(1+\frac{10}{T}\Big)(T+2)(T+T) < 1.1\end{aligned}$$

for $T \geqslant 100$.

Finally, suppose $d > \frac{T}{2}$. Then on the boundary of $[-d, T+d]^2$,

$$M\mu_T(v) \leqslant \max_{0 \leqslant n \leqslant T} \frac{(n+1)T}{(2(d+n))^2}.$$

Again if the maximum is attained when $n \geqslant \frac{T}{2}$, then $(T+1)^{-2}\alpha|\{M\mu_T \geqslant \alpha\}| \leqslant 1$ for every $\alpha > 0$. If the maximum is attained when $0 \leqslant n < \frac{T}{4}$, then

$$M\mu_T(v) \leqslant \frac{(\frac{T}{4}+1)T}{(2d)^2},$$

so

$$\frac{1}{(T+1)^2}\alpha|\{M\mu_T \geqslant \alpha\}| \leqslant \Big(\frac{T+2d}{T+1}\Big)^2 \frac{(T+4)T}{16d^2} \leqslant \frac{(T+4)T}{(T+1)^2} < 1.1$$

for $T \geqslant 100$. And if the maximum is attained when $\frac{T}{4} \leqslant n < \frac{T}{2}$, then

$$M\mu_T(v) \leqslant \frac{(\frac{T}{2}+1)T}{(2(\frac{T}{4}+d))^2},$$

so

$$\begin{aligned}\frac{1}{(T+1)^2}\alpha|\{M\mu_T \geqslant \alpha\}| &\leqslant \Big(\frac{T+2d}{T+1}\Big)^2 \frac{(\frac{T}{2}+1)T}{(\frac{T}{2}+2d)^2} \\ &= \frac{2(T+2)T}{(T+1)^2}\Big(\frac{T+2d}{T+4d}\Big)^2 \leqslant 2\Big(\frac{T+T}{T+2T}\Big)^2 = \frac{8}{9}.\end{aligned}$$

This completes the proof of Step 5 and of Proposition 1.2. □

□

Remark 1.3. Probably the simplest way of seeing that $c_2 > \big(\frac{1+2^{1/2}}{2}\big)^2$ is to modify μ_T so that the horizontal spacing of the Dirac deltas is $\frac{3}{2}$ instead of 1. More precisely, fix $\alpha = 1$ and define ν_T by placing one Dirac delta at each of the points in $((\frac{3}{2})\mathbb{Z} \times \mathbb{Z}) \cap ([0, (\frac{3}{2})T] \times [0, T])$. Then it is clear that for each pair n_1, n_2 with $0 \leqslant n_1$, $n_2 < T$ we have $[(\frac{3}{2})n_1, (\frac{3}{2})(n_1+1)] \times [n_2, n_2+1] \subset \{M\nu_T \geqslant 1\}$, from which $c_2 \geqslant \frac{3}{2}$ quickly follows. The same construction shows that in $\mathbb{R}^n$, $c_n \geqslant \frac{3}{2}$: Use spacings of length $\frac{3}{2}$ on the first axis, and of length 1 on all the others. The proof of the next proposition utilizes a more refined version of this basic idea, achieving slightly better bounds.

Proposition 1.4. *Let c_n be the best constant appearing in the weak type $(1,1)$ inequality for the Hardy-Littlewood maximal function. Then for every $n \geqslant 2$ we have*

$$c_n \geqslant \frac{3}{2}\Big(\frac{1+2^{1/n}}{2}\Big)^{n-1} - \frac{1}{2}\left(\Big(\frac{3}{2}(1+2^{1/n}) - 2\Big(\frac{3}{2}\Big)^{1/n}\Big)^{n-1} - \Big(\frac{3-2^{1/n}}{2}\Big)^{n-1}\right).$$

Therefore $\liminf_n c_n \geqslant \frac{47\sqrt{2}}{36}$.

Proof. Set $\alpha = 1$. Given any positive integer n, for $i = 1, \ldots, n$ write $a_1^n = \frac{3}{2}$ and $a_i^n = \frac{1+2^{1/n}}{2}$ if $i \geqslant 2$. We define in $\mathbb{R}^n$ the measure $\nu_T := \sum_{i=1}^{(T+1)^n} \delta_{x_i}$, where T is a large natural number and $x_i = (a_1^n m_{i1}, \ldots, a_n^n m_{in})$ with $m_{ij} \in \mathbb{N}$, $0 \leqslant m_{ij} \leqslant T$. As in the proof of Proposition 1.2, we have that if we divide the number of rectangles of the form $\prod_{i=1}^{n} [a_i^n m_i, a_i^n(m_i + 1)]$ which are contained in $\prod_{i=1}^{n} [a_i^n, a_i^n(T-1)]$ by the number of masses in ν_T, the quotient tends to 1 as $T \to \infty$ (since $\lim_{T\to\infty} \frac{(T-2)^n}{(T+1)^n} = 1$). This fact, together with translation invariance, entails that

$$|\{M\nu_T \geqslant 1\} \cap ([\tfrac{3}{2}, 3] \times [(1 + 2^{1/n})/2, 1 + 2^{1/n}]^{n-1})| \leqslant c_n.$$

By considering centered squares of sidelength either 1 or $2^{1/n}$, one easily checks that $([\frac{3}{2}, 2] \cup [\frac{5}{2}, 3]) \times [(1 + 2^{1/n})/2, 1 + 2^{1/n}]^{n-1} \subset \{M\nu_T \geqslant 1\}$. Let $A := (2, \frac{5}{2}) \times [(1 + 2^{1/n})/2, 1 + 2^{1/n}]^{n-1}$. Suppose $x = (x_1, \ldots, x_n) \in A$. If for every $i = 2, \ldots, n$, $x_i \in [2^{1/n}, (3 + 2^{1/n})/2]$, then the cube C_x centered at x of sidelength 2 contains 2^n Dirac deltas, so $M\nu_T(x) \geqslant 1$. The same conclusion follows if for some i between 2 and n, $x_i \in \big[\frac{1+2^{1/n}}{2}, \big(\frac{3}{2}\big)^{1/n}\big] \cup \big[3\frac{1+2^{1/n}}{2} - \big(\frac{3}{2}\big)^{1/n}, 1 + 2^{1/n}\big]$: Then the cube C_x, centered at x and of sidelength $2\big(\frac{3}{2}\big)^{1/n}$, contains $2^n + 2^{n-1}$ Dirac deltas. Denote by b_n the measure of all the points in A which satisfy neither the first nor the second condition considered above. Then

$$b_n = \frac{1}{2}\left(\left(\frac{3}{2}(1 + 2^{1/n}) - 2\Big(\frac{3}{2}\Big)^{1/n}\right)^{n-1} - \left(\frac{3 + 2^{1/n}}{2} - 2^{1/n}\right)^{n-1}\right).$$

Therefore

$$c_n \geqslant |\{M\nu_T \geqslant 1\} \cap ([\tfrac{3}{2}, 3] \times [(1 + 2^{1/n})/2, 1 + 2^{1/n}]^{n-1})| \geqslant \frac{3}{2}\Big(\frac{1 + 2^{1/n}}{2}\Big)^{n-1} - b_n.$$

Letting $n \to \infty$ we obtain

$$\liminf_n c_n \geqslant \frac{3}{2}\sqrt{2} - \frac{7}{36}\sqrt{2} = \frac{47\sqrt{2}}{36}.$$

□

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