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BASIC SUBGROUPS IN ABELIAN GROUP RINGS

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Abstract. Suppose R is a commutative ring with identity of prime characteristic p and G is an arbitrary abelian p-group. In the present paper, a basic subgroup and a lower basic subgroup of the p-component $U_p(RG)$ and of the factor-group $U_p(RG)/G$ of the unit group $U(RG)$ in the modular group algebra RG are established, in the case when R is weakly perfect. Moreover, a lower basic subgroup and a basic subgroup of the normed p-component $S(RG)$ and of the quotient group $S(RG)/G_p$ are given when R is perfect and G is arbitrary whose G/G_p is p-divisible. These results extend and generalize a result due to Nachev (1996) published in Houston J. Math., when the ring R is perfect and G is p-primary. Some other applications in this direction are also obtained for the direct factor problem and for a kind of an arbitrary basic subgroup.

Keywords: basic and lower basic subgroups, units, modular abelian group rings

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1. Introduction

Let RG be the group ring of an abelian group G over a commutative ring R with unity of prime characteristic p, let $U(RG)$ be the group of all units in RG with a subgroup $V(RG)$ of the normalized (i.e. whose coefficient sum is equal to 1) units, and let $U_p(RG)$ and $S(RG)$ be their p-torsion parts, respectively. For G an abelian group, G_p denotes its p-primary component and for R a commutative ring, we let $U(R)$ denote the group of all invertible elements of R, $U_p(R)$ denotes its p-component and $N(R) = \bigcup\limits_{n=1}^{\infty} R(p^n)$ denotes the nilradical of R (i.e. the ideal of all nilpotent elements) (cf. [1]). We shall follow essentially the notation and terminology from the abelian group theory used in [6].

The main purpose of this paper is to obtain a (lower) basic subgroup of $U_p(RG) = S(RG) \times U_p(R)$, $S(RG)$, $U_p(RG)/G$ and $S(RG)/G_p$ under some minimal restrictions on R and G. These restrictions are that R must be weakly perfect, that is, $R^{p^i} = R^{p^{i+1}}$ for any fixed natural i, i.e. R^{p^i} is perfect, and moreover $N(R^{p^i}) = 0$ and G must contain only p-torsions; or R is perfect and G/G_p is p-divisible for G arbitrary. In this aspect N. Nachev gave a basic subgroup of $S(RG)$ [11] provided R is perfect and G is p-primary. Moreover, this result of Nachev was extended and generalized in [3] by using another technique and a different method of proof to find a criterion for the p-group $S(RG; B) = 1 + I(RG; B)$ to be basic in $S(RG)$ where B is a p-subgroup of $G (B \leqslant G_p)$ and $I(RG; B)$ is a relative augmentation ideal of RG with respect to B, generated by the elements $1 - b$, where b varies in B. Some other facts on basic subgroups of $S(RG)$ can be found in [2, 4]. As an example, in [4] we stated and proved that if B is a p-basic subgroup of G (so B_p is basic in G_p), then $S(RG; B_p) \subseteq B^*$, the basic subgroup of $S(RG)$. That is why probably $B^* = S(RG; B_p) + J$, where J is a special selected ideal of RG. Besides (see [3]), $S(RG; B_p) = B^*$ assuming R perfect and G/G_p p-divisible. On the other hand, the direct factor problem for basic subgroups is considered, and the form of an arbitrary basic subgroup of $U_p(RG)$ and $S(RG)$ is discussed, too. Thus we begin with the following central section.

2. A construction of basic subgroups

The next statement, part of which was announced in [2], plays an important role in this work:

Main Theorem. *Suppose G is an abelian p-group and R is an abelian ring with identity of prime characteristic p so that there is $i \in \mathbb{N}_0$ for which $R^{p^i} = R^{p^{i+1}}$ and R^{p^i} has no nilpotents. Then if B is (proper) basic in G, the subgroup $S(RG; B) + R(p^i)G$ is (proper) basic in $U_p(RG)$, and conversely. Moreover, B is a direct factor of $S(RG; B) + R(p^i)G$, and $S(RG; B) + R(p^i)G$ is a lower basic subgroup if and only if $\inf_{n \in \mathbb{N}} \max(|R^{p^n}|, |G^{p^n}|) = \max(|R^{p^i}|, |G/B|)$ (in particular $|R^{p^i}| > |G^{p^i}|$). Under the above assumptions, $(S(RG; B) + R(p^i)G)G/G$ is basic in $U_p(RG)/G$.*

The next result is a strong generalization and extension of a theorem due to Nachev ([11, 2, 3]). Moreover, our proof is absolutely different from that in [11]. Its advantage is clear. Thus we can formulate

Theorem. *Assume that R is an abelian unitary perfect ring of prime* char $R = p$ *and G is an abelian group such that G/G_p is p-divisible. If B_p is (proper) basic in G_p, then $S(RG; B_p)$ is (proper) basic in $S(RG)$ and conversely, and B_p is its direct factor. Moreover, $S(RG; B_p)$ is a lower basic subgroup of $S(RG)$ if and only if $\max(|R|, |G/B_p|) = \inf_{n \in \mathbb{N}} \max(|R|, |G^{p^n}|)$ (in particular $|R| > |G^p|$). By the above assumptions, $S(RG; B_p)G_p/G_p$ is basic in $S(RG)/G_p$.*

We continue with

3. Proofs of the theorems

Now, we can attack the main assertions stated above. However, before proving the main theorem we need some preliminaries. The next claim is well-known and documented [9], but for the convenience of the reader we give a standard proof.

Proposition 1. *Let P and L be commutative rings with 1. Moreover, let G and A be abelian groups. If $\varphi_P\colon P \to L$ is a ring-epimorphism and $\psi_G\colon G \to A$ is a group-epimorphism, then $\Phi_{PG}\colon PG \to LA$ defined by $\Phi_{PG}\Big(\sum\limits_k \alpha_k g_k\Big) = \sum\limits_k \varphi_P(\alpha_k)\psi_G(g_k)$, where $\alpha_k \in P$ and $g_k \in G$, is a ring-epimorphism.*

Proof. It is evident that Φ_{PG} is a correctly defined map which is a surjection. Now, suppose $x = \sum\limits_k \alpha_k g_k \in PG$ and $x' = \sum\limits_k \alpha'_k g_k \in PG$ ($\alpha'_k \in P$). Then

$$\begin{aligned}
\Phi_{PG}(x + x') &= \Phi_{PG}\Big(\sum_k (\alpha_k + \alpha'_k) g_k\Big) = \sum_k \varphi_P(\alpha_k + \alpha'_k)\psi_G(g_k) \\
&= \sum_k [\varphi_P(\alpha_k) + \varphi_P(\alpha'_k)]\psi_G(g_k) = \sum_k \varphi_P(\alpha_k)\psi_G(g_k) + \sum_k \varphi_P(\alpha'_k)\psi_G(g_k) \\
&= \Phi_{PG}(x) + \Phi_{PG}(x').
\end{aligned}$$

Further, $xx' = \sum\limits_{k,l} \alpha_k \alpha'_l g_k g_l$ and

$$\begin{aligned}
\Phi_{PG}(xx') &= \sum_{k,l} \varphi_P(\alpha_k \alpha'_l)\psi_G(g_k g_l) = \sum_{k,l} \varphi_P(\alpha_k)\varphi_P(\alpha'_l)\psi_G(g_k)\psi_G(g_l) \\
&= \Big[\sum_k \varphi_P(\alpha_k)\psi_G(g_k)\Big]\Big[\sum_k \varphi_P(\alpha'_k)\psi_G(g_k)\Big] = \Phi_{PG}(x) \cdot \Phi_{PG}(x'),
\end{aligned}$$

which gives the result. □

Corollary 2. *The map $\Phi_{RG}\colon RG \to R^{p^i}(G/H)$ where $i \in \mathbb{N}_0$, defined as $\Phi_{RG}\Big(\sum\limits_k r_k g_k\Big) = \sum\limits_k r_k^{p^i} g_k H$, is a ring-epimorphism with the kernel*

$$\ker \Phi = I(RG; H) + R(p^i)G.$$

Thus

$$RG/(I(RG; H) + R(p^i)G) \cong R^{p^i}(G/H).$$

In particular, if G is a p-torsion, then

$$\Phi_{U_p(RG)}\colon\ U_p(RG) \to U_p(R^{p^i}(G/H))$$

defined as above under the restriction on $U_p(RG)$, is a surjection (a group-epimorphism) with the kernel $1+\ker\Phi = 1 + I(RG;H) + R(p^i)G$. Thus

$$U_p(RG)/(1 + I(RG;H) + R(p^i)G) \cong U_p(R^{p^i}(G/H)).$$

P r o o f. The Frobenious epimorphism $R \to R^{p^i}$ along with the natural epimorphism $G \to G/H$ induce a group ring-epimorphism $RG \to R^{p^i}(G/H)$ according to Proposition 1. Now we shall calculate the kernel of this map. Assume $x \in I(RG;H) + R(p^i)G$. Hence $x = \sum\limits_{k,l} r_{kl}g_{kl}(1-h_k) + \sum\limits_{j}\alpha_j a_j$, where $r_{kl} \in R$, $\alpha_j \in R(p^i)$; $g_{kl} \in G$, $a_j \in G$, $h_k \in H$. Since Φ is a homomorphism, it is a routine matter to see that

$$\Phi(x) = \Phi\biggl(\sum_{k,l} r_{kl}g_{kl}(1-h_k)\biggr) + \Phi\biggl(\sum_{j}\alpha_j a_j\biggr) = 0.$$

So, $x \in \ker\Phi$ and immediately we conclude $I(RG;H) + R(p^i)G \subseteq \ker\Phi$.

For the converse, take $y \in \ker\Phi$. Consequently, $y = \sum\limits_k r_k g_k$ ($r_k \in R$, $g_k \in G$) and $\Phi(y) = \sum\limits_k r_k^{p^i} g_k H = 0$. Without loss of generality we may assume that $g_1H = \ldots = g_{m-1}H$ and $g_mH \neq \ldots \neq g_tH \neq g_mH$ ($1 \leqslant k \leqslant t$, m is fixed such that $1 \leqslant m \leqslant t$). Therefore

$$r_1^{p^i} + \ldots + r_{m-1}^{p^i} = (r_1 + \ldots + r_{m-1})^{p^i} = 0$$

and $r_m^{p^i} = \ldots = r_t^{p^i} = 0$. Thus we derive $r_1 + \ldots + r_{m-1} \in R(p^i)$ and $r_m \in R(p^i), \ldots, r_t \in R(p^i)$. Furthermore,

$$\begin{aligned} y &= r_1g_1(1 - g_{m-1}g_1^{-1}) + r_2g_2(1 - g_{m-1}g_2^{-1}) + \ldots + r_{m-2}g_{m-2}(1 - g_{m-1}g_{m-2}^{-1}) \\ &\quad + (r_1 + r_2 + \ldots + r_{m-1})g_{m-1} + r_mg_m + \ldots + r_tg_t \in I(RG;H) + R(p^i)G, \end{aligned}$$

which gives that $\ker\Phi \subseteq I(RG;H) + R(p^i)G$. Finally, $\ker\Phi = I(RG;H) + R(p^i)G$ as claimed.

Now we can apply the well-known "*theorem for the homomorphisms*" to obtain $RG/\ker\Phi \cong R^{p^i}(G/H)$.

Let us assume that G is p-primary. Choose $\sum\limits_k r_k^{p^i} g_kH \in U_p(R^{p^i}(G/H))$. Hence $\Bigl(\sum\limits_k r_k^{p^i} g_kH\Bigr)^{p^s} = H$ for any fixed natural s. We may select s so that $g_k^{p^s} = 1$.

Consequently, $\sum_k r_k^{p^{i+s}} = 1$ and so

$$\sum_k r_k^{p^{i+s}} g_k^{p^{i+s}} = \Big(\sum_k r_k g_k\Big)^{p^{i+s}} = 1.$$

That is why $\sum_k r_k g_k \in U_p(RG)$. The converse has a similar proof and so Φ is a map of $U_p(RG)$ onto $U_p(R^{p^i}(G/H))$. Clearly $\ker \Phi_{U_p(RG)} = 1 + \ker \Phi_{RG} = 1 + I(RG;H) + R(p^i)G$. The proof is completed. $\square$

Lemma 3. *For every ordinal δ the following identities are true:*
(a) $(G_p)^{p^\delta} = (G^{p^\delta})_p$, $[N(R)]^{p^\delta} = N(R^{p^\delta})$,
(b) $U^{p^\delta}(R) = U(R^{p^\delta})$, $[U_p(R)]^{p^\delta} = U_p(R^{p^\delta})$,
(c) $U^{p^\delta}(RG) = U(R^{p^\delta}G^{p^\delta})$, $[U_p(RG)]^{p^\delta} = U_p(R^{p^\delta}G^{p^\delta})$,
(d) $S^{p^\delta}(RG) = S(R^{p^\delta}G^{p^\delta})$.

Proof. (a): It is obvious and so we omit the details.

(b), (c) and (d): Now we observe that it is sufficient to show only that $U^{p^\delta}(R) = U(R^{p^\delta})$ holds. From this, in view of the first equality of (a) and of the simple facts that $\operatorname{char} RG = p$ and $(RG)^{p^\delta} = R^{p^\delta}G^{p^\delta}$, it will follow that all the other relations are fulfilled.

In the sequel our conclusions are based of the standard transfinite induction on δ and so we will consider only the case for $\delta = 1$. Put $\delta = 1$ and choose $x \in U^p(R)$. Hence $x = r^p$ where $r \in U(R)$. Thus there exists $r' \in R$ such that $r \cdot r' = 1$. Obviously $r^p \cdot (r')^p = 1$ and $x \in U(R^p)$, i.e. $U^p(R) \subseteq U(R^p)$.

Conversely, putting $y \in U(R^p)$ we conclude $y = \alpha^p$ where $\alpha \in R$ and $\alpha^p \cdot \alpha'^p = 1$ for some $\alpha' \in R$. Consequently, $\alpha \cdot \alpha^{p-1} \cdot \alpha'^p = 1$, i.e. $\alpha \cdot \alpha'' = 1$ where $\alpha^{p-1} \cdot \alpha'^p = \alpha'' \in R$. Finally, $\alpha \in U(R)$ and so $y \in [U(R)]^p$, which gives $U(R^p) \subseteq U^p(R)$ and completes the proof. $\square$

Lemma 4. *Suppose that there exists $i \in \mathbb{N}_0$ such that R^{p^i} contains no nilpotents, i.e. $N(R^{p^i}) = 0$. Then $N(R) = R(p^i)$ and $[R(p^i)]^{p^n} = R^{p^n}(p^i)$ for each $n \in \mathbb{N}_0$.*

Proof. For $n \geqslant i$ the proof is obvious, because $[R(p^i)]^{p^n} = 0$ and moreover $R^{p^n}(p^i) = 0$, $R^{p^n} \subseteq R^{p^i}$. Now consider the case when $n < i$. By definition $R(p^i) = \{r \in R \mid r^{p^i} = 0\}$ and $R^{p^n}(p^i) = \{\alpha^{p^n} \mid \alpha \in R \text{ and } \alpha^{p^{n+i}} = 0\}$. Clearly $[R(p^i)]^{p^n} \subseteq R^{p^n}(p^i)$. For the converse, choose $x \in R^{p^n}(p^i)$. Therefore $x = \alpha^{p^n}$ and $(\alpha^{p^i})^{p^n} = 0$, i.e. $x = \alpha^{p^n}$ and $\alpha^{p^i} = 0$. Finally, $x \in [R(p^i)]^{p^n}$, as desired.

Now assume $r \in R$ and $r^{p^t} = 0$ for $t > i$. Hence $r^{p^{t+i}} = 0$, i.e. $(r^{p^i})^{p^t} = 0$. Thus $r^{p^i} = 0$, and we may conclude that $R(p^t) = R(p^i)$, i.e. $N(R) = R(p^i)$. The proof of the lemma is complete. $\square$

Remark. By the same argument as above $N(R^{p^i}) = 0 \Leftrightarrow R(p^i) = R(p^{i+1}) \Leftrightarrow N(R) = R(p^i)$.

Recall that for $H \leqslant G$ we have defined $S(RG;H) = 1 + I_p(RG;H) = \{1 + z \mid z^{p^t} = 0$ for some natural t when $z \in I(RG;H)\}$.

Lemma 5. *Assume* $1 \in L \leqslant R$ *and* $B \leqslant H \leqslant G$, $A \leqslant G$. *Then*

(∗) $I(RG;H) \cap LA = I(LA; A \cap H)$ *and* $HS(RG;B) \cap S(LA) = (A_p \cap H)S(LA; A \cap B)$,

(∗∗) $(I(RG;H) + R(p^i)G) \cap LA = I(LA; A \cap H) + L(p^i)A$ *and* $G(S(RG;H) + R(p^i)G) \cap U_p(LA) = A_p(S(LA; A \cap H) + L(p^i)A)$.

P r o o f. (∗) Indeed, $x = \sum\limits_k r_k g_k \in I(RG;H)$ if and only if $\sum\limits_k r_k g_k H = 0$, i.e. $\sum\limits_{g_k \in \overline{g}H} r_k = 0$ for any $\overline{g} \in G$. Hence we can write

$$z = \sum_k r_k g_k = \sum_k \alpha_k a_k \in I(RG;H) \cap LA,$$

where $\sum\limits_k r_k g_k H = 0$ ($r_k \in R$, $\alpha_k \in L$; $g_k \in G$, $a_k \in A$). Because $\overline{g} \in A$ and $g_k = a_k \in A$ we deduce $\overline{g}H \cap A = \overline{g}(H \cap A)$ and

$$\sum_{g_k \in \overline{g}H \cap A} r_k = \sum_{g_k \in \overline{g}(H \cap A)} r_k = \sum_{a_k \in \overline{g}(H \cap A)} r_k = 0,$$

i.e. $z \in I(LA; A \cap H)$. The converse is elementary. To verify the second relation we observe that each element of the left hand-side is $\sum\limits_{a \in A} \alpha_a a = h \sum\limits_{g \in G} r_g g$, where $\alpha_a \in L$, $r_g \in R$ and $h \in H$. Moreover,

$$\sum_{g \in \overline{g}B} r_g = \begin{cases} 0, & \overline{g} \notin B, \\ 1, & \overline{g} \in B \end{cases}$$

for each $\overline{g} \in G$, and $\sum\limits_{a \in A} \alpha_a = 1$. Observe that $\alpha_a = r_g$ and $a = hg$. Since $\sum\limits_{a \in A} \alpha_a a \in S(LA)$, there is $a_p \in A_p$. Since clearly $a_p \in H$ $\left(\sum\limits_{g \in B} r_g = \sum\limits_{a \in A} \alpha_a = 1,\ g = ah^{-1} \in B \subseteq H\right)$, it follows that $\sum\limits_{a \in A} \alpha_a a = a_p \sum\limits_{a \in A} \alpha_a a a_p^{-1}$ has the coefficient sum

$$\sum_{a \in \overline{a}B} \alpha_a = \begin{cases} 0, & \overline{a} \notin B, \\ 1, & \overline{a} \in B \end{cases}$$

for every $\overline{a} \in A$ and as above we conclude the equality holds.

(**) We can proceed analogously to the above. To demonstrate this, choose z on the left hand-side. Write $z = \sum_k \alpha_k a_k$ where $\sum_{a_k \in \overline{a}H} \alpha_k^{p^i} = 0$ for any $\overline{a} \in A$. But $\overline{a}H \cap A = \overline{a}(H \cap A)$ and consequently

$$\sum_{a_k \in \overline{a}(H \cap A)} \alpha_k^{p^i} = 0,$$

i.e. $z \in I(LA; A \cap H) + L(p^i)A$. The converse is trivial.

For the last relation note that every element on the left hand-side is $\sum_{a \in A} \alpha_a a = g' \sum_{g \in G} r_g g$, where $\alpha_a \in L$, $r_g \in R$ and $g' \in G$. Besides,

$$\sum_{g \in \overline{g}H} r_g^{p^i} = \begin{cases} 1, & \overline{g} \in H, \\ 0, & \overline{g} \notin H \end{cases}$$

for all $\overline{g} \in G$. It is easily seen that $\alpha_a = r_g$ and $a = g'g$ ($g \in \overline{g}H \iff a \in \overline{a}H$ for each $\overline{a} \in A$). Because $\sum_{a \in A} \alpha_a a \in U_p(LA)$, there is $a_p \in A_p$. Consequently, $\sum_{a \in A} \alpha_a a = a_p \sum_{a \in A} \alpha_a a a_p^{-1}$ has the Frobenious coefficient sum equal to

$$\sum_{a \in \overline{a}A} \alpha_a^{p^i} = \begin{cases} 1, & \overline{a} \in H, \\ 0, & \overline{a} \notin H \end{cases}$$

for every $\overline{a} \in A$. This completes the proof in general. So the lemma is shown. □

Part of the next statement is an expansion of facts well-documented and mentioned in [3,5], and is included here in details for the sake of completeness and the convenience of the reader.

Theorem 6. *Let H be a pure p-subgroup of G. Then $S(RG;H)/H$ and $[S(RG;H) + R(p^i)G]/H$ are direct sums of cyclic groups provided that H is. Thus H is a direct factor of $S(RG;H)$ and $S(RG;H) + R(p^i)G$ with a direct sum of cyclics complements.*

P r o o f. Indeed, we can write [6], $H = \bigcup_{n=1}^{\infty} H_n$, $H_n \subseteq H_{n+1}$ and $H_n \cap H^{p^n} = 1$. Since $H \cap G^{p^n} = H^{p^n}$, we have $H_n \cap G^{p^n} = 1$. Furthermore $S(RG;H) = \bigcup_{n=1}^{\infty} S(RG;H_n)$, where $S(RG;H_n) \subseteq S(RG;H_{n+1})$, and by virtue of (*) and (c) we compute

$$\begin{aligned} S(RG;H_n) \cap S^{p^n}(RG;H) \subseteq S(RG;H_n) \cap S^{p^n}(RG) &= S(RG;H_n) \cap S(R^{p^n}G^{p^n}) \\ &= S(R^{p^n}G^{p^n}; G^{p^n} \cap H_n) = 1. \end{aligned}$$

Moreover, $S(RG;H)/H = \bigcup_{n=1}^{\infty} [S(RG;H_n)H/H]$ and Lemma 5 ensures that

$$\begin{aligned}(S(RG;H_n)H) \cap [S(R^{p^n}G^{p^n};H^{p^n})H] &= H((S(RG;H_n)H) \cap S(R^{p^n})G^{p^n};H^{p^n}))\\ &= HS(R^{p^n}G^{p^n};G^{p^n} \cap H_n) = H.\end{aligned}$$

Thus the well-known Kulikov criterion in [6] is applicable to obtain that $S(RG;H)/H$ is a direct sum of cyclics, as stated. Finally, by what we have shown above,

$$([S(RG;H) + R(p^i)G]/H)^{p^i} \cong S(R^{p^i}G^{p^i};H^{p^i})/H^{p^i}$$

is a direct sum of cyclics since H^{p^i} is a direct sum of cyclics and is pure in G^{p^i}. Consequently [6], so is the desired group. Besides, H is pure in $S(RG;H)$ and the latter is pure in $S(RG)$ [3]. So H is pure in $S(RG)$, whence it is pure in $S(RG;H) + R(p^i)G \subseteq S(RG)$ and the Kulikov theorem [6, Theorem 28.2] is applicable. The proof is complete. □

Now we are in position to begin with

P r o o f of Main Theorem. We shall establish that the p-group $B^* = S(RG;B) + R(p^i)G$ satisfies the three necessary and sufficient conditions for a basic subgroup given in [6]. From this it will follow immediately that B^* is a basic subgroup of $U_p(RG)$ and we are done since the converse is apparent.

1) *A direct sum of cyclic groups.* First consider the problem of the decomposition of B^* into a direct sum of cyclics. According to ([6], p. 111, Proposition 18.3) B^* is a direct sum of cyclics if and only if $(B^*)^{p^i} = S(R^{p^i}G^{p^i};B^{p^i})$ (see [3]) is a direct sum of cyclics. But B is a direct sum of cyclics and B is pure in G. Hence $B^{p^i} \subseteq B$ is a direct sum of cyclics and B^{p^i} is pure in G^{p^i}. Therefore Theorem 6 or ([3], Theorem 2) applies to prove our claim.

2) *Purity.* Secondly, consider the question of the purity of B^* in $U_p(RG)$. We shall show that if B is pure in G, then for every $n \in \mathbb{N}$ we have

$$[S(RG;B) + R(p^i)G] \cap U_p^{p^n}(RG) = \begin{cases} S(R^{p^n}G^{p^n};B^{p^n}) + R^{p^n}(p^i)G^{p^n}, & n < i,\\ S(R^{p^n}G^{p^n};B^{p^n}), & n \geqslant i.\end{cases}$$

In fact, we can apply Lemma 3 and Lemma 5 together with the facts that $B \cap G^{p^n} = B^{p^n}$ and

$$R(p^i) \cap R^{p^n} = R^{p^n}(p^i) = \begin{cases} R^{p^n}(p^i), & n < i,\\ 0, & n \geqslant i.\end{cases}$$

Thus owing to Lemma 4 (more specially, $R^{p^n}(p^i) = [R(p^i)]^{p^n}$ or to $N(R) = R(p^i)$ combined with (a)) and also to [6], we arrive at the conclusions in this case.

3) *Divisibility.* Because $U_p(RG) = S(RG) \times U_p(R)$, by virtue of Lemma 3 and [3, Corollary 2] we get that $U_p(RG)$ is divisible if and only if R is perfect and G is divisible (this assertion follows also obviously). Thus we are ready to show that $U_p(RG)/B^*$ is divisible provided G/B is divisible and R^{p^i} is perfect. Indeed, owing to Corollary 2, $U_p(RG)/B^* \cong U_p(R^{p^i}(G/B))$. So, the above conclusions imply that $U_p(R^{p^i}(G/B))$ is divisible as required. The direct factor property follows by utilizing Theorem 6. This completes the proof of the first part of the theorem in general.

The fact that

$$\big(S(RG;B) + R(p^i)G\big)G/G \cong (S(RG;B) + R(p^i)G) \\ /\ B\big([S(RG;B) + R(p^i)G] \cap G = B\big)$$

is basic in $U_p(RG)/G$ follows by making use of Corollary 2 plus the fact that an epimorphic image of a divisible group is divisible [6], Lemmas 3 and 5, Theorem 6 and conclusions similar to the above. Finally, we obtain a lower basic subgroup. In fact, Corollary 2 and [12, 13] guarantee that

$$\begin{aligned}\operatorname{rank}(U_p(RG)/S(RG;B) + R(p^i)G) &= \operatorname{rank} U_p(R^{p^i}(G/B)) = \operatorname{rank} S(R^{p^i}(G/B)) \\ &= \max(|R^{p^i}|, |G/B|).\end{aligned}$$

On the other hand, Lemma 3 and [6, 9] yield

$$\begin{aligned}\operatorname{fin\,rank} U_p(RG) &= \inf_{n\in\mathbb{N}} \operatorname{rank}(U_p^{p^n}(RG)) = \inf_{n\in\mathbb{N}} \operatorname{rank}(U_p(R^{p^n}G^{p^n}))[p] \\ &= \inf_{n\in\mathbb{N}} |U_p(R^{p^n}G^{p^n})[p]| = \inf_{n\in\mathbb{N}} \max(|R^{p^n}|, |G^{p^n}|),\end{aligned}$$

completing the proof of the second part. The theorem is proved. □

We continue with

Proof of Theorem. The fact that $S(RG;B_p)$ is basic in $S(RG)$ as well as the converse situation follow by virtue of [3]. The direct factor property holds by virtue of Theorem 6. Moreover, $S(RG;B_p)G_p/G_p \cong S(RG;B_p)/B_p$ is indeed basic in $S(RG)/G_p$ according to Lemmas 3, 5 and Theorem 6 along with conclusions analogous to those in [3]. The lower basic subgroup can be established as in Main Theorem. The proof is complete. □

Remark. P. Hill in [8] has shown that $V(RG)/G$ has a special basis (called a ν-basis or more properly, a Hill-basis) provided R is a perfect field and G is an abelian p-group. Our two theorems give another basis to that of Hill, which, however, determines the group $V(RG)/G$ more completely for a more general coefficient ring R (cf. [6]). Thus the long-standing direct factor problem due to May [9] whether $V(RG)/G$ is totally projective, is well-examined.

Corollary 7 ([11], [3]). *Suppose R is perfect and $N(R)=0$. Then if B is a basic subgroup of a p-group G, then $S(RG;B)$ is basic in $S(RG)$.*

Proof. Follows by application of our main theorem, since $R=R^{p^i}$ implies $R^{p^i}=R^{p^{i+1}}$ and also $N(R^{p^i})=0$ and $R(p^i)=0$. Therefore $1+I(RG;B)+R(p^i)G$ reduces to $1+I(RG;B)=S(RG;B)$, which gives the result. □

Remark. The above corollary is true even if $N(R)\neq 0$ (cf. [11] or [3] and the second central theorem stated in Section 2).

Corollary 8. *Suppose R is weakly perfect of exponent i such that its maximal perfect subring has no nilpotents and B is basic in the p-torsion G. If $B_{S(RG)}$ and $B_{U_p(R)}$ are the basic subgroups of $S(RG)$ and $U_p(R)$ respectively, then*

$$B_{S(RG)}\cong[1+I(RG;B)+R(p^i)G]/B_{U_p(R)}.$$

Proof. Since $U_p(RG)=S(RG)\times U_p(R)$, it follows from [6] that $B_{U_p(RG)}=B_{S(RG)}\times B_{U_p(R)}$. But as we see, $B_{U_p(RG)}=1+I(RG;B)+R(p^i)G$, and so apparently the above isomorphism holds. The proof is complete. □

4. Applications

In this section we proceed by proving the following ring-theoretic assertion:

Example 9. Let R be finite. Then R is perfect if and only if $N(R)=0$.

Proof. The sufficiency is well-known, but we will prove it. In fact, $\varphi\colon R\to R^{p^i}$ defined by $\varphi(r)=r^{p^i}$ $(r\in R)$ is an isomorphism and so $|R|=|R^{p^i}|$, i.e. $R=R^{p^i}$.

Conversely, using Lemma 3, R perfect yields $N(R)$ is perfect, i.e. $N(R)=[N(R)]^{p^n}$ for every $n\in\mathbb{N}$. But R is finite, hence $N(R)$ is. Thus $R(p^m)=R(p^{m+1})$ for any $m\in\mathbb{N}$ and consequently $N(R)=R(p^m)$ (see Lemma 4 and also the remark formulated below). Finally, $N(R)=[R(p^m)]^{p^n}=0$ when $n\geqslant m$. This completes the proof. □

Remark. By the same argument as above, R perfect and $N(R)$ finite yield $N(R)=0$.

Of some interest and importance is the calculation of a power of the basic subgroup which is given in the sequel.

Proposition 10. *If G is arbitrary with G/G_p p-divisible and R is abelian unitary perfect of prime* $\operatorname{char} R=p$, *or G is p-primary and R is unitary abelian weakly perfect* *of prime* $\operatorname{char} R = p$ *such that its maximal perfect subring is without nilpotents, then the basic subgroup of the infinite groups* $S(RG)$ *or* $U_p(RG)$ *has cardinality* $\max(|R|, |G|)$.

Corollary 11. *Under the above assumptions on* R *and* G *the following holds:* $S(RG)$ *or* $U_p(RG)$ *has a countable basic subgroup* $\Longleftrightarrow$ *both* R *and* G *are countable.*

An easy reformulation is given by the following

Corollary 12. *If either* R *or* G *is uncountable, then* $S(RG)$ *and* $U_p(RG)$ *contain uncountable basic subgroups.*

Remark. When the set of all basic subgroups of the abelian p-primary group G is finite, then N. Nachev in [11] proved that not every basic subgroup of $S(RG)$ is of the above kind provided R is perfect.

Now we will give a further supplement to this result, namely

Claim 13. *Assume that* G *and* R *are as in Proposition* 10. *If* G_p *or* G *contains an infinite number of different basic subgroups, then the set of all different basic subgroups of* $S(RG)$ *or* $U_p(RG)$*, respectively, has cardinality* $\max(|R|, |G|)^{\max(|R|,|G|)}$. *In particular, there are basic subgroups of* $S(RG)$ *and* $U_p(RG)$ *that are not of these kinds.*

P r o o f. By virtue of the second theorem from Section 2, $S(RG; B_p)$ is basic in $S(RG)$ for all choices of the basic B_p. Moreover, $S(RG; B_p) = S(RG; B'_p) \Longleftrightarrow B_p = B'_p$ for other basic B'_p in G_p. Hence $S(RG)$ has an infinite set of distinguished basic subgroups. Applying [10, 7] or [6, p. 178, Exercise 3], the power of the last set is $|S(RG)|^{|S(RG;B_p)|} = [\max(|R|, |G|)]^{\max(|R|,|G|)}$, owing to Proposition 10 and [9] as well. On the other hand the cardinality of the set of all distinguished basic subgroups of G_p is $|G_p|^{|B_p|}$ [10, 7, 6]. Therefore there is a real possibility to have $[\max(|R|, |G|)]^{\max(|R|,|G|)} > |G_p|^{|B_p|}$. Indeed, take $|R| > |G|$ to be a strong limit cardinal, whence $|R|^{|R|} > |R| > |G_p|^{|B_p|}$, which verifies the claim. The conclusions for the other case are identical. This is the end of the proof. $\square$

Remark. If G is p-torsion infinite reduced with a basic subgroup B and $|R| \leqslant |G|$, then we have $|G|^{|G|} = |G|^{|B|}$ if and only if $|G| = |B|$ provided the generalized **CH** holds. In fact, if $|G| = \aleph_0$, then $|G| = |B|$. Let now $|G| = \aleph_{\alpha+1} = 2^{\aleph_\alpha}$ for some ordinal α, then $|G|^{|G|} = 2^{|G|}$ and $|G|^{|B|} = 2^{\max(|B|,\aleph_\alpha)}$. Thus $2^{|G|} = 2^{\max(|B|,\aleph_\alpha)}$ yields $|G| = |B|$ as desired. We mention that such groups G with $|G| = |B|$ are known to be starred.

In the conclusion we note that [6], $|G| \leqslant |B|^{\aleph_0}$.

5. Open problems

Here are two actual questions which immediately arise. First, what is the situation for B^* when $N(R^{p^i}) \neq 0$. The second question is what is the basic subgroup of $U_p(RG)$ (or of $S(RG)$) in the general case when R is not even weakly perfect, i.e. when $R^{p^i} = R^{p^{i+1}}$ and i is a non-natural ordinal.

Of some interest and importance is the problem of the calculation of lower and upper basic subgroups in $U_p(RG)$ and $S(RG)$. The same is valid also for $U_p(RG)/G_p$ and $S(RG)/G_p$ for a large modular coefficient ring R. And finally, what is the basic (lower and upper) subgroup of $S(KG)$ and $S(KG)/G$ when G is p-primary and K is the first kind field with respect to p ($\operatorname{char} K \neq p$)?

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