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Ioannis K. Argyros

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LOCAL CONVERGENCE THEOREMS OF NEWTON'S METHOD FOR NONLINEAR EQUATIONS USING OUTER OR GENERALIZED INVERSES

Ioannis K. Argyros, Lawton

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Abstract. We provide local convergence theorems for Newton's method in Banach space using outer or generalized inverses. In contrast to earlier results we use hypotheses on the second instead of the first Fréchet-derivative. This way our convergence balls differ from earlier ones. In fact we show that with a simple numerical example that our convergence ball contains earlier ones. This way we have a wider choice of initial guesses than before. Our results can be used to solve undetermined systems, nonlinear least squares problems and ill-posed nonlinear operator equations.

Keywords: Newton's method, Banach space, Fréchet-derivative, local convergence, outer inverse, generalized inverse

MSC 2000: 65J15, 47H17, 49D15

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the equation

$$F'(x_0)^{\#}F(x) = 0, \tag{1}$$

where F is a twice Fréchet-differentiable operator defined on an open convex subset of a Banach space E_1 with values in a Banach space E_2, and $x_0 \in D$. Here, $F'(x) \in L(E_1, E_2)$ the space of bounded linear operators from E_1 into E_2, denotes the Fréchet-derivative of F evaluated at $x \in D$. $F''(x) \in L\big(E_1, L(E_1, E_2)\big)$ $(x \in D)$ denotes the second Fréchet-derivative of F evaluated at $x \in D$ [5], [12]. Operator

$F'(x)^{\#}$ $(x \in D)$ denotes an outer inverse of $F'(x)$ $(x \in D)$. Many authors have provided local and semilocal results for the convergence of Newton's method to x^* using hypotheses on the first Fréchet-derivative [2], [3], [6]–[14]. Recently, we provided semilocal convergence theorems using hypotheses on the second Fréchet-derivative [4], [5].

Here we provide local convergence theorems for Newton's method using outer or generalized inverses given by

$$x_{n+1} = x_n - F'(x_n)^{\#}F(x_n) \quad (n \geqslant 0) \quad (x_0 \in D). \tag{2}$$

Our Newton-Kantorovich type convergence hypothesis is different from the corresponding famous condition used in the above-mentioned works (see Remark 1(b)), unless if the Lipschitz constant for the second Fréchet-derivative is zero (see Remark 1(b)). Hence, our results have theoretical and practical value. In fact we show using a simple numerical example that our convergence ball contains earlier ones. This way, we have a wider choice of initial guesses than before. Our results can be used to solve undetermined systems, nonlinear least squares problems and ill-posed nonlinear operator equations [1]–[10], [12], [14].

2. Preliminaries

In this section we restate some of the definitions and lemmas given in the elegant paper [9].

Let $A \in L(E_1, E_2)$. A linear operator $B\colon E_2 \to E_1$ is called an inner inverse of A if $ABA = A$. A linear operator B is an outer inverse of A if $BAB = B$. If B is both an inner and an outer inverse of A, then B is called a generalized inverse of A. There exists a unique generalized inverse $B = A^{\dagger}_{P,Q}$ satisfying $ABA = A$, $BAB = B$, $BA = I - P$, and $AB = Q$, where P is a given projector on E_1 onto $N(A)$ (the null set of A) and Q is a given projector of E_2 onto $R(A)$ (the range of A). In particular, if E_1 and E_2 are Hilbert spaces, and P, Q are orthogonal projectors, then $A^{\dagger}_{P,Q}$ is called the Moore-Penrose inverse of A.

We will need five lemmas of Banach-type and perturbation bounds for outer inverses and for generalized inverses in Banach spaces. The Lemmas 1–5 stated here correspond to Lemmas 2.2–2.6 in [9] respectively. See also [14] for a comprehensive study of inner, outer and generalized inverses.

Lemma 1. *Let $A \in L(E_1, E_2)$ and $A^{\#} \in L(E_2, E_1)$ be an outer inverse of A. Let $B \in L(E_1, E_2)$ be such that $\|A^{\#}(B-A)\| < 1$. Then $B^{\#} = (I + A^{\#}(B-A))^{-1}A^{\#}$ is a* *bounded outer inverse of B with $N(B^{\#}) = N(A^{\#})$ and $R(B^{\#}) = R(A^{\#})$. Moreover, the following perturbation bounds hold:*

$$\|B^{\#} - A^{\#}\| \leqslant \frac{\|A^{\#}(B-A)A^{\#}\|}{1 - \|A^{\#}(B-A)\|} \leqslant \frac{\|A^{\#}(B-A)\|\,\|A^{\#}\|}{1 - \|A^{\#}(B-A)\|}$$

and

$$\|B^{\#}A\| \leqslant (1 - \|A^{\#}(B-A)\|)^{-1}.$$

Lemma 2. *Let $A, B \in L(E_1, E_2)$ and $A^{\#},\ B^{\#} \in L(E_2, E_1)$ be outer inverses of A and B, respectively. Then $B^{\#}(I - AA^{\#}) = 0$ if and only if $N(A^{\#}) \subseteq N(B^{\#})$.*

Lemma 3. *Let $A \in L(E_1, E_2)$ and suppose E_1 and E_2 admit the topological decompositions $E_1 = N(A) \oplus M$, $E_2 = R(A) \oplus S$. Let $A^{\dagger}$ $(= A^{\dagger}_{M,S})$ denote the generalized inverse of A relative to these decompositions. Let $B \in L(E_1, E_2)$ satisfy*

$$\|A^{\dagger}(B-A)\| \leqslant 1$$

and

$$(I + (B-A)A^{\dagger})^{-1}B \quad \text{maps } N(A) \text{ into } R(A).$$

Then $B^{\dagger} = B^{\dagger}_{R(A^{\dagger}),N(A^{\dagger})}$ exists and is equal to

$$B^{\dagger} = A^{\dagger}(I + TA^{\dagger})^{-1} = (I + A^{\dagger}T)^{-1}A^{\dagger},$$

where $T = B - A$. Moreover, $R(B^{\dagger}) = R(A^{\dagger})$, $N(B^{\dagger}) = N(A^{\dagger})$ and $\|B^{\dagger}A\| \leqslant (1 - \|A^{\dagger}(B-A)\|)^{-1}$.

Lemma 4. *Let $A \in L(E_1, E_2)$ and $A^{\dagger}$ be the generalized inverse of Lemma 3. Let $B \in L(E_1, E_2)$ satisfy the conditions $\|A^{\dagger}(B-A)\| < 1$ and $R(B) \subseteq R(A)$. Then the conclusion of Lemma 3 holds and $R(B) = R(A)$.*

Lemma 5. *Let $A \in L(E_1, E_2)$ and $A^{\dagger}$ be a bounded generalized inverse of A. Let $B \in L(E_1, E_2)$ satisfy the condition $\|A^{\dagger}(B-A)\| < 1$. Define $B^{\#} = (I + A^{\dagger}(B - A))^{-1}A^{\dagger}$. Then $B^{\#}$ is a generalized inverse of B if and only if $\dim N(B) = \dim N(A)$ and $\operatorname{codim} R(B) = \operatorname{codim} R(A)$.*

Let $A \in L(E_1, E_2)$ be fixed. Then, we will denote the set on nonzero outer inverses of A by

$$\Delta(A) = \{B \in L(E_2, E_1) :\ BAB = B, \quad B \neq 0\}.$$

3. Convergence Analysis

In [5], we showed the following semilocal convergence theorem for Newton's method (2) using outer inverses for Fréchet-differentiable operators.

Theorem 1. *Let $F\colon D \subseteq E_1 \to E_2$ be a twice Fréchet-differentiable operator. Assume:*

(a) *There exist an open convex subset D_0 of D, $x_0 \in D_0$, a bounded outer inverse $F'(x_0)^{\#}$ of $F'(x_0)$, and constants $a, b, \eta \geqslant 0$ such that for all $x, y \in D_0$ the following conditions hold:*

$$\|F'(x_0)^{\#}(F''(x) - F''(y))\| \leqslant a\|x - y\|, \tag{3}$$

$$\|F'(x_0)^{\#}F(x_0)\| \leqslant \eta, \tag{4}$$

$$\|F'(x_0)^{\#}F''(x_0)\| \leqslant b, \tag{5}$$

and

$$3\eta a^2 \leqslant [b^2 + 2a]^{3/2} - [3ba + b^3]. \tag{6}$$

Define the real polynomial f by

$$f(t) = \eta - t + \frac{b}{2}t^2 + \frac{a}{6}t^3, \tag{7}$$

and denote by t^*, t^{**} $(t^* \leqslant t^{**})$ the nonnegative zeros of p.

(b) *Assume more*

$$\overline{U}(x_0, t^*) = \{x \in E_1 : \|x - x_0\| \leqslant t^*\} \subseteq D_0. \tag{8}$$

Then,

(i) *Newton's method $\{x_n\}$ $(n \geqslant 0)$ generated by (2) with*

$$F'(x_n)^{\#} = \big[I + F'(x_0)^{\#}\big(F'(x_n) - F'(x_0)\big)\big]^{-1}F'(x_0)^{\#} \quad (n \geqslant 0)$$

is well defined, remains in $U(x_0, t^*)$ and converges to a solution $x^* \in \overline{U}(x_0, t^*)$ of equation $F'(x_0)^{\#}F(x) = 0$;

(ii) *The following error bounds hold for all $n \geqslant 0$*

$$\|x_{n+1} - x_n\| \leqslant t_{n+1} - t_n \tag{9}$$

and

$$\|x_n - x^*\| \leqslant t^* - t_n, \tag{10}$$

where $\{t_n\}$ $(n \geqslant 0)$ *is a monotonically increasing sequence generated by*

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}; \tag{11}$$

(iii) *Equation* $F'(x_0)^{\#}$ *has a unique solution in* $\tilde{U} \cap \{R(F'(x_0)^{\#}) + x_0\}$, *where*

$$\tilde{U} = \begin{cases} \overline{U}(x_0, t^*) \cap D_0 & \text{if } t^* = t^{**}, \\ U(x_0, t^{**} \cap D_0 & \text{if } t^* < t^{**}, \end{cases} \tag{12}$$

and

$$R(F'(x_0)^{\#}) + x_0 := \{x + x_0\colon\ x \in R(F'(x_0)^{\#})\}.$$

We provide a local convergence theorem for Newton's method $\{x_n\}$ $(n \geqslant 0)$ generated by (2) for twice Fréchet-differentiable operators.

Theorem 2. *Let* $F\colon\ D \subseteq E_1 \to E_2$ *be a twice Fréchet-differentiable operator. Assume:*

(a) $F''(x)$ *satisfies a Lipschitz condition*

$$\|F''(x) - F''(y)\| \leqslant a_0\|x - y\| \quad \text{for all } x, y \in D; \tag{13}$$

(b) *There exists* $x^* \in D$ *such that* $F(x^*) = 0$ *and*

$$\|F''(x^*)\| \leqslant b_0; \tag{14}$$

(c) *Let*

$$r_0 = \frac{2}{pb_0 + \sqrt{(pb_0)^2 + 2a_0p}} \quad \text{for some } p > 0 \tag{15}$$

be such that $U(x^*, r_0) \subseteq D$;

(d) *There exists an* $F'(x^*)^{\#} \in \Delta(F'(x^*))$ *such that*

$$\|F'(x^*)^{\#}\| \leqslant p, \tag{16}$$

and for any $x \in U(x^*, r_1)$, *where for given* $\varepsilon_0 > 1$

$$r_1 = \frac{2(1 - \varepsilon_0^{-1})}{pb_0 + \sqrt{(pb_0)^2 + 2(1 - \varepsilon_0^{-1})pa_0}}, \tag{17}$$

the set $\Delta(F'(x))$ *contains an element of minimal mean.*

Then, there exists $U(x^*,r)\subseteq D$ *with* $r\in(0,r_1)$ *such that for any* $x_0\in U(x^*,r)$, *Newton's method* $\{x_n\}$ $(n\geqslant 0)$ *generated by* (2) *for*

$$F'(x_0)^{\#}\in\operatorname{argmin}\bigl\{\|B\|\colon\ B\in\Delta\bigl(F'(x_0)\bigr)\bigr\}$$

with $F'(x_n)^{\#}=\bigl[I+F'(x_0)^{\#}\bigl(F'(x_n)-F'(x_0)\bigr)\bigr]^{-1}F'(x_0)^{\#}$, *converges to* $y\in U(x_0,r_0)\cap\bigl\{R\bigl(F'(x_0)^{\#}\bigr)+x_0\bigr\}$ *such that* $F'(x_0)^{\#}F(y)=0$. *Here, we denote*

$$R\bigl(F'(x_0)^{\#}\bigr)+x_0=\bigl\{x+x_0\colon\ x\in R\bigl(F'(x_0)^{\#}\bigr)\bigr\}.$$

P r o o f. (i) We first define parameter ε by

$$\varepsilon\in(0,\min\{\varepsilon_1,\varepsilon_2\}],$$

where,

(18)
$$\varepsilon_1=\frac{[(p\varepsilon_0(b_0+a_0r_1)^2+2p\varepsilon_0a_0]^{3/2}-[3p\varepsilon_0(b_0+a_0r_1)p\varepsilon_0a_0+(p\varepsilon_0)^3(b_0+a_0r_1)^3]}{3(p\varepsilon_0)^3a_0^2}$$

and

$$\varepsilon_2=\frac{[1-\frac{p\varepsilon_0(b_0+a_0r_1)(r_0-r_1)}{2}-\frac{p\varepsilon_0a_0}{6}(r_0-r_1)^2](r_0-r_1)}{p\varepsilon_0}. \tag{19}$$

We will use Theorem 1. Operator F is continuous at x^*. Hence, there exists $U(x^*,r)\subseteq D$, $r\in(0,r_1)$, such that

$$\|F(x)\|\leqslant\varepsilon\quad\text{for all } x\in U(x^*,r_1). \tag{20}$$

Using the identity,

$$F'(x)-F'(x^*)=\int_0^1\{F''[x^*+t(x-x^*)]-F''(x^*)\}\,\mathrm{d}t\,(x-x^*)+F''(x^*)(x-x^*), \tag{21}$$

conditions (13), (14), (15), and (16), we get in turn

$$\begin{aligned}\bigl\|F'(x^*)^{\#}\bigl(F'(x)-F'(x^*)\bigr)\bigr\|&\leqslant\|F'(x^*)^{\#}\|\Bigl\{\int_0^1\|F''[x^*+t(x-x^*)]\\&\qquad-F''(x^*)\|\,\|x-x^*\|\,\mathrm{d}t+\|F''(x^*)\|\,\|x-x^*\|\Bigr\}\\&\leqslant p\Bigl[\frac12a_0r_0^2+b_0r_0\Bigr]<1,\end{aligned}$$

by the choice of r_0.

It follows from Lemma 1 that

$$F'(x)^{\#} = \big[I + F'(x^*)^{\#}\big(F'(x) - F'(x^*)\big)\big]^{-1} F'(x^*)^{\#}, \tag{22}$$

is an outer inverse of $F'(x)$, and

$$\|F'(x)^{\#}\| \leqslant \frac{\|F'(x^*)^{\#}\|}{1 - p\big[\frac{1}{2}a_0 r_1^2 + b_0 r_1\big]} \leqslant p\varepsilon_0, \tag{23}$$

by the choice of r_1 and ε_0. That is, for any $x_0 \in U(x^*, r)$, the outer inverse

$$F'(x_0)^{\#} \in \operatorname{argmin}\big\{\|B\|\colon\ B \in \Delta\big(F'(x_0)\big)\big\} \quad \text{and} \quad \|F'(x_0)^{\#}\| \leqslant p\varepsilon_0.$$

Set,

$$b = p\varepsilon_0[b_0 + a_0 r_1] \quad \text{and} \quad a = p\varepsilon_0 a_0. \tag{24}$$

We can then obtain for all $x, y \in D$

$$\begin{gathered}\big\|F'(x_0)^{\#}\big(F''(x) - F''(x)\big)\big\| \leqslant p\varepsilon_0\|F''(x) - F''(y)\| \leqslant p\varepsilon_0 a_0\|x - y\| = a\|x - y\|,\\ \|F'(x_0)^{\#}F''(x_0)\| \leqslant p\varepsilon_0\|F''(x_0)\| \leqslant p\varepsilon_0[b_0 + a_0 r_1] = b \quad \text{(by (13))},\end{gathered}$$

and

$$3\|F'(x_0)^{\#}F(x_0)\|a^2 \leqslant 3p\varepsilon_0\varepsilon(p\varepsilon_0 a_0)^2 \leqslant (b^2 + 2a)^{3/2} - (3ba + b^3),$$

by the choice of ε and ε_1. Hence, there exists a minimum positive zero $t^* < r_1$ of polynomial f given by (7). It also follows from (15), (17) and the choice of ε_2 that $f(r_0 - r_1) \leqslant 0$. That is,

$$r_1 + t^* \leqslant r_0. \tag{25}$$

Hence, for any $x \in U(x_0, t^*)$ we have

$$\|x^* - x\| \leqslant \|x_0 - x^*\| + \|x_0 - x\| \leqslant r_1 + t^* \leqslant r_0 \quad \text{(by (25))}. \tag{26}$$

It follows from (26) that $U(x_0, t^*) \subseteq U(x^*, r_0) \subseteq D$. The hypotheses of Theorem 1 hold at x_0. Consequently Newton's method $\{x_n\}$ $(n \geqslant 0)$ stays in $U(x_0, t^*)$ for all $n \geqslant 0$ and converges to a solution y of equation $F'(x_0)^{\#}F(x) = 0$.

That completes the proof of Theorem 1. $\square$

In the next theorem we examine the order of convergence of Newton method $\{x_n\}$ $(n \geqslant 0)$.

Theorem 3. *Under the hypotheses of Theorem* 2,

$$\|y - x_{n+1}\| \leqslant \frac{\frac{1}{6}a\|x_n - y\| + \frac{b}{2}}{1 - b\|x_n - y\| - \frac{a}{2}\|x_n - y\|^2}\|y - x_n\|^2 \quad \text{for all } n \geqslant 0, \tag{27}$$

and if $y \in U(x_0, r^*)$, *where*

$$r_2 = \frac{12}{9b + \sqrt{81b^2 + 76a}}, \tag{28}$$

then, sequence $\{x_n\}$ $(n \geqslant 0)$ *converges to* y *quadratically.*

P r o o f. We first note that $r_2 < r_0$. By Lemma 1 and (22) we get $R\big(F'(x_0)^{\#}\big) = R\big(F'(x_n)^{\#}\big)$ $(n \geqslant 0)$. We have

$$x_{n+1} - x_n = F'(x_n)^{\#}F(x_n) \in R\big(F'(x_n)^{\#}\big) \quad (n \geqslant 0),$$

from which it follows

$$x_{n+1} \in R\big(F'(x_n)^{\#}\big) + x_n = R\big(F'(x_{n-1})^{\#}\big) + x_n = R\big(F'(x_0)^{\#}\big) + x_0,$$

and $y \in R\big(F'(x_n)^{\#}\big) + x_{n+1}$ $(n \geqslant 0)$. That is we conclude that

$$y \in R\big(F'(x_0)^{\#}\big) + x_0 = R\big(F'(x_n)^{\#}\big) + x_0,$$

and

$$\begin{aligned} F'(x_n)^{\#}F'(x_n)(y - x_{n+1}) &= F'(x_n)^{\#}F'(x_n)(y - x_0) - F'(x_n)^{\#}F'(x_n)(x_{n+1} - x_0) \\ &= y - x_{n+1}. \end{aligned}$$

We also have by Lemma 2 $F'(x_n)^{\#} = F'(x_n)^{\#}F'(x_0)F'(x_0)^{\#}$. By $F'(x_0)^{\#}F(y) = 0$ and $N\big(F'(x_0)^{\#}\big) = N\big(F'(x_n)^{\#}\big)$, we get $F'(x_n)^{\#}F(y) = 0$. Using the estimate

$$\begin{aligned} &\|y - x_{n+1}\| = \|F'(x_n)^{\#}F'(x_n)(y - x_{n+1})\| \\ &\quad = \big\|F'(x_n)^{\#}F'(x_n)\big[y - x_n + F'(x_n)^{\#}(F(x_n) - F(y))\big]\big\| \\ &\quad \leqslant \|F'(x_n)^{\#}F'(x_0)\| \left\| F'(x_0)^{\#}\left\{\int_0^1 [F''[x_n + t(y - x_n)] - F''(x^*)](1 - t)\,\mathrm{d}t\right\}\right. \\ &\qquad \left. \times (y - x_n)^2 + \frac{1}{2}F''(x^*)(y - x_n)^2 \right\| \\ &\quad \leqslant \frac{\frac{1}{6}a\|x_n - y\| + \frac{b}{2}}{1 - b\|x_n - y\| - \frac{a}{2}\|x_n - y\|^2}\|y - x_n\|^2 \quad (n \geqslant 0), \end{aligned}$$

which shows (27) for all $n \geqslant 0$. By the choice of r_2 and (27) there exists $\alpha \in [0,1)$ such that $\|y - x_{n+1}\| \leqslant \alpha\|y - x_n\|$ $(n \geqslant 0)$, which together with (27) show that $x_n \to y$ as $n \to \infty$ quadratically.

That completes the proof of Theorem 3. $\square$

We provide a result corresponding to Theorem 2 but involving generalized instead of outer inverses.

Theorem 4. *Let F satisfy the hypotheses of Theorems 2 and 3 except* (d) *which is replaced by*

(d)′ *the generalized inverse $F'(x^*)$ exists, $\|F'(x^*)^\dagger\| \leqslant p$,*

$$\dim N(F'(x)) = \dim N(F'(x^*)) \tag{29}$$

and

$$\operatorname{codim} R(F'(x)) = \operatorname{codim} R(F'(x^*)) \tag{30}$$

for all $x \in U(x^*, r_1)$.

Then, the conclusions of Theorems 2 and 3 hold with

$$F'(x_0)^\# \in \{B\colon\ B \in \Delta(F'(x_0)), \|B\| \leqslant \|F'(x_0)^\dagger\|\}. \tag{31}$$

P r o o f. In Theorem 2 we showed that the outer inverse $F'(x)^\# \in \operatorname{argmin}\big\{\|B\|\colon B \in \Delta\big(F'(x)\big)\big\}$ for all $x \in U(x^*, r)$, $r \in (0, r_1)$ and $\|F'(x)^\#\| \leqslant p\varepsilon_0$. We must show that under (d)′ the outer inverse

$$F'(x)^\# \in \{B\colon\ B \in \Delta(F'(x)), \|B\| \leqslant \|F'(x)^\dagger\|\}$$

satisfies $\|F'(x)^\#\| \leqslant p\varepsilon_0$. As in (21), we get

$$\big\|F'(x^*)^\dagger\big(F'(x) - F'(x^*)\big)\big\| \leqslant p\Big[\frac{1}{2}a_0 r_0^2 + b_0 r_0\Big] < 1.$$

Moreover, by Lemma 5

$$F'(x)^\dagger = \big[I + F'(x^*)^\dagger\big(F'(x) - F'(x^*)\big)\big]^{-1} F'(x^*)^\dagger \tag{32}$$

is the generalized inverse of $F'(x)$. Furthermore, by Lemma 1 as in (23) $\|F'(x)^\dagger\| \leqslant p\varepsilon_0$. That is the outer inverse

$$F'(x_0)^\# \in \big\{B\colon\ B \in \Delta\big(F'(x_0)\big), \|B\| \leqslant \|F'(x_0)^\dagger\|\big\}$$

satisfies $\|F'(x_0)^\#\| \leqslant p\varepsilon_0$, provided that $x_0 \in U(x^*, r)$.

The rest follows exactly as in Theorems 2 and 3.

That completes the proof of Theorem 4. $\square$

Remark 1. (a) We note that Theorem 1 was proved in [5] with the weaker condition

$$\left\|F'(x_0)^{\#}\left(F''(x)-F''(x_0)\right)\right\| \leqslant a_1\|x-x_0\|$$

replacing (3).

(b) Our conditions differ from the corresponding ones in [10] (see, for example, Theorem 3.1) unless if $a=0$, in which case our condition (6) becomes the Newton-Kantorovich hypothesis (3.3) in [10, p. 450]:

$$K\eta \leqslant \frac{1}{2}, \tag{33}$$

where K is such that

$$\left\|F'(x_0)^{\#}\left(F'(x)-F'(y)\right)\right\| \leqslant k\|x-y\| \tag{34}$$

for all $x, y \in D$. Similarly (if $a=0$), our r_0 equals the radius of convergence in Theorem 3.2 [10, p. 450].

(c) In Theorem 3.2 [10] the condition

$$\|F'(x)-F'(y)\| \leqslant c_0\|x-y\| \quad \text{for all } x, y \in D \tag{35}$$

was used instead of (34). The ball used there is $U(x^*, r^*)$, (corresponding to $U(x^*, r_0)$) where

$$r^* = \frac{1}{c_0 p}. \tag{36}$$

Finally, for convergence $x_0 \in U(x^*, r_1^*)$, where

$$r_1^* = \frac{1}{3} r^*. \tag{37}$$

The results obtained here can be used to solve undetermined systems, nonlinear least squares problems and ill-posed nonlinear operator equations [1], [2], [5]–[10], [12], [14]. As another possible area of applications we consider operator F satisfying an autonomous differential equation of the form

$$F'(x) = P(F(x)) \quad (x \in D), \tag{38}$$

where $P\colon E_2 \to E_1$ is a known Fréchet-differentiable operator [6], [13]. Using (38) we get $F'(x^*) = P(F(x^*)) = P(0)$, and $F''(x^*) = F'(x^*)Q'(F(x^*)) = P(0)P'(0)$. That is, without knowing x^* we can use the results obtained here.

Below we consider such a case. For simplicity we have taken $F'(x)^{\#} = F'(x)^{-1}$ $(x \in D)$.

Example. Let $E_1 = E_2 = \mathbb{R}$, $D = U(0,1)$, and define functions F and P on D by

$$F(x) = \mathrm{e}^x - 1 \quad \text{and} \quad P(x) = x + 1. \tag{39}$$

Note that with the above choices of F, and P condition (38) holds. Using Theorems 1–3, Remark 1 and Theorems 5 and 6, we obtain, for $\varepsilon_0 = 2$: $c_0 = a_0 = e$, $b_0 = p = 1$, $a = c = 5.4365637$, $b = 3.8565696$, $r^* = 0.367898$, $r_0 = 0.5654448$, $r_1^* = 0.1226265$, $r_1 = 0.3414969$, and $r_2 = 0.1573525$. That is, in all cases our convergence balls contain the corresponding ones in [10]. Hence, our Theorems provide a wider choice of initial guesses x_0 than Theorem 3.2 in [10]. This observation is important in numerical computations [1], [2], [6]–[14].

Remark 2. Methods/routines of how to construct the appropriate actions of the required outer generalized inverses of the derivative can be found at a great variety in the elegant paper [1].

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Author's address: Cameron University, Department of Mathematics, Lawton, OK 73505, U.S.A.