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UNIFORM CONVERGENCE OF THE GENERALIZED BIEBERBACH POLYNOMIALS IN REGIONS WITH ZERO ANGLES

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Abstract. Let C be the extended complex plane; $G \subset C$ a finite Jordan with $0 \in G$; $w = \varphi(z)$ the conformal mapping of G onto the disk $B(0; \varrho_0) := \{w\colon |w| < \varrho_0\}$ normalized by $\varphi(0) = 0$ and $\varphi'(0) = 1$. Let us set $\varphi_p(z) := \int_0^z [\varphi'(\zeta)]^{2/p}\,\mathrm{d}\zeta$, and let $\pi_{n,p}(z)$ be the generalized Bieberbach polynomial of degree n for the pair $(G, 0)$, which minimizes the integral $\iint\limits_G |\varphi_p'(z) - P_n'(z)|^p\,\mathrm{d}\sigma_z$ in the class of all polynomials of degree not exceeding $\leqslant n$ with $P_n(0) = 0$, $P_n'(0) = 1$. In this paper we study the uniform convergence of the generalized Bieberbach polynomials $\pi_{n,p}(z)$ to $\varphi_p(z)$ on $\overline{G}$ with interior and exterior zero angles and determine its dependence on the properties of boundary arcs and the degree of their tangency.

Keywords: conformal mapping, Quasiconformal curve, Bieberbach polynomials, complex approximation

MSC 2000: 30C30, 30E10, 30C70

1. Introduction and main result

Let C be the extended complex plane; G a finite Jordan domain with $0 \in G$; $L := \partial G$, $\Omega := \operatorname{ext}\overline{G}$; $w = \varphi(z)$ the conformal mapping of G onto the disk $B(0; \varrho_0) := \{w : |w| < \varrho_0\}$ normalized by $\varphi(0) = 0$, $\varphi'(0) = 1$; where $\varrho_0 = \varrho_0(0, G)$ is called the conformal radius of G with respect to 0.

It is well known [22, p. 435] that the unique function minimizing the integral

$$\|f\|_{L_p^1(G)} := \|f'\|_{L_p(G)} := \left(\iint\limits_G |f'(z)|^p\,\mathrm{d}\sigma_z\right)^{1/p}, \quad p > 0 \tag{1.1}$$

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in the class of all analytic functions in G normalized by $f(0)=0,\ f'(0)=1$ is the function

$$\varphi_p(z) := \int_0^z \left[\varphi'(\zeta)\right]^{2/p} \mathrm{d}\zeta, \quad z \in G.$$

Let $p>0$. Using a method similar to the one given in [13, p. 137], it is seen that there exists a polynomial $\pi_{n,p}(z)$ furnishing a minimum to the integral $\|\varphi_p - P_n\|_{L^1_p(G)}$ in the class of polynomials $P_n(z)$ of degree not exceeding n normalized by $P_n(0)=0, P_n'(0)=1$, and if $p>1$ these polynomials $\pi_{n,p}(z)$ are determined uniquely [13, p. 142]. We call such a polynomial $\pi_{n,p}(z)$ the n-th generalized Bieberbach polynomial for the pair $(G,0)$ as in [16]. In the case of $p=2$, let us emphasize that $\pi_{n,2}(z)$ coincides with the Bieberbach polynomial for the pair $(G,0)$, see, for example, [14].

If G is a Carathéodory region, then $\|\varphi_p - \pi_{n,p}\|_{L^1_p(G)} \to 0$ for $n\to\infty$ [28, p. 63], and so the sequence $\{\pi_{n,p}(z)\}$ converges uniformly to $\varphi_p(z)$ on compact subsets of G. Our purpose is to extend the uniform convergence of the sequence $\{\pi_{n,p}(z)\}$ to $\varphi_p(z)$ on $\overline{G}$, and to find a constant $\gamma=\gamma(G,p)>0$ satisfying the inequality

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} := \max_{z\in\overline{G}} \{|\varphi_p(z) - \pi_{n,p}(z)|\} \tag{1.2}$$

depending on the properties of G.

In the case of $p=2$, the estimate (1.2) has been studied in [17], [20], [25], [29] when L satisfies certain smoothness conditions, and in [2], [5], [8], [9], [10], [14], [15], [18], [24] for L having some zero or non zero angles. In the case of $p>2$ the existence of a $\gamma>0$ satisfying (1.2) for some regions with quasiconformal boundary has been investigated in [16]. In the case of $p \geqslant p(L) > 1$, when L is quasiconformal and additionally satisfies certain conditions, constants $\gamma>0$ satisfying (1.2) and explicitely depending on geometric properties of the boundary L have been studied in [4] and [6]. It is well known, however, that even though the qusiconformal curves have many properties, they cannot contain zero angles.

In this paper, we propose to study the estimate (1.2) in domains with certain exterior and interior zero angles.

We begin with some definitions. Throughout this paper, $c, c_1, c_2, \ldots$ denote positive and $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ sufficiently small positive constants which depend on G in general.

Definition 1. [19, p. 97] We say that a Jordan curve L is K-quasiconformal if there exists a K-quasiconformal mapping f of a domain $D \supset L$ such that $f(L)$ is a circle.

Definition 2. A Jordan arc ℓ is called K-quasiconformal when ℓ is a part of some closed K-quasiconformal curve.

Definition 3. [3] We say that $G \subset PQ(K,\alpha,\beta)$, $K \geqslant 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$, if $L := \partial G$ can be expressed as the union of a finite number of K-quasiconformal arcs, $K = \max\limits_{1\leqslant j\leqslant m} \{K_j\}$, connecting at points $z_0, z_1, \ldots, z_m$, such that L is a locally K-quasiconformal at z_0 and in the local co-ordinate system (x,y) with origin at z_j, $1 \leqslant j \leqslant m$, the following conditions are satisfied:

a) for $j = \overline{1,p}$

$$\left\{z = x + \mathrm{i}y\colon\ c_1 x^{1+\alpha} \leqslant y \leqslant c_2 x^{1+\alpha}, \quad 0 \leqslant x \leqslant \varepsilon_1\right\} \subset C\overline{G},$$
$$\{z = x + \mathrm{i}y\colon\ |y| \geqslant \varepsilon_2 x, \quad 0 \leqslant x \leqslant \varepsilon_1\} \subset \overline{G},$$

b) for $j = \overline{p+1,m}$

$$\left\{z = x + \mathrm{i}y\colon\ c_3 x^{1+\beta} \leqslant y \leqslant c_4 x^{1+\beta}, \quad 0 \leqslant x \leqslant \varepsilon_3\right\} \subset \overline{G},$$
$$\{z = x + \mathrm{i}y\colon\ |y| \geqslant \varepsilon_4 x, \quad 0 \leqslant x \leqslant \varepsilon_3\} \subset C\overline{G}$$

for some constants $-\infty < c_1 < c_2 < \infty$, $-\infty < c_3 < c_4 < \infty$, $\varepsilon_i > 0$, $i = \overline{1,4}$.

It is clear from Definition 3 that each domain $G \subset PQ(K,\alpha,\beta)$ may have p exterior and $m-p$ interior zero angles. If a domain G does not have exterior zero angles ($p = 0$) (interior zero angles ($p = m$)), then we write $G \subset PQ(K,0,\beta)$ ($G \subset PQ(K,\alpha,0)$).

If a domain G does not have such angles ($\alpha = \beta = 0$), then G is bounded by a K-quasiconformal curve.

We introduce the following notation:

$$p_1 := p_1(K) := \frac{\sqrt{(K^2+1)^2 + 32K^4} - K^2 - 1}{2K^2};$$

$\beta_0 \in \left[0, \sqrt{2}-1\right)$ and $\tilde{\beta}_0 \in \left[\dfrac{p_1}{2} - 1, \sqrt{2}-1\right)$ are arbitrary numbers;

$$\beta_1 := \beta_1(p;K) := \frac{K^2}{pK^2 + 3K^2 + 1};$$

$$\beta_2 := \beta_2(p;K) := \frac{\sqrt{\left[p(2K^2+1) - 2\right]^2 + 16(p+4)(K^2+1)} + p\left(2K^2+1\right) - 2}{8(K^2+1)} - 1;$$

$$K^* := \max\{K\colon\ \beta_1 < \beta_2\}.$$

Theorem 1. *Let $p \geqslant 2(1+\beta_0)$, and assume that $G \subset PQ(K,\alpha,\beta)$ for some $K \geqslant 1$, $\alpha < 2/p$ and $\beta < \min\{\beta_0; 2/(p+2)\}$, if $\beta_0 > 0$, and $\beta = 0$, if $\beta_0 = 0$. Then, for any $n \geqslant 3$,*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leqslant c_1 \begin{cases} \sqrt{\ln\ln n}\,(\ln n)^{(2\alpha-1)/(2\alpha)}, & p = 2, \\ (\ln n)^{(\alpha p-2)/(2\alpha p)}, & p > 2. \end{cases}$$

Theorem 2. *Let $p \geqslant 2(1+\beta_0)$, for $\beta_0 \neq 0$, and assume that $G \subset PQ(K,0,\beta)$ for some $K \geqslant 1$, $\beta < \min\{\beta_0;\beta_1\}$. Then, for any $n \geqslant 2$,*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leqslant c_2 n^{-\gamma},$$

for each γ with $0 < \gamma < \frac{1}{pK^2}$.

Theorem 3. *Let $2 < p < p_1$, and assume that $G \subset PQ(K,0,\beta)$ for some $K \geqslant 1$, $\frac{p}{2} - 1 < \beta < \min\left\{\beta_1; \frac{1+2K^2(p-2)}{4K^2}\right\}$. Then, for any $n \geqslant 2$,*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leqslant c_3 n^{-\gamma},$$

for each γ with $0 < \gamma < \frac{1-2K^2(2\beta+2-p)}{pK^2}$.

Theorem 4. *Let $p \geqslant 2(1+\tilde{\beta}_0)$, and assume that $G \subset PQ(K,0,\beta)$ for some $K \geqslant 1$, $\beta_1 < \beta < \min\left\{\tilde{\beta}_0; \frac{2}{p+2}\right\}$. Then, for any $n \geqslant 2$,*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leqslant c_4 n^{-\gamma},$$

for each γ with $0 < \gamma < \frac{2-(p+2)\beta}{p(1+\beta)(K^2+1)}$.

Theorem 5. *Let $2 - \frac{1}{2K^2} < p < 2$, and assume that $G \subset PQ(K,0,\beta)$ for some $K \geqslant 1$, $\beta < \min\left\{\beta_1; \frac{2(p-1)}{p+2}; \frac{1+2K^2(p-2)}{4K^2}\right\}$. Then, for any $n \geqslant 2$,*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leqslant c_5 n^{-\gamma},$$

for each γ with $0 < \gamma < \frac{1-2K^2(2\beta+2-p)}{pK^2}$.

Theorem 6. *Let $\frac{3}{2} < p < 2$, and assume that $G \subset PQ(K,0,\beta)$ for some $1 \leqslant K < K^*$, $\beta_1 < \beta < \beta_2$. Then, for any $n \geqslant 2$,*

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \leqslant c_3 n^{-\gamma},$$

for each γ with $0 < \gamma < \frac{2-(p+2)\beta}{p(1+\beta)(K^2+1)} - \frac{2}{p}(2\beta+2-p)$.

Remarks.

a) In the case of $p = 2$, Theorem1 is the same one as in [8] or [9];
b) Theorems 1–6 extend the results in [4, 6] and [16] to domains bounded by a piecewise quasiconformal curve with zero angles.
c) The statements of Theorems 2, 3, 5, 6 are also correct in the case of $p = 2$. However, much better results were obtained in [5].

2. Some auxiliary facts

Let $G \subset C$ be finite domain bounded by a Jordan curve L and let $w = \Phi(z)$ ($w = \tilde{\varphi}(z)$) be the conformal mapping of $\Omega(G)$ onto $\tilde{\Omega} := \{w\colon |w| > 1\}$ ($\{w\colon |w| < 1\}$) normalized by $\Phi(\infty) = \infty$, $\lim\limits_{z\to\infty} \Phi(z)/z > 0$ ($\tilde{\varphi}(0) = 0$, $\tilde{\varphi}'(0) > 0$).

For $t > 0$, let $L_t := \{z\colon |\Phi(z)| = t,\ t > 1,\ |\tilde{\varphi}(t)| = t, t < 1\}$, $G_t = \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$.

Let L be a K-quasiconformal curve. Then there exists a quasiconformal reflection α across L such that $\alpha(G) = \Omega$, $\alpha(\Omega) = G$, α fixes the points of L and α satisfies the condition

$$|\alpha(z) - z'| \approx |z - z'|, \quad z' \in L, \tag{2.1}$$

in some neighbourhood of L [7, p. 76] (see [14, Lemma 1]).

(Here and throught this paper the symbols " $a \approx b$ " and " $a \prec b$ " stand for $c_1 a \leqslant b \leqslant c_2 a$ and $a \leqslant c_1 b$ for some c_1, c_2 respectively.)

Lemma 2.1. [1] *Let L be a K-quasiconformal curve; $z_1 \in L$, $z_2, z_3 \in G \cap \{z\colon |z - z_1| \leqslant c_1 d(z_1, L_{R_0})\}$, $w_j = \tilde{\varphi}(z_j)$ ($z_2, z_3 \in G \cap \{z\colon |z - z_1| \leqslant c_2 d(z_1, L_{r_0})\}$, $w_j = \Phi(z_j)$), $j = 1, 2, 3$. Then,*

1) *the statements $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \approx |z_1 - z_3|$ and $|w_1 - w_2| \approx |w_1 - w_3|$;*
2) *if $|z_1 - z_2| \prec |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2}$$

and, consequently, for any $z_3 \in L_{R_0}$ ($z_3 \in L_{r_0}$)

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}$$

where $1 < R_0 \leqslant 2$ and $r_0 = R_0^{-1}$ are fixed constants.

Lemma 2.2. [3, 11] *Let L be a K-quasiconformal curve. Then for every $z \in L$ and $z_0 \in G$ there exist an arc $\ell(z, z_0)$ in G joining z to z_0 and having the following properties:*

i) *$d(\zeta, L) \approx |\zeta - z|$ for every $\zeta \in \ell(z, z_0)$,*
ii) *for every $\zeta_1, \zeta_2 \in \ell(z, z_0)$, if $\tilde{\ell}(\zeta_1, \zeta_2)$is the subarc of $\ell(z, z_0)$, then $\operatorname{mes} \tilde{\ell}(\zeta_1, \zeta_2) \prec |\zeta_1 - \zeta_2|$.*

Lemma 2.3. [9] *Let L be a K-quasiconformal curve. Then for every rectifiable arc $\ell \subset G$ $\operatorname{mes} \ell \approx \operatorname{mes} \alpha(\ell)$.*

Lemma 2.4. [5] *Let L be a K-quasiconformal curve, $G_\varepsilon = \{z\colon\ z \in G, d(z, L) < \varepsilon\}$. Then*

$$\operatorname{mes} \varphi(G_\varepsilon) \prec \varepsilon^\delta, \quad \delta = \frac{K^2 + 1}{2K^2}.$$

Note 2.1. If G is an arbitrary continuum, then we can chose $\delta = \frac{1}{2}$ [26, p. 181].

3. Some properties of the domains $G \in PQ(K, \alpha, \beta)$

Suppose that a domain $G \in PQ(K, \alpha, \beta)$ is given. For the sake of simplicity, but without loss of generality, we assume that $\alpha > 0$, $\beta > 0$; $p = 1$, $m = 2$, $z_1 = 1$, $z_2 = -1$; $(-1, 1) \subset G$ and let the local co-ordinate axes in Definition 1 be parallel to OX and OY in the co-ordinate system; $L^1 := \{z\colon\ z \in L, \operatorname{Im} z \geqslant 0\}$, $L^2 \colon = \{z\colon\ z \in L, \operatorname{Im} z \leqslant 0\}$. Then z_0 is taken as an arbitrary point on L^2 (or on L^1 subject to the chosen direction).

We recall that the domain $G \in PQ(K, \alpha, \beta)$ has interior and exterior zero angles in the neighbourhood of the points $z_1 = 1$ and $z_2 = -1$, respectively. Therefore, following the argument in [9], we can say that the function $w = \Phi(z)(w = \tilde{\varphi}(z))$ for the domain $G \in PQ(K, \alpha, \beta)$ satisfies the conditions described in Lemma 2.1 in the neighbourhood of the point $z_2 = -1$ $(z_1 = 1)$. So, we can easily get from Lemma 2.1

$$\begin{aligned}
d(z, L) &\prec (|\tilde{\varphi}(z)| - 1)^{K^{-2}};\ |z - 1| \prec |\tilde{\varphi}(z) - \tilde{\varphi}(1)|^{K^{-2}}, \\
&\qquad \forall z \in M_1 := \{z \in G\colon\ |z + 1| > \varepsilon_1\}, \\
d(z, L) &\prec (|\Phi(z)| - 1)^{K^{-2}};\ |z + 1| \prec |\Phi(z) - \Phi(-1)|^{K^{-2}}, \\
&\qquad \forall z \in M_2 := \{z \in \Omega\colon\ |z - 1| > \varepsilon_2\} > \varepsilon_2\}.
\end{aligned} \tag{3.1}$$

On the other hand, using the properties of the functions $w = \Phi(z)$and $w = \tilde{\varphi}(z)$ in the neighbourhood of the point $z_1 = 1$ and $z_2 = -1$, respectively, (see [9, 12]) we obtain

$$|z - 1| \prec [-\ln |\Phi(z) - \Phi(1)|]^{-\alpha^{-1}}, \quad |z + 1| \prec [-\ln |\tilde{\varphi}(z) - \tilde{\varphi}(-1)|]^{-\beta^{-1}}. \tag{3.2}$$

4. Estimation of the C-norm of polynomials

Lemma 4.1. *Let G be Jordan domain, such that for every $z \in L$ there exist an arc $\gamma(z,0)$ in G joining 0 to z and having the following properties:*

i) $\operatorname{mes}\gamma(\zeta_1,\zeta_2) \prec |\zeta_1-\zeta_2|$ *for every* $\zeta_1,\zeta_2 \in \gamma(z,0)$,

ii) *there exist a monotone increasing function $f(t)$ such that $d(\zeta,L) \succ f(|\zeta-z|)$ for every $\zeta \in \gamma(z,0)$. Then, for all polynomials $P_n(z)$, $\deg P_n \leqslant n$, $P_n(0)=0$,*

$$\|P_n\|_{C(\overline{G})} \prec \left\{\int_{\varepsilon n^{-2}}^{c} f^{-2/p}(t)\,\mathrm{d}t\right\} \|P_n'\|_{L_p(G)}\,, \quad p>0.$$

Proof. When $p=2$ the proof is given in [9]. In the case $p \neq 2$, we use the familiar methot given in [9]. Let $z \in L$ be an arbitrary point. For an $\varepsilon>0$ small enough, if $\gamma_1 := \{\zeta\colon\ \zeta \in \gamma(z,0),\ |\zeta-z|<\varepsilon n^{-2}\}$ and $\gamma_2=\gamma(z;0)\setminus\gamma_1$, then

$$|P_n(z)| \leqslant \left|\ \int\limits_{\gamma(z;0)} P_n'(\zeta)\,\mathrm{d}\zeta\right| \leqslant \int_{\gamma_1} |P_n'(\zeta)|\,|\mathrm{d}\zeta| + \int_{\gamma_2} |P_n'(\zeta)|\,|\mathrm{d}\zeta|\,.$$

It is well known that $\|P_n'\|_{C(\overline{G})} \leqslant c_1 n^2 \|P_n\|_{C(\overline{G})}$ and

$$|P_n'(\xi)|^p \leqslant \frac{1}{d^2(\xi,L)}\|P_n'\|_{L_p}^p$$

for all $\xi \in G$ and $p>0$, by the mean-value property [22, p. 432]. Therefore, since $\operatorname{mes}\gamma_1 \leqslant c_2\varepsilon n^{-2}$ for a $c_2>0$ which is independent of ε, we obtain

$$\begin{aligned}|P_n'(z)| &\leqslant c_1 n^2 \|P_n\|_{C(\overline{G})}\int_{\gamma_1}|\mathrm{d}\zeta| + c_3\|P_n'\|_{L_p}\int_{\gamma_2} d^{-\frac{2}{p}}(\zeta)\,\mathrm{d}\zeta\\ &\leqslant \varepsilon c_1 c_2 \|P_n\|_{C(\overline{G})} + c_4\|P_n'\|_{L_p}\int_{\varepsilon n^{-2}}^{c} f^{-\frac{2}{p}}(t)\,\mathrm{d}t.\end{aligned}$$

Using the maximum modulus principle and choosing ε such that $\varepsilon c_1 c_2<1$, the proof is complete. □

Corollary 4.1. *Let $G \in PQ(K,\alpha,\beta)$ for some $K \geqslant 1$, $\alpha \geqslant 0$, $\beta>0$. Then*

$$\|P_n\|_{C(\overline{G})} \prec A_n \|P_n'\|_{L_p(G)},$$

where

$$A_n = \begin{cases} n^{\frac{2}{p}(2\beta+2-p)}, & p<2(\beta+1);\\ \ln n, & p=2(\beta+1);\\ c, & p>2(\beta+1).\end{cases}$$

5. Estimate of L_p-norm for some Cauchy integrals

Let G be an arbitrary Jordan domain and $\gamma \in \Omega$ an arc rectifiable except for one of its endpoints of its $z_0 \in L$ which satisfies the following conditions:

i) $\operatorname{mes} \gamma(\zeta_1, \zeta_2) \prec |\zeta_1 - \zeta_2|$, for all $\zeta_1, \zeta_2 \in \gamma$;

ii) there exists a monotone increasing function $g(t)$ such that $d(\zeta, L) \succ g(|\zeta - z_0|)$ for all $\zeta \in \gamma$.

Lemma 5.1. *Let $l > 0$. .Let on the arc γ a measurable function $f(z)$ be given such that there exists a monotone increasing function $\nu(t)$, $\nu(0) = 0$, with $|f(\zeta)| \prec \nu(|\zeta - z_0|)$ for all $\zeta \in \gamma$. Then, for the function*

$$F_\gamma(z) = \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta, \quad z \notin \gamma,$$

the following estimate is satisfied:

$$\begin{aligned} \|F_\gamma'\|^2 &\prec \ell^{\frac{4(1-p)}{p}} \left[\int_0^\ell \nu(t) \, \mathrm{d}t \right]^2 \\ &+ \begin{cases} \ell^{\frac{2(2-p)}{p}} \displaystyle\int_0^{c\ell} \nu^2(t) \left[\frac{1}{t} + \frac{h_{0,1}(t)}{t^2} + h_{2,1}(t) \right] \mathrm{d}t, 1 < p < 2 \\ \displaystyle\int_0^{c\ell} \nu^2(t) \left[t^{\frac{4-3p}{p}} + \frac{1}{t^2} h_{0,\frac{p}{2}}^{\frac{2}{p}}(t) + h_{p,\frac{p}{2}}^{\frac{2}{p}}(t) \right] \mathrm{d}t \quad p \geqslant 2, \end{cases} \end{aligned}$$

where

$$h_{\lambda,\mu}(t) := \int_0^t \frac{r^{1-\lambda} \, \mathrm{d}r}{g^\mu(r)}.$$

P r o o f. For each $\zeta \in \gamma$, let us set $G = \bigcup_{i=1}^{4} G^i$, where

$$\begin{aligned} G^1 &= \{z \colon |z - z_0| \geqslant 2\ell\} \cap G; \\ G^2 &= \{z \colon 2|\zeta - z_0| \leqslant |z - z_0| < 2\ell\} \cap G; \\ G^3 &= \left\{z \colon \tfrac{1}{2}|\zeta - z_0| \leqslant |z - z_0| < 2|\zeta - z_0|\right\} \cap G; \\ G^4 &= \left\{z \colon |z - z_0| < \tfrac{1}{2}|\zeta - z_0|\right\} \cap G. \end{aligned}$$

Then,

$$\|F_\gamma'\|_{L_p(G)}^p = \sum_1^4 \iint\limits_{G^k} \left| \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right|^p \mathrm{d}\sigma_z. \tag{5.1}$$

For all $z \in G^1$ we have

$$|\zeta - z| \geqslant ||z - z_0| - |\zeta - z_0|| \geqslant ||z - z_0| - \ell| \geqslant |z - z_0| - \frac{|z - z_0|}{2} = \frac{|z - z_0|}{2},$$

and

$$\begin{aligned}(5.2)\qquad \iint\limits_{G^1} \left|\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta\right|^p \mathrm{d}\sigma_z &\prec \iint\limits_{G^1} \frac{\mathrm{d}\sigma_z}{|z - z_0|^{2p}} \left[\int_\gamma \nu(|\zeta - z_0|)|\,\mathrm{d}\zeta|\right]^p \\ &\prec \int_{2\ell}^{c} \frac{\mathrm{d}r}{r^{2p-1}} \left[\int_0^\ell \nu(t)\,\mathrm{d}t\right]^p \prec \ell^{2(1-p)} \left[\int_0^\ell \nu(t)\,\mathrm{d}t\right]^p, \quad p > 1.\end{aligned}$$

For the estimate of the integral on G^k, $k = 2, 3, 4$, we first apply Minkowski's inequality to the interior integral

$$\begin{aligned}(5.3)\qquad &\iint\limits_{G^k} \left[\int_\gamma \frac{|f(\zeta)|^2|\,\mathrm{d}\zeta|}{|\zeta - z|^2 d(z, \gamma)}\right]^{\frac{p}{2}} \left[d(z, \gamma) \int_\gamma \frac{|\,\mathrm{d}\zeta|}{|\zeta - z|^2}\right]^{\frac{p}{2}} \mathrm{d}\sigma_z \\ &\prec \iint\limits_{G^k} \left[\int_\gamma \frac{\nu^2(|\zeta - z_0|)|\,\mathrm{d}\zeta|}{|\zeta - z|^2 d(z, \gamma)}\right]^{\frac{p}{2}} \mathrm{d}\sigma_z =: J_k.\end{aligned}$$

After that, using the generalized Minkowski's inequality [27, p. 286] for $p > 2$, and Hölder's inequality for $1 < p \leqslant 2$, we obtain

$$\begin{aligned}(5.4)\qquad J_k &\prec \left(\iint\limits_{G^k} \mathrm{d}\sigma_z\right)^{1 - \frac{p}{2}} \left\{\iint\limits_{G^k} \left[\int_\gamma \frac{\nu^2(|\zeta - z_0|)|\,\mathrm{d}\zeta|}{|\zeta - z|^2 d(z, \gamma)}\right] \mathrm{d}\sigma_z\right\}^{\frac{p}{2}} \\ &= \left(\iint\limits_{G^k} \mathrm{d}\sigma_z\right)^{1 - \frac{p}{2}} \left\{\int_\gamma \nu^2(|\zeta - z_0|)|\,\mathrm{d}\zeta| \iint\limits_{G^k} \frac{\mathrm{d}\sigma_z}{|\zeta - z|^2 d(z, \gamma)}\,\mathrm{d}\sigma_z\right\}^{\frac{p}{2}}, \\ &\qquad 1 < p \leqslant 2;\end{aligned}$$

$$(5.5)\qquad J_k \prec \left\{\int_\gamma \left[\iint\limits_{G^k} \left(\frac{\nu^2(|\zeta - z_0|)}{|\zeta - z|^2 d(z, \gamma)}\right)^{\frac{p}{2}}\right]^{2/p} |\,\mathrm{d}\zeta|\right\}^{p/2}, \; p > 2.$$

i) $k = 2$. Since for all $\zeta \in \gamma$ and $z \in G^2$

$$\begin{aligned}|\zeta - z| &\geqslant \big||z - z_0| - |\zeta - z_0|\big| \geqslant \frac{|z - z_0|}{2}, \\ d(z, L) &\geqslant \big||z - z_0| - |\zeta - z_0|\big| \geqslant \frac{|z - z_0|}{2},\end{aligned}$$

we get

$$J_2 \prec \ell^{2(1-\frac{p}{2})}\left[\int_0^{c\ell} \nu^2(t)\,\mathrm{d}t \int_t^{c\ell} \frac{\mathrm{d}r}{r^2}\right]^{\frac{p}{2}} \prec \ell^{2(1-\frac{p}{2})}\left[\int_0^{\ell} \frac{\nu^2(t)}{t}\,\mathrm{d}t\right]^{\frac{p}{2}}, \quad 1 < p \leqslant 2;$$

$$J_2 \prec \left\{\int_0^{\ell} \nu^2(t)\left[\iint\limits_{G^2} \frac{\mathrm{d}\sigma_z}{|z-z_0|^{\frac{3p}{2}}}\right]^{\frac{2}{p}} \mathrm{d}t\right\}^{\frac{p}{2}} \prec \left\{\int_0^{\ell} \nu^2(t)\left[\int_t^{c\ell} r^{1-\frac{3p}{2}}\,\mathrm{d}r\right]^{\frac{2}{p}} \mathrm{d}t\right\}^{\frac{p}{2}}$$

$$\prec \left\{\int_0^{c\ell} \nu^2(t)\, t^{\frac{4-3p}{p}}\,\mathrm{d}t\right\}^{\frac{p}{2}}, \quad p > 2.$$

ii) $k = 3$. In this case $|\zeta - z| \geqslant |\zeta - z_0|$; $\quad d(z, L) \succ g(|\zeta - z_0|) \succ g(|z - z_0|)$, and we obtain

$$J_3 \prec \ell^{2(1-\frac{p}{2})}\left[\int_0^{c\ell} \frac{\nu^2(t)}{t^2} \int_{t/2}^{2t} \frac{r\,\mathrm{d}r}{g(r)}\,\mathrm{d}t\right]^{\frac{p}{2}} \prec \ell^{2(1-\frac{p}{2})}\left[\int_0^{c\ell} \frac{\nu^2(t)}{t^2} \int_0^{t} \frac{r\,\mathrm{d}r}{g(r)}\,\mathrm{d}t\right]^{\frac{p}{2}}, \ 1 < p < 2;$$

$$J_3 \prec \left\{\int_0^{c\ell} \frac{\nu^2(t)}{t^2}\left[\int_{t/2}^{2t} \frac{r\,\mathrm{d}r}{g^{p/2}(r)}\right]^{\frac{p}{2}} \mathrm{d}t\right\}^{\frac{p}{2}} \prec \left\{\int_0^{c\ell} \frac{\nu^2(t)}{t^2}\left[\int_0^{t} \frac{r\,\mathrm{d}r}{g^{p/2}(r)}\right]^{\frac{2}{p}} \mathrm{d}t\right\}^{\frac{p}{2}}, \ p \geqslant 2.$$

iii) $k = 4$. In this case $|\zeta - z| \geqslant |\zeta - z_0| - |z - z_0| \geqslant \frac{|\zeta - z_0|}{2}$ and we have analogously

$$J_4 \prec \ell^{2(1-\frac{p}{2})}\left[\int_0^{c\ell} \nu^2(t) \int_0^{t} \frac{\mathrm{d}r}{r g(r)}\,\mathrm{d}t\right]^{\frac{p}{2}}, \quad 1 < p \leqslant 2;$$

$$J_4 \prec \left\{\int_0^{c\ell} \nu^2(t)\left[\iint\limits_{G^k} \frac{\mathrm{d}\sigma_z}{|z-z_0|^p g^{p/2}(|z-z_0|)}\right]^{\frac{2}{p}} \mathrm{d}t\right\}^{\frac{p}{2}}$$

$$\prec \left\{\int_0^{c\ell} \nu^2(t)\left[\int_0^{t} \frac{r^{1-p}\,\mathrm{d}r}{g^{p/2}(r)}\right]^{\frac{2}{p}} \mathrm{d}t\right\}^{\frac{p}{2}}, \quad p > 2.$$

Using the estimates of the terms J_k, $k = 2, 3, 4$, introduced above from (5.1)–(5.4) and using the evident inequality

$$|a+b|^p \leqslant \begin{cases} |a|^p + |b|^p, & p \leqslant 1, \\ 2^{p-1}\left(|a|^p + |b|^p\right), & p > 1 \end{cases}$$

we finish the proof. $\square$

Corollary 5.1. *Let $G \in PQ(K,\alpha,0)$ for some $K \geqslant 1$, $\alpha > 0$; $\nu(t) = t^{1-\frac{1}{p}}$. Then*

$$\left\|F'_\gamma\right\|_{L_p(G)} \prec \begin{cases} |\ln \ell|^{\frac{1}{p}} \cdot \ell^{\frac{2-\alpha p}{2p}}, & \alpha < 2(1-\frac{1}{p});\ 1 < p \leqslant 2, \\ \ell^{\frac{2-\alpha p}{2p}}, & \alpha < \frac{2}{p};\ p > 2. \end{cases}$$

Corollary 5.2. *Let $G \in PQ(K,0,\beta)$ for some $K \geqslant 1$, $\beta > 0$; $\nu(t) = t^{1-\frac{1+\beta}{p}}$. Then*

$$\left\|F'_\gamma\right\|_{L_p(G)} \prec \begin{cases} |\ln \ell|^{\frac{1}{p}} \cdot \ell^{\frac{2-(2+p)\beta}{2p}}, & \beta < \frac{2(p-1)}{2+p};\ 1 < p \leqslant 2, \\ \ell^{\frac{2-(2+p)\beta}{2p}}, & \beta < \frac{2}{2+p};\ p > 2. \end{cases}$$

6. Continuity of the function φ_p on $\overline{G}$

Lemma 6.1. *Let $p > 1$ and assume that $G \in PQ(K,\alpha,\beta)$ for some $K \geqslant 1$, $\alpha \geqslant 0$; $\beta < p-1$. Then the function*

$$\varphi_p(z) := \int_0^z \left[\varphi'(\zeta)\right]^{2/p} \mathrm{d}\zeta$$

can be extented to $\overline{G}$ continuously.

Proof. It is clear that $\varphi_p(z)$ is uniformly continuous on every compact subset of G for all $p > 0$. Let us show that $\varphi_p(z)$ is continuous on G. Let z and ζ be two arbitrary points in G. Which are close to L and $w := \varphi(z)$, $\tau := \varphi(\zeta)$. Without loss of generality, we assume that $|w| \leqslant |\tau|$ and $\arg w \leqslant \arg \tau$. Let us set

$$\begin{aligned} \Gamma_1(\tau,w) &:= \left\{t\colon\ \arg t = \arg w,\ |w| \geqslant |t| \geqslant |w| - |w-\tau|\right\}, \\ \Gamma_2(\tau,w) &:= \left\{t\colon\ \arg w \leqslant \arg t \leqslant \arg \tau,\ |t| = |w| - |w-\tau|\right\}, \\ \Gamma_3(\tau,w) &:= \left\{t\colon\ \arg t = \arg \tau,\ |\tau| \geqslant |t| \geqslant |w| - |w-\tau|\right\}. \end{aligned}$$

Then $\Gamma(\tau,w) := \bigcup\limits_{j=1}^{3} \Gamma_j(\tau,w)$ is an arc joining w to τ in $B(0;\varrho_0)$ and $\tilde{\Gamma}(z,\zeta) := \psi(\Gamma(\tau,w))$ (here ψ is the inverse function of φ) is an arc in G joining z and ζ. Since $\varphi_p'(z)$ is analytic in G (we consider the branch of $\left[\varphi_p'(\zeta)\right]^{2/p}$ which takes the value 1 at the point 0), it follows that

$$|\varphi_p(z) - \varphi_p(\zeta)| \leqslant \int\limits_{\tilde{\Gamma}(z,\zeta)} |\varphi'(\xi)|^{\frac{2}{p}} |\,\mathrm{d}\xi|. \tag{6.1}$$

Let ξ be a point on $\tilde{\Gamma}(z,\zeta)$ and $d(\xi,L)$ be the distance from ξ to L. Then if $B(\xi) := B(\xi, d(\xi,L))$, using the mean-value property [22, p. 432] we get

$$|\varphi'(\xi)|^{2/p} \prec d^{-2/p}(\xi,L)\{\operatorname{mes} \varphi(B(\xi))\}^{1/p},\ p > 0. \tag{6.2}$$

Using the Lemma 2.4 (see Note 2.1) and [3, Lemma 4] we have

$$(6.3)\qquad |\varphi'(\xi)|^{2/p} \prec d^{-2/p}(\xi, L)\{\operatorname{mes} B(\xi)\}^{1/(2p)} \prec d^{-1/p}(\xi, L).$$

From (6.1), (6.3) and using the [1, Lemma 3] we obtain

$$(6.4)\quad |\varphi_p(z) - \varphi_p(\zeta)| \prec \int\limits_{\Gamma(w,\tau)} d^{-\frac{1}{p}}(\psi(t), L)\,|\psi'(t)|\,|\mathrm{d}t| \approx \int\limits_{\Gamma(w,\tau)} d^{1-\frac{1}{p}}(\psi(t), L)\frac{|\mathrm{d}t|}{\varrho_0 - |t|},$$

If $|\xi + 1| \geqslant c\varepsilon_3$, then we get

$$(6.5)\qquad |\varphi_p(z) - \varphi_p(\zeta)| \prec \int\limits_{\Gamma(w,\tau)} (\varrho_0 - |t|)^{\frac{p-1}{pK^2}}\,|\mathrm{d}t| \prec |\tau - w|^{\frac{p-1}{pK^2}} \prec |z - \zeta|^{\frac{p-1}{pK^4}}$$

by (3.1), and if $|\xi + 1| < c\varepsilon_3$, then we also have

$$(6.6)\qquad |\varphi_p(z) - \varphi_p(\zeta)| \prec \int\limits_{\Gamma(w,\tau)} |\xi + 1|^{\frac{p-1}{p}(\beta+1)}\frac{|\mathrm{d}t|}{\varrho_0 - |t|}$$

$$\prec \int\limits_{\Gamma(w,\tau)} \left(\ln\frac{1}{\varrho_0 - |t|}\right)^{-\frac{p-1}{p\beta}(\beta+1)} \frac{|\mathrm{d}t|}{\varrho_0 - |t|}$$

$$\prec \left(\ln\frac{1}{|\tau - w|}\right)^{-\lambda} \prec \left(\ln\frac{1}{|z - \zeta|}\right)^{-\lambda}, \quad \lambda := \frac{(p-1)(\beta+1)}{p\beta} - 1 > 0$$

by (3.2), (6.4) and (6.3) complete the proof. □

Corollary 6.1. *Let $p > 1$ and $G \in PQ(K, \alpha, \beta)$. Then, for all $z \in L$, $\zeta \in G$*

$$|\varphi_p(z) - \varphi_p(\zeta)| \prec |z - \zeta|^{1-(1+\beta)\frac{1}{p}}.$$

Proof. From (6.1) and (6.3) we get

$$|\varphi_p(z) - \varphi_p(\zeta)| \prec \int\limits_{\tilde{\Gamma}(z,\zeta)} d^{-\frac{1}{p}}(\xi, L)|\,\mathrm{d}\xi|.$$

Since $G \in PQ(K, \alpha, \beta)$, the domain G satisfies the conditions of Lemma 4.1. Therefore,

$$\int\limits_{\tilde{\Gamma}(z,\zeta)} d^{-\frac{1}{p}}(\xi, L)|\,\mathrm{d}\xi| \prec \int\limits_{\tilde{\Gamma}(z,\zeta)} f^{-\frac{1}{p}}(|\xi - z|)|\,\mathrm{d}\xi| \prec \int\limits_0^{c|z-\zeta|} f^{-\frac{1}{p}}(t)\,\mathrm{d}t \prec |z - \zeta|^{1-(1+\beta)\frac{1}{p}}.$$

□

Suppose that a domain $G \in PQ(K, \alpha, \beta)$, $\alpha > 0$, $\beta > 0$ is given. For the sake of simplicity, but without loss of generality, we take the domain G as in the beginning of Section 3.

Each L^j, $j = 1, 2$ is a K_j-quasiconformal arc. Let $\alpha_j(.)$ be the quasiconformal reflection across L^j. Let us also set

$$\gamma_1^1 := \left\{ z = x + \mathrm{i}y \colon\ y = \frac{2c_1 + c_2}{3}\,(x-1)^{1+\alpha} \right\};$$
$$\gamma_1^2 := \left\{ z = x + \mathrm{i}y \colon\ y = \frac{c_1 + 2c_2}{3}\,(x-1)^{1+\alpha} \right\};$$
$$\gamma_2^1 := \alpha_1 \left\{ z = x + \mathrm{i}y \colon\ y = \frac{2c_3 + c_4}{3}\,(x+1)^{1+\beta} \right\};$$
$$\gamma_2^2 := \alpha_2 \left\{ z = x + \mathrm{i}y \colon\ y = \frac{c_3 + 2c_4}{3}\,(x+1)^{1+\beta} \right\},$$

where the constants $c_j, j = \overline{1,4}$ are those from the definition of the class $PQ(K, \alpha, \beta)$. It is easy to check from Lemma 2.3 that $\operatorname{mes} \gamma_j^i(\zeta_1, \zeta_2) \prec |\zeta_1 - \zeta_2|$ for all $\zeta_1, \zeta_2 \in \gamma_j^i$, $i, j = 1, 2$.

Let $0 < \varepsilon < 1$ be small enough and $R := 1 + cn^{\varepsilon - 1}$. Let us choose points z_j^i, $i, j = 1, 2$ such that they are in the intersection of L_R and γ_j^i and are the first such points in $\tilde{L}_R^1 := \{z : z \in L_R, \operatorname{Im} z \geqslant 0\}$ or $\tilde{L}_R^2 := L_R \setminus \tilde{L}_R^1$ (according to the motion on L_R). These points divide L_R into four parts: $L_R^1 := L_R^1(z_1^1, z_2^1)$ with the endpoints z_1^1 and z_2^1, $L_R^2 := L_R^2(z_2^2, z_1^2)$, $L_R^3 := L_R^3(z_1^2, z_1^1)$, $L_R^4 := L_R^4(z_2^1, z_2^2)$, $L_R := \bigcup_{j=1}^{4} L_R^j$; $\gamma_i^j(R)$ is a subarc of γ_i^j joining points ± 1 with z_i^j; $\Gamma_R^j := \gamma_1^j(R) \cup \gamma_2^j(R) \cup L_R^j$; $U_j := \operatorname{int}\bigl(\Gamma_R^j \cup L^j\bigr)$, $i, j = 1, 2$.

We extend the function φ_p to $U_1 \cup U_2$ in the following way

$$\tilde{\varphi}_p(z) := \begin{cases} \varphi_p(z), & z \in \overline{G}, \\ (\varphi_p \circ \alpha_j)(z) & z \in U_j. \end{cases} \tag{7.1}$$

Then

$$\tilde{\varphi}_{p,\overline{z}}(z) = \begin{cases} 0, & z \in G, \\ (\varphi'_{p,\alpha_j} \circ \alpha_j)(z)\alpha_{j,\overline{z}}, & z \in U_j. \end{cases} \tag{7.2}$$

From the Cauchy-Pompeiu formula [19, p. 148], we get

$$\varphi_p(z) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\Gamma_R^1 \cup \Gamma_R^2} \frac{\tilde{\varphi}_p(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{\pi} \iint\limits_{U_1 \cup U_2} \frac{\tilde{\varphi}_{p,\overline{\zeta}}(\zeta)}{\zeta - z}\, \mathrm{d}\sigma_\zeta, \quad z \in G.$$

Then, using the above notations we obtain

(7.3)

$$\varphi_p(z) = \frac{1}{2\pi\mathrm{i}}\int_{L_R} \frac{f_p(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \sum_{i,j=1}^{2} \frac{1}{2\pi\mathrm{i}} \int\limits_{\gamma_i^j(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p\left((-1)^i\right)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{\pi}\iint\limits_{U_1\cup U_2} \frac{\tilde{\varphi}_{p,\overline{\zeta}}(\zeta)}{\zeta - z}\,\mathrm{d}\sigma_\zeta,$$

where

$$f_p(\zeta) := \begin{cases} \tilde{\varphi}_p(\zeta), & \zeta \in L_R^1 \cup L_R^2, \\ \varphi_p(1), & \zeta \in L_R^3, \\ \tilde{\varphi}_p(-1), & \zeta \in L_R^4. \end{cases}$$

Lemma 7.1. *Let $p > 1$ and assume that $G \in PQ(K, \alpha, \beta)$ for some $K \geqslant 1$, $0 < \alpha < \min\left\{2\left(1 - \frac{1}{p}\right); \frac{2}{p}\right\}$, $\beta < p_0 := \min\left\{p - 1; \frac{2(p-1)}{p+2}; \frac{2}{p+2}\right\}$. Then for any $n \geqslant 3$*

$$\|\varphi_p - \pi_{n,p}\|_{L_p^1(G)} \prec \begin{cases} \sqrt[p]{\ln\ln n}\,(\ln n)^{\frac{\alpha p - 2}{2\alpha p}}, & 1 < p \leqslant 2, \\ (\ln n)^{\frac{\alpha p - 2}{2\alpha p}}, & p > 2. \end{cases} \tag{7.4}$$

Lemma 7.2. *Let $p > 1$ and assume that $G \in PQ(K, 0, \beta)$ for some $K \geqslant 1$, $\beta < p_0$. Then, for any $n \geqslant 2$ and arbitrary small $\varepsilon > 0$*

$$\|\varphi_p - \pi_{n,p}\|_{L_p^1(G)} \prec \begin{cases} \left(\dfrac{1}{n}\right)^{\frac{1-\varepsilon}{pK^2}}, & \beta < \min\left\{p_0; \dfrac{K^2 - 1}{3K^2 + pK^2 + 1}\right\}, \\ \left(\dfrac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}}, & \dfrac{K^2 - 1}{3K^2 + pK^2 + 1} \leqslant \beta < p_0. \end{cases} \tag{7.5}$$

P r o o f. The proofs of Lemmas 7.1, 7.2 are similar, we give them together. Since the first term in the (7.3) is analytic in $\overline{G}$, there is a polynomial $P_n(z)$ of degree not exceeding n [23, p. 142] such that

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{L_R} \frac{f_p(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta - P_n'(z)\right| \prec \frac{1}{n}, \quad z \in \overline{G}. \tag{7.6}$$

So, from (7.3) we get

$$\begin{aligned} &(7.7)\ \left\|\varphi_p' - P_n'\right\|_{L_p(G)} \\ &\quad\prec \frac{1}{n} + \sum_{i,j=1}^{2} \left\| \int\limits_{\gamma_i^j(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p\left((-1)^i\right)}{(\zeta - z)^2}\,\mathrm{d}\zeta \right\|_{L_p(G)} + \left\| \iint\limits_{U_1\cup U_2} \frac{\tilde{\varphi}_{p,\overline{\zeta}}(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\sigma_\zeta \right\|_{L_p(G)} \\ &\quad =: \frac{1}{n} + \sum_{k=1}^{5} J_k. \end{aligned}$$

For all $p > 1$ and $\beta < p - 1$,

$$(7.8)\quad |\tilde{\varphi}_p(\zeta) - \varphi_p(-1)| = |\varphi_p(\alpha_j(\zeta)) - \varphi_p(-1)| \prec |\zeta + 1|^{1-\frac{1+\beta}{p}}, \quad \zeta \in \gamma_j^1(R),$$

$$(7.9)\quad |\tilde{\varphi}_p(\zeta) - \varphi_p(1)| = |\varphi_p(\alpha_j(\zeta)) - \varphi_p(1)| \prec |\zeta - 1|^{1-\frac{1}{p}}, \quad \zeta \in \gamma_j^2(R),$$

from (2.1) and Corollary 6.1. Thus, for each $\alpha < \min\left\{2\left(1 - \frac{1}{p}\right); \frac{2}{p}\right\}, \beta < p_0$ we obtain

$$(7.10)\quad \left\| \int_{\gamma_i^1(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(-1)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right\|_{L_p(G)} \prec \begin{cases} |\ln \ell_{j,1}|^{\frac{1}{p}} \, \ell_{j,1}^{\frac{2-(p+2)\beta}{2p}}, & 1 < p \leqslant 2, \\ \ell_{j,1}^{\frac{2-(p+2)\beta}{2p}}, & p > 2, \end{cases}$$

$$(7.11)\quad \left\| \int_{\gamma_i^2(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(1)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right\|_{L_p(G)} \prec \begin{cases} |\ln \ell_{j,2}|^{\frac{1}{p}} \, \ell_{j,2}^{\frac{2-\alpha p}{2p}}, & 1 < p \leqslant 2, \\ \ell_{j,2}^{\frac{2-\alpha p}{2p}}, & p > 2, \end{cases}$$

from Corollary 5.1, 5.2, where $\ell_{j,i} = \operatorname{mes} \gamma_j^i(R)$, $i, j = 1, 2$. On the other hand, according to [21, Lemma 9] we have

$$d(z_j, L^j) \prec n^{-\frac{2}{K^2+1}}.$$

Then, from (2.1), (3.1) and (3.2) we get

$$(7.12)\quad \ell_{j,i} \prec \left|z_j^i - (-1)^i\right| \prec \begin{cases} d^{\frac{1}{1+\beta}}(z_j^i, L^j) \prec \left(\dfrac{1}{n}\right)^{\frac{2-\varepsilon}{(1+\beta)(K^2+1)}}, & i = 1, \ \forall \varepsilon > 0, \\ (\ln n)^{-\alpha^{-1}}, & i = 2. \end{cases}$$

Thus, it follows from (7.10) and (7.11) that

$$(7.13)\quad \left\| \int_{\gamma_i^1(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(-1)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right\|_{L_p(G)} \prec \left(\frac{1}{n}\right)^{\frac{2-(p+2)\beta-\varepsilon}{p(1+\beta)(K^2+1)}}, \quad p > 1,$$

$$(7.14)\quad \left\| \int_{\gamma_i^2(R)} \frac{\tilde{\varphi}_p(\zeta) - \varphi_p(1)}{(\zeta - z)^2} \, \mathrm{d}\zeta \right\|_{L_p(G)} \prec \begin{cases} \sqrt[p]{\ln \ln n} \, (\ln n)^{\frac{\alpha p - 2}{2\alpha p}}, & 1 < p \leqslant 2 \\ (\ln n)^{\frac{\alpha p - 2}{2\alpha p}}, & p > 2. \end{cases}$$

Since the Hilbert transformation

$$(Tf)(z) := -\frac{1}{\pi} \iint \frac{f(\zeta)}{(\zeta - z)^2} \, \mathrm{d}\sigma_\zeta$$

is a bounded linear operator from L_p into itself for $p>1$, and

$$\iint\limits_{U_1\cup U_2}\left|\tilde{\varphi}_{p,,\overline{\zeta}}(\zeta)\right|^p d\sigma_\zeta \approx \iint\limits_{U_1\cup U_2}\left|\varphi'(\alpha_j(\zeta))\right|^2 d\sigma_\zeta$$

$$\prec \sum_{j=1}^{2}\iint\limits_{\alpha(U_j)}\left|\varphi'(\zeta)\right|^2 d\sigma_\zeta \prec \sum_{j=1}^{2}\operatorname{mes}\varphi\left(\alpha_j(U_j)\right),$$

by (7.2) and (2.1), the Calderon-Zigmund inequality [7, p. 89] shows that

$$J_5 \prec \left(\sum_{j=1}^{2}\operatorname{mes}\varphi\left(\alpha_j(U_j)\right)\right)^{\frac{1}{p}}. \tag{7.15}$$

For sufficiently large c and small $\varepsilon_0<\frac{1}{2}$, let us set:

$$V_1^j := \left\{\zeta\colon\ \zeta\in\alpha_j(U_j), |\zeta-1|\leqslant c(\ln n)^{-\alpha^{-1}}\right\};$$
$$V_2^j := \alpha_j(U_j)\setminus V_1^j,\quad j=1,2,\quad \alpha>0;$$

$$U_{\varepsilon_0} := \{\zeta\colon\ |\zeta+1|\leqslant\varepsilon_0\};\quad \tilde{V}_j^1 := U_j\cap U_{\varepsilon_0},\quad j=1,2,\quad \alpha=0.$$

Then, by Lemma 2.4, we obtain

$$\operatorname{mes}\varphi(V_1^j)\prec(\ln n)^{-\alpha^{-1}};\operatorname{mes}\varphi(\alpha_j(\tilde{V}_j^1))\prec n^{\frac{\varepsilon-2}{K^2+1}\delta}=n^{\frac{\varepsilon-1}{K^2}};$$

$$\operatorname{mes}\varphi(\alpha_j(U_j\setminus\tilde{V}_j^1))\prec n^{\frac{\varepsilon-1}{K^2}},$$

and

$$J_5\prec\begin{cases}(\ln n)^{-\frac{1}{\alpha p}}, & \alpha>0,\\ n^{\frac{\varepsilon-1}{pK^2}}, & \alpha=0.\end{cases} \tag{7.16}$$

From (7.7), (7.13), (7,14) and (7.16) we get

$$\left\|\varphi_p'-P_n'\right\|_{L_p(G)}\prec\begin{cases}\sqrt[p]{\ln\ln n}\,(\ln n)^{\frac{\alpha p-2}{2\alpha p}}, & \alpha>0,\ 1<p\leqslant 2,\\ (\ln n)^{\frac{\alpha p-2}{2\alpha p}}, & \alpha>0,\ p>2,\\ n^{\frac{\varepsilon-2+(p+2)\beta}{p(1+\beta)(K^2+1)}}+n^{\frac{\varepsilon-1}{pK^2}}, & \alpha=0,\ p>1\end{cases} \tag{7.17}$$

for arbitrary small $\varepsilon>0$. Now, if $\tilde{P}_n(z):=P_n(z)-P_n(0)+z[1-P_n'(0)]$, then we can easily see that (7.17) is also satisfied for $\tilde{P}_n(z)$ and $\tilde{P}_n(0)=0,\tilde{P}_n'(0)=1$. Thus, we can obtain the proof of Lemmas 7.1, 7.2 by considering the extremal property of $\pi_{np}(z)$. □

8. The proof of Theorems 1–6

We use the familiar method given in [4, 9, 24].

Lemma 8.1. *Suppose that G is a Jordan domain and*

$$\{\alpha_n\}\downarrow,\ \{\beta_n\}\uparrow,\ \{\gamma_n := \alpha_n\beta_n\}\downarrow, \quad n\to\infty,$$

are sequences such that the following is satisfied: if

$$\|\varphi_p - \pi_{n,p}\|_{L^1_p(G)} \prec \alpha_n, \quad n = 2, 3, \ldots,$$

then

$$\|P_n\|_{C(\overline{G})} \prec \beta_n \|P'_n\|_{L_p(G)}, \quad n = 1, 2, \ldots$$

for all polynomials $P_n(z)$ of degree not exceeding n with $P_n(0) = 0$; and, in addition, there exists a sequence of indices $\{n_k\}_{k=1}^{\infty}$ such that $\beta_{n_{k+1}} \leqslant c\beta_{n_k}$, $\gamma_{n_{k+1}} \leqslant \varepsilon\gamma_{n_k}$, $\forall k = 1, 2, \ldots$ for some $c \geqslant 1$, $0 < \varepsilon < 1$. Then

$$\|\varphi_p - \pi_{n,p}\|_{C(\overline{G})} \prec \gamma_n.$$

The proof of this lemma is given similar to those of lemma 15 in [9]. Therefore, by taking α_n from Lemmas 7.1, 7.2, β_n from Corollary 4.1 and, in addition, combining this with $G \in PQ(K, \alpha, \beta)$ in the cases $\alpha = 0$ or $\beta = 0$, we prove Theorems 1–6.

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