H. G. Ince

Cesàro wedge and weak Cesàro wedge FK-spaces

Czechoslovak Mathematical Journal, Vol. 52 (2002), No. 1, 141–154

Persistent URL: `http://dml.cz/dmlcz/127708`

Czechoslovak Mathematical Journal, 52 (127) (2002), 141–154

CESÀRO WEDGE AND WEAK CESÀRO WEDGE FK-SPACES

H. G. İnce, Ankara

(Received February 17, 1999)

Abstract. In this paper we deal with Cesàro wedge and weak Cesàro wedge FK-spaces, and give several characterizations. Some applications of these spaces to general summability domains are also studied.

Keywords: FK-space, wedge FK-space, weak wedge FK-space, compact operator, matrix mapping

MSC 2000: 46A35, 46A45, 47B37, 40C05

1. Introduction

The aim of this paper is to study topological sequence spaces in which the arithmetic means of coordinate vectors converge to zero. Naturally, our work has been largely influenced by that of Bennett [2].

In Section 2 we give the notation and terminology. Section 3 is devoted to Cesàro wedge FK-spaces, and some characterizations are given. Section 4 deals with weak Cesàro wedge FK-spaces. In Section 5 we give some applications of (weak) Cesàro wedge FK-spaces to general summability domains. We also obtain some results regarding the (weak) Cesàro wedgeness of particular summability domains.

2. Notation and preliminaries

Let w denote the space of all real or complex-valued sequences. It can be topologized with the seminorms $p_i(x) = |x_i|$ $(i = 1, 2, \ldots)$, and any vector subspace of w is called a sequence space. A sequence space X, with a vector space topology τ, is a K-space provided that the inclusion mapping $I\colon (X, \tau) \to w$ is continuous. If, in addition, τ is complete, metrizable and locally convex then (X,τ) is called an FK-space. So an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals are continuous. The basic properties of such spaces may be found in [12], [13] and [15].

By m, c, c_0 we denote respectively the spaces of all bounded sequences, convergent sequences and the null sequences with $\|x\|_\infty = \sup\limits_n |x_n|$; bs, cs and ℓ^p $(1 \leqslant p < \infty)$ will denote the space of all bounded series, convergent series and all absolutely p-summable sequences, respectively. As usual, ℓ^1 is denoted simply by ℓ. We shall also need the sequence spaces

$$bv := \left\{x \in w\colon\ \sum_{j=1}^{\infty} |x_j - x_{j+1}| < \infty\right\}$$

and $bv_0 = bv \cap c_0$; the space

$$h := \left\{x \in w\colon\ \lim_j x_j = 0 \text{ and } \sum_{j=1}^{\infty} j|\Delta x_j| < \infty\right\}$$

is normed with the norm

$$\|x\|_h = \sum_{j=1}^{\infty} j|\Delta x_j| + \sup_j |x_j|,$$

where $\Delta x_j = x_j - x_{j+1}$. The spaces

$$\sigma_0 := \left\{x \in w\colon\ \lim_n \frac{1}{n} \sum_{j=1}^{n} x_j = 0\right\}$$

and

$$\sigma_\infty := \left\{x \in w\colon\ \sup_n \frac{1}{n} \left|\sum_{j=1}^{n} x_j\right| < \infty\right\}$$

are normed with the norm

$$\|x\| = \sup_n \frac{1}{n} \left|\sum_{j=1}^{n} x_j\right|$$

(see [1], [4]).

Throughout the paper e denotes the sequence of ones, $(1,1,\ldots,1\ldots)$; δ^j $(j = 1,2,\ldots)$, the sequence $(0,0,\ldots,0,1,0,\ldots)$ with the one in the j-th position; $\varphi := \ell\text{-hull}\{\delta^k\colon\ k \in \mathbb{N}\}$. Let X be an FK-space containing φ. Then $X^f := \{\{f(\delta^k)\}\colon\ f \in X'\}$, where X' is the topological dual of X.

$x \in X$ is said to have AK if $x^{(n)}$ converges to x in τ, where

$$x^{(n)} = \sum_{j=1}^{n} x_j \delta^j = (x_1, x_2, \ldots, x_n, 0, 0, \ldots).$$

If each $x \in X$ has AK, then (X, τ) is called an AK-space. The space X is said to have AD if φ is dense in X ([13], p. 59).

Let $z = (z_k) \in w$ be such that $z_k \neq 0$ for every $k = 1, 2, \ldots$ Then

$$V_0(z) := \left\{ x \in c_0 \colon \sum_{k=1}^{\infty} |z_k|\, |\Delta x_k| < \infty \right\}$$

is an $FK - AK$ space [5], with the norm

$$\|x\|_{V_0(z)} = \sum_{k=1}^{\infty} |z_k|\, |\Delta x_k|.$$

Finally, $s = \{s_n\}_{n=1}^{\infty}$ always denotes a strictly increasing sequence of non-negative integers with $s_1 = 0$. We shall also be interested in spaces of the form

$$c|s| = \left\{ x \in c_0 \colon \sup_n \sum_{j=s_n+1}^{s_{n+1}} j|\Delta x_j| < \infty \right\}$$

which become an FK-space with the norm

$$\|x\|_{c|s|} = \sup_n \sum_{j=s_n+1}^{s_{n+1}} j|\Delta x_j|,$$

and $h \subset c|s| \subset c_0 \subset m$.

Following Bennett [2] we say that a K-space (X, τ) is a wedge space if $\delta^j \to 0$ in τ, and a weak wedge space if $\delta^j \to 0$ weakly in X.

In a seminar held at Ankara University during the summer of 1996, Prof. Bennett of Indiana University (USA) introduced the concept of the Cesàro wedge space. Motivated by his talks we introduce the weak Cesàro wedge space; and for (weak) Cesàro wedge spaces we study results analogous to those given by Bennett [2].

This section is devoted to Cesàro wedge spaces.

Definiton 3.1. If (X,τ) is a K-space containing φ and

$$\frac{e^{(n)}}{n}=\frac{1}{n}\sum_{k=1}^{n}\delta^k=\Big(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n},0,0\ldots\Big)\to 0 \quad \text{in } \tau$$

then (X,τ) is called a Cesàro wedge space or simply C_1-wedge space. Every wedge space is a C_1-wedge space but the converse is not true. For example, c_0, c, m, bv, bv_0 and ℓ_p $(p>1)$ are C_1-wedge spaces but not wedge spaces.

We will now give a result related to C_1-wedge FK-spaces. However, we first require

Lemma 3.2. *Suppose* $\lim\limits_j z_j^n/j=0$ $(n=1,2,\ldots)$. *Then there exists* $z\in w$ *with* $\lim\limits_j z_j/j=0$ *such that* $\lim\limits_j z_j^n/z_j=0$ $(n=1,2,\ldots)$.

Moreover, for any such z *we have* $V_0(z)\subseteq\bigcap\limits_{n=1}^{\infty}V_0(z^n)$.

P r o o f. To prove the lemma we use the technique given by Bennett [2].

We may choose a sequence $\{j_k\}_{k=0}^{\infty}$ of positive integers such that

$$1=j_0<j_1<\ldots<j_k<j_{k+1}<\ldots$$

and

$$\max_{1\leqslant n\leqslant k}\left|\frac{z_j^n}{j}\right|<\frac{1}{4^k}\qquad (j\geqslant j_k;\;\; k=1,2,\ldots).$$

Define $z\in w$ as follows:

$$z_j=\frac{1}{2^k}\quad (j_k\leqslant j<j_{k+1};\;\; k=0,1,2,\ldots).$$

It is clear that $\lim\limits_j z_j/j=0$ and, fixing n, we get

$$\left|\frac{z_j^n}{z_j}\right|=\left|\frac{z_j^n/j}{z_j/j}\right|<\frac{1}{2^k}\text{ whenever } j_k\leqslant j<j_k \text{ and } k\geqslant n.$$

Hence $\lim\limits_j z_j^n/z_j=0$ for each n. The second part of the claim follows from the preceding inequality. $\square$

Theorem 3.3. *The following conditions are equivalent for an FK-space (X,τ):*

(i) *X is a Cesàro wedge space.*

(ii) *X contains $V_0(z)$ for some $z \in w$ such that $z_j = o(j)$.*

(iii) *X contains $c|s|$ for some s and the inclusion mapping $I\colon (c|s|, \|\cdot\|_{c|s|}) \to (X,\tau)$ is compact.*

(iv) *X contains h and the inclusion mapping $I\colon (h, \|\cdot\|_h) \to (X,\tau)$ is compact.*

Proof. (i) $\Rightarrow$ (ii). Let $\{r_n\}_{n=1}^{\infty}$ be a defining family of seminorms for the topology τ and let

$$z_j^n := r_n(e^{(j)}) = r_n\biggl(\sum_{i=1}^{j} \delta^i\biggr); \quad j, n = 1, 2, \ldots$$

Then $\lim\limits_j z_j^n/j = 0$ $(n = 1, 2, \ldots)$ since X is a C_1-wedge space. Suppose $x \in \bigcap\limits_{n=1}^{\infty} V_0(z^n)$. Then $x \in c_0$ and

$$\sum_{j=1}^{\infty} \biggl| r_n\biggl(\sum_{i=1}^{j} \delta^i\biggr)\biggr| |\Delta x_j| = \sum_{j=1}^{\infty} r_n\biggl(\Delta x_j \sum_{i=1}^{j} \delta^i\biggr) < \infty; \quad n = 1, 2, \ldots$$

Since X is complete, $\sum\limits_{j=1}^{\infty} \Delta x_j \sum\limits_{i=1}^{j} \delta^i$ converges in (X,τ) to, say, z. But, since $X \subset w$ and X is an FK-space, $\sum\limits_{j=1}^{\infty} \Delta x_j \sum\limits_{i=1}^{j} \delta^i$ converges to x in w, i.e.

$$p_i\biggl(x - \sum_{j=1}^{p} \Delta x_j \sum_{i=1}^{j} \delta^i\biggr) \to 0; \quad p \to \infty, \ i = 1, 2, \ldots$$

Consequently $z = x$ and so $\bigcap\limits_{n=1}^{\infty} V_0(z^n) \subset X$. Choosing $z \in w$ such that $z_j = o(j)$ as in Lemma 3.2 we get $V_0(z) \subset X$.

(ii) $\Rightarrow$ (iii). Assume $V_0(z) \subset X$ for some z with $z_j = o(j)$. Let $s_0 = 0$ and let $\{s_n\}_{n=1}^{\infty}$ denote a strictly increasing sequence satisfying

$$\frac{|z_j|}{j} \leqslant \frac{1}{2^n}, \quad j \geqslant s_n; \ n = 2, 3, \ldots \tag{1}$$

Let $x \in c|s|$. Suppose $m, p \in \mathbb{N}$, $m \leqslant p$. Then using (1) we get

$$\begin{aligned}\sum_{j=s_m+1}^{s_{p+1}} |z_j|\,|\Delta x_j| &= \sum_{n=m}^{p} \sum_{j=s_n+1}^{s_{n+1}} \frac{|z_j|}{j} j |\Delta x_j| \\ &\leqslant \|x\|_{c|s|} \sum_{n=m}^{p} \frac{1}{2^n} \to 0 \quad \text{as} \quad m, p \to \infty.\end{aligned}$$

Hence $\sum_{j=1}^{\infty} |z_j|\,|\Delta x_j| < \infty$. It follows that $c|s| \subset V_0(z)$ and so $c|s| \subset X$. Now let $U \subset c|s|$ be such that $\|x\|_{c|s|} \leqslant M$ for all $x \in U$. Observe that $U \subset V_0(z)$. For $s_n < m \leqslant s_{n+1}$ and $x \in U$, by (1) we get

$$\|x - x^{(m)}\|_{V_0(z)} = \sum_{j=m+1}^{\infty} |z_j|\,|\Delta x_j| \leqslant \sum_{i=n}^{\infty} \sum_{j=s_i+1}^{s_{i+1}} \frac{z_j}{j} j|\Delta x_j| \leqslant \|x\|_{c|s|} \sum_{i=n}^{\infty} \frac{1}{2^i}.$$

Hence $x^{(m)} \to x$ in $(V_0(z), \|\cdot\|_{V_{0(z)}})$ uniformly on U. Since $V_0(z)$ is an AK-space then by Lemma 2 of [2] U is relatively compact in $V_0(z)$. Since the inclusion mapping $I\colon V_0(z) \to X$ is continuous, $I(U) = U$ is relatively compact in X. Thus the inclusion mapping $I\colon c|s| \to X$ is compact.

(iii) $\Rightarrow$ (iv). This follows immediately since the inclusion mapping $I\colon h \to c|s|$ is continuous.

(iv) $\Rightarrow$ (i). The set $Z = \{e^{(n)}/n\colon n = 1, 2, \ldots\}$ is a bounded subset of h. Since the inclusion mapping $I\colon h \to X$ is compact, $I(Z) = Z$ is τ-relatively compact in X. Thus, by Theorem 2.3.11 of [6], $e^{(n)}/n \to 0$ in (X, τ) since it converges to zero in w. $\square$

Theorem 3.4. *Let $z \in \sigma_0$. Then $z^\beta := \big\{x \in w\colon \sum_k z_k x_k$ converges$\big\}$ is a C_1-wedge FK-space.*

Proof. z^β is an FK-space with $p_n(x) = |x_n|$ $(n = 1, 2, \ldots)$, $P_0(x) = \sup_m \left|\sum_{k=1}^{m} z_k x_k\right|$ ([13], Theorem 4.3.7). It is clear that $p_n(e^{(r)}/r) \to 0$ for each n. To complete the proof we must show that $P_0(x) = \max_{1 \leqslant m \leqslant r} \frac{1}{r}\left|\sum_{k=1}^{m} z_k\right| \to 0$ as $r \to \infty$. Let $z \in \sigma_o$. Choose a sequence $\{\nu_N\}$ of natural numbers for which

$$\frac{\nu_N}{\nu_{N-1}} \geqslant 2^N \quad \text{and} \quad \frac{1}{\nu}\left|\sum_{k=1}^{\nu} z_k\right| \leqslant 2^{-N} \ \ (\forall \nu \geqslant \nu_N).$$

Then for any $N > 2$ take $r \geqslant \nu_N$; hence we have

(i) $\frac{1}{r}\left|\sum_{k=1}^{r} z_k\right| \leqslant 2^{-N}$ for $m = r$,

(ii) $\frac{m}{r}\frac{1}{m}\left|\sum_{k=1}^{m} z_k\right| \leqslant 2^{-N} \sup_m \frac{1}{m}\left|\sum_{k=1}^{m} z_k\right|$ for $m < \nu_{N-1}$,

(iii) $\frac{m}{r}\frac{1}{m}\left|\sum_{k=1}^{m} z_k\right| \leqslant 2^{-(N-1)}$ for $\nu_{N-1} \leqslant m < r$.

Thus $P_0(e^{(r)}/r) = \max\Big\{\sup_{m<\nu_{N-1}} \frac{1}{r}\left|\sum_{k=1}^{m} z_k\right|, \sup_{\nu_{N-1}\leqslant m<r} \frac{1}{r}\left|\sum_{k=1}^{m} z_k\right|, \frac{1}{r}\left|\sum_{k=1}^{r} z_k\right|\Big\}$ which tends to zero as $r \to \infty$, hence the result. $\square$

Corollary 3.5. *The intersection of all C_1-wedge FK-spaces is h.*

P r o o f. Let Y be the intersection of all C_1-wedge FK-spaces. Then by Theorems 3.3 and 3.4 we get

$$h \subset Y \subset \bigcap \{z^\beta \colon\ z \in \sigma_0\} = \sigma_0^\beta .$$

Since $\sigma_0^\beta = h$ (see [4]) the result follows immediately. □

Theorem 3.6. *Let $z_n = o(n)$. Then $V_0(z)$ is a C_1-wedge FK-space.*

P r o o f. Observe that $\|e^{(n)}/n\|_{V_0(z)} = |z_n|/n$, from which the conclusion follows at once. □

Corollary 3.7. $\bigcap\limits_{z_n = o(n)} V_0(z) = h.$

P r o o f. If $z_n = o(n)$, then by Theorem 3.6, $V_0(z)$ is a C_1-wedge FK-space. Now Theorem 3.3 (iv) yields that $h \subset \bigcap\limits_{z_n = o(n)} V_0(z)$. The reverse inclusion follows from Theorem 3.3 (ii) and Corollary 3.5. □

Theorem 3.8.

(i) *An FK-space which contains a C_1-wedge FK-space must be a C_1-wedge space.*

(ii) *A closed subspace, containing φ, of a C_1-wedge FK-space is a C_1-wedge FK-space.*

(iii) *A countable intersection of C_1-wedge FK-spaces is a C_1-wedge FK-space.*

P r o o f. The proof is easily obtained from the elementary properties of FK-spaces (see, e.g., [13], Chapter 4). □

Remark. h is not a C_1-wedge space. Hence, it follows from Corollary 3.5 that there is no smallest C_1-wedge space.

Now consider the surjection $T\colon\ w \to w$ given by

$$T(x) = \{x_1, x_1 + x_2, \dots, x_1 + x_2 + \dots + x_n, \dots\} . \tag{2}$$

Then $T^{-1}(x) = \{x_1, x_2 - x_1, \dots, x_n - x_{n-1}, \dots\}$. Let $X \subset w$ and let (X, τ) be an FK-space. The space $T^{-1}(X)$ equipped with the FK-topology τ' given by the seminorms q_k, $q_k(x) = d_k(T(x))$, where $\{d_k\}$ is the sequence of seminorm which generates the topology τ on X. In addition, $T\colon\ (T^{-1}(X), \tau') \to (X, \tau)$ is a topological isomorphism (see [2], and [6], pp. 253–254).

Theorem 3.9. *If X is a C_1-wedge FK-space then $X \cap (bs \setminus cs_0) \neq \emptyset$, where* $cs_0 = \Big\{x\colon \lim\limits_n \sum\limits_{k=1}^{n} x_k = 0\Big\}$.

P r o o f. It is clear that cs is not a C_1-wedge space and so by Theorem 3.8 (i), nor is $X \cap cs$. Theorem 3.8 (ii) shows that $X \cap cs$ is not closed in X. Consider the one-to-one mapping T of w onto itself given by (2). T maps X onto an FK-space, say F. Since $X \cap cs$ is not closed in X,

$$T(X \cap cs) = T(X) \cap T(cs) = F \cap c$$

is not closed in F. Since c_0 is of codimension 1 in c, it follows from [8, Chapter 15.8 (3)] that $c_0 \cap X$ is not closed in X. Hence $F \cap (m \setminus c_0) \neq \emptyset$ by Corollary 1 (i) of [10], and so

$$T^{-1}(F \cap (m \setminus c_0)) = X \cap T^{-1}(m \setminus c_0) = X \cap (bs \setminus cs_0) \neq \emptyset.$$

□

Theorem 3.10. *If X is a C_1-wedge FK-space then $bs \cap X$ is a non separable subspace of bs.*

P r o o f. As in the proof of Theorem 3.9 we see that $X \cap cs$ is not closed in X. So $T(X \cap cs) = T(X) \cap T(cs) = F \cap c$ is not closed in F. Now Theorem 8 of [1] implies that $m \cap F$ is a non-separable subspace of m. Since $T^{-1}(m \cap F) = T^{-1}(m) \cap T^{-1}(F) = bs \cap E$, one can easily conclude that $bs \cap E$ is a non-separable subspace of bs. □

4. Weak Cesàro wedge FK-spaces

In this section we will study weak Cesàro wedge FK-spaces.

Definition 4.1. A K-space X containing φ is called a weak Cesàro wedge space or simply a weak C_1-wedge space, if $e^{(n)}/n \to 0$ weakly in X.

It is easily seen that every weak wedge space is a weak C_1-wedge space but the converse does not hold. For example bv_0 is a weak C_1-wedge space but not a weak wedge space.

Theorem 4.2. *An FK-space X is a weak C_1-wedge space if and only if X contains h and the inclusion mapping $I\colon h \to X$ is weakly compact.*

P r o o f. If (X,τ) is a weak C_1-wedge space then, for all $f \in X'$,

$$f\left(\frac{e^{(n)}}{n}\right) = \frac{1}{n}\sum_{k=1}^{n} f(\delta^k) \to 0,$$

and so $\{f(\delta^k)\} \in \sigma_\infty$. Hence $X^f \subset \sigma_\infty$. Since $\sigma_\infty = h^f$, [4], and h is an AD-space it follows from Theorem 8.6.1 in [13] (see also [11], Theorem 4) that $h \subset X$, which implies that the inclusion mapping $I\colon\ h \to X$ is continuous. Using the fact that h is an AK-space ([4]), we have for all $x \in h$ and $f \in X'$ that

$$f\left(\sum_{k=1}^{\infty} x_k \delta^k\right) = \sum_{k=1}^{\infty} x_k f(\delta^k) = \langle I(x), f\rangle = \langle x, \{f(\delta^k)\}\rangle.$$

Since $\{f(\delta^k)\} \in \sigma_0$ and $\sigma_0' = h$ (see [4]), I is $(\sigma(h,\sigma_0)-\sigma(X,X'))$-continuous. By the Banach-Alaoglu theorem ([9], p. 61), the set $B = \{x\colon\ x \in h,\ \|x\|_h \leqslant 1\}$ is $\sigma(h,\sigma_0)$-compact and hence $I(B) = B$ is $\sigma(X,X')$-compact. Thus the inclusion mapping $I\colon\ h \to X$ is weakly compact.

Conversely, let $h \subset X$ and let $I\colon\ h \to X$ be weakly compact. Then $B = \{x\colon\ x \in h,\ \|x\|_h \leqslant 1\}$ is $\sigma(X,X')$-relatively compact. Thus, by Theorem 2.3.11 of [6], $e^{(n)}/n \to 0$ in $\sigma(X,X')$ since it converges to zero in w. □

Corollary 4.3. *The intersection of all weak C_1-wedge FK-spaces is h.*

P r o o f. Considering Theorems 3.4 and 4.2 and using the idea which we have used in Corollary 3.5, one can get the proof. □

We note in passing that Theorem 3.8 also holds for weak C_1-wedge FK-spaces.

Remark. It follows from Corollary 4.3 that there is no smallest weak C_1-wedge space since h is not a weak C_1-wedge space.

Recall that the intersection of all weak wedge (also wedge) FK-spaces is ℓ, ([2]), and that $h \subset \ell$.

Theorem 4.4. *If X is a (weak) C_1-wedge FK-space then $X \cap (cs/\ell) \neq \emptyset$.*

P r o o f. It is clear that ℓ is not a weak C_1-wedge space and so by an analogue of Theorem 3.8 (i) for weak C_1-wedge spaces, nor is $\ell \cap X$. Again an analogue of Theorem 3.8 (ii) for weak C_1-wedge spaces shows that $\ell \cap X$ is not closed in X. Then Theorem 2 (i) of [1] implies $X \cap (cs/\ell) \neq \emptyset$.

The proof is the same also for C_1-wedge FK-spaces. □

5. Applications of Cesàro wedge and weak Cesàro wedge spaces to general summability domains

In this section we shall be concerned with matrix transformations $y = Ax$, where $x, y \in w$, $A = \{a_{ij}\}_{i,j=1}^{\infty}$ is an infinite matrix with complex coefficients and $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$ $(i = 1, 2 \ldots)$.

The sequence $\{a_{ij}\}_{j=1}^{\infty}$ is called the i-th row of A and is denoted by r^i $(i = 1, 2, \ldots)$; similarly, the j-th column of A, $\{a_{ij}\}_{i=1}^{\infty}$ is denoted by k^j $(j = 1, 2, \ldots)$.

For an FK-space (E, q) we consider the summability domain E_A defined by $E_A = \{x \in w\colon\ Ax \in E\}$. An important result of Zeller ([14], Theorem 4.10) (see also [13]) asserts that E_A is an FK-space when topologized by means of the seminorms

(a) $p_i(x) = |x_i|$ $(i = 1, 2, \ldots)$,

(b) $h_i(x) = \sup\limits_m \Bigl|\sum\limits_{j=1}^{m} a_{ij}x_j\Bigr|$ $(i = 1, 2, \ldots)$,

(c) $(q \circ A)(x) = q(Ax)$.

The following theorem is an application of Theorem 3.3 to summability domains.

Theorem 5.1. *Let E be an FK-space and A a matrix. Then the following conditions are equivalent:*

(i) *E_A is a C_1-wedge space.*

(ii) *$h \subset E_A$, $r^i \in \sigma_0$ for all i, and the mapping $A\colon\ h \to E$ is compact.*

(iii) *$k^j \in E$ for all j, and $\beta_n \to 0$ as $n \to \infty$ in E, where*

$$\beta_n = \Bigl\{\frac{1}{n}\sum_{j=1}^{n} a_{ij}\colon\ i \geqslant 1\Bigr\}.$$

P r o o f. (i) $\Rightarrow$ (ii). From Theorem 3.3, (i) $\Rightarrow$ (iv), $h \subset E_A$ and the inclusion mapping $I\colon\ h \to E_A$ is compact. Also by Theorem 4.2.8 in [13], $A\colon\ E_A \to E$ is continuous. Then $A\colon\ h \to E$ which may be regarded as a composition of $I\colon\ h \to E_A$ with $A\colon\ E_A \to E$, must be compact. Since $e^{(j)}/j \to 0$ in E_A by hypothesis we get $A(e^{(j)}/j) \to 0$ in E. Therefore

$$P_i\colon\ \Bigl(A\Bigl(\frac{e^{(j)}}{j}\Bigr)\Bigr) = \Bigl(A\Bigl(\frac{1}{j}\sum_{k=1}^{j}\delta^k\Bigr)\Bigr)_i = \Bigl(\frac{1}{j}\sum_{k=1}^{j} a_{ij}\Bigr)_i \to 0$$

as $j \to \infty$ $(i = 1, 2, \ldots)$, where $P_i\colon\ E \to K$ $(= \mathbb{R}$ or $\mathbb{C})$ is defined by $P_i(u) = u_i$. Hence $r^i \in \sigma_0$, $\forall i \geqslant 1$.

(ii) $\Rightarrow$ (iii). Obviously $k^j = A(\delta^j) \in E$ $(j = 1, 2, \ldots)$ since $\delta^j \in h$ $(j = 1, 2, \ldots)$. Observe that $B = \{e^{(n)}/n\colon\ n = 1, 2, \ldots\}$ is bounded in h, and by hypothesis $A\colon\ h \to$

E is compact, hence $A(B) = \{A(e^{(n)}/n)\colon\ n = 1,2,\ldots\}$ is relatively compact in E. Since $r^i \in \sigma_0$ for all i, by Theorem 2.3.11 of [6] we have

$$A\Big(\frac{e^{(n)}}{n}\Big) = \Big(\Big(\frac{1}{n}\sum_{j=1}^{n} a_{ij}\Big)_i\Big) \to 0 \text{ in } E.$$

(iii) $\Rightarrow$ (i). By hypothesis $k^j = A(\delta^j) \in E$ $(j = 1,2,\ldots)$. Clearly we have, for fixed i, that $p_i(e^{(n)}/n) \to 0$ $(n\to\infty)$. Since $A\Big(\frac{1}{n}\sum\limits_{j=1}^{n}\delta^j\Big) \to 0$ $(n\to\infty)$ in E, we also have $(q\circ A)(e^{(n)}/n) = q\Big(A\Big(\frac{1}{n}\sum\limits_{j=1}^{n}\delta^j\Big)\Big) \to 0$ $(n\to\infty)$.

We now show that $h_i(e^{(n)}/n) = \max\limits_{1\leqslant m\leqslant n}\frac{1}{n}\Big|\sum\limits_{j=1}^{m} a_{ij}\Big| \to 0$ $(i = 1,2,\ldots)$. Since $A\Big(\frac{1}{n}\sum\limits_{j=1}^{n}\delta^j\Big) \to 0$ $(n\to\infty)$ in E and E is an FK-space, the same holds also in w. Hence

$$p_i\Big(A\Big(\frac{1}{n}\sum_{j=1}^{n}\delta^j\Big)\Big) = \Big|\frac{1}{n}\sum_{j=1}^{n} a_{ij}\Big| \to 0 \quad (n\to\infty,\ i = 1,2,\ldots).$$

Using the idea that we have used in Theorem 3.4, one can show that $h_i(e^{(n)}/n) \to 0$ $(n\to\infty,\ i = 1,2,\ldots)$. Thus E_A is a C_1-wedge space. $\square$

Corollary 5.2. *$m_A(c_A, (c_0)_A)$ is a C_1-wedge space if and only if the following conditions are satisfied:*

(i) $\sup\limits_i |a_{ij}| < \infty$ $(j = 1,2,\ldots)$
   ($\lim\limits_i a_{ij}$ *exists,* $\lim\limits_i a_{ij} = 0$ *respectively*);

(ii) $\lim\limits_{n\to\infty}\sup\limits_i\Big|\frac{1}{n}\sum\limits_{j=1}^{n} a_{ij}\Big| = 0.$

P r o o f. This is just Theorem 5.1, (i) $\Leftrightarrow$ (iii), with $E = m(c, c_0)$. $\square$

The following theorem is an application of Theorem 4.2 to summability domains.

Theorem 5.3. *Let E be an FK-space and A a matrix. Then the following conditions are equivalent:*

(i) *E_A is a weak C_1-wedge space.*

(ii) *$h \subset E_A$, $r^i \in \sigma_0$ for all i, and the mapping $A\colon\ h \to E$ is weakly compact.*

(iii) *$k^j \in E$ for all j, and the sequence $\{\beta_n\colon\ n \geqslant 1\}$ converges weakly to zero in E, where*

$$\beta_n = \Big\{\frac{1}{n}\sum_{j=1}^{n} a_{ij}\colon\ i \geqslant 1\Big\}.$$

P r o o f. (i) $\Rightarrow$ (ii). Theorem 4.2 implies $h \subset E_A$ and the inclusion mapping $I\colon h \to E_A$ is weakly compact. Also $A\colon E_A \to E$ is weakly continuous. Thus $A\colon h \to E$, where $A = A \circ I$ is weakly compact. Since $e^{(n)}/n \to 0$ (weakly) in E_A by hypothesis, we have $A(e^{(n)}/n) \to 0$ (weakly) in E. Since the coordinate functionals $P_i\colon E \to K$, defined by $P_i(x) = x_i$, are continuous,

$$P_i\Big(A\Big(\frac{e^{(n)}}{n}\Big)\Big) = \Big(A\Big(\frac{1}{n}\sum_{j=1}^{n}\delta^j\Big)\Big)_i = \frac{1}{n}\sum_{j=1}^{n} a_{ij} \to 0 \quad (n \to \infty), \ \ (i = 1, 2, \ldots).$$

Thus $r^i \in \sigma_0$, $(i = 1, 2, \ldots)$.

(ii) $\Rightarrow$ (iii). As in the proof of Theorem 5.1, (ii) $\Rightarrow$ (iii), $A(B) = \{A(e^{(n)}/n)\colon n = 1, 2, \ldots\}$ is weakly relatively compact in E. Thus, since $r^i \in \sigma_0$ $(i = 1, 2, \ldots)$, by Theorem 2.3.11 of [6] we have

$$A\Big(\frac{1}{n}\sum_{j=1}^{n}\delta^j\Big) = \Big(\Big(\frac{1}{n}\sum_{j=1}^{n} a_{ij}\Big)_i\Big) \to 0 \quad \text{(weakly) in } E.$$

(iii) $\Rightarrow$ (i). The condition $k^j = A(\delta^j) \in E$ implies that $\varphi \subset E_A$. By Theorem 4.4.2 in [13], $f \in E_A'$ if and only if $f(x) = \alpha x + g(Ax)$ for all $x \in E_A$, where $\alpha \in w_A^\beta$, $g \in E'$, $\alpha x = \sum\limits_{k=1}^{\infty} \alpha_k x_k$ and $|\alpha x| \leqslant M h_i(x)$ $(i = 1, 2, \ldots)$ for some $M > 0$. Since $A(e^{(n)}/n) \to 0$ (weakly) in E by hypothesis, we have for all $g \in E'$ that $g\big(A(e^{(n)}/n)\big) \to 0$ $(n \to \infty)$. As in the proof of Theorem 5.1, (iii) $\Rightarrow$ (i), we get $h_i(e^{(n)}/n) \to 0$. Hence it follows from the inequality $\big|\alpha(e^{(n)}/n)\big| \leqslant M h_i(e^{(n)}/n)$ $(i = 1, 2, \ldots)$ that $\alpha(e^{(n)}/n) \to 0$ $(n \to \infty)$. We have already shown that $f(e^{(n)}/n) \to 0$ $(n \to \infty)$, from which the result follows. $\square$

Corollary 5.4. *m_A is a weak C_1-wedge space if and only if the following conditions are satisfied:*

(i) $\sup\limits_{i,n}\Big|\frac{1}{n}\sum\limits_{j=1}^{n} a_{ij}\Big| < \infty$;

(ii) *given $\varepsilon > 0$ and an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers, there exists L (depending only on ε and $\{n_k\}_{k=1}^{\infty}$) such that*

$$\sup_i \min_{1 \leqslant r \leqslant L}\Big|\frac{1}{n_{k_r}}\sum_{j=1}^{n_{k_r}} a_{ij}\Big| < \varepsilon.$$

P r o o f. This follows by putting $E = m$ in Theorem 5.3, (i) $\Leftrightarrow$ (iii), and using the characterization of weak sequential convergence in m given in [3], IV, 6.3, p. 281. $\square$

Our next result follows immediately from Theorems 5.1 and 5.3.

Theorem 5.5. *Let E be an FK-space such that weakly convergent sequences are convergent in the FK-topology and let A be a matrix. Then E_A is a C_1-wedge space if and only if it is a weak C_1-wedge space.*

In particular, Theorem 5.5 holds when $E = \ell$, bv.

Corollary 5.6. *The following conditions are equivalent for any matrix A:*
(i) *ℓ_A is a (weak) C_1-wedge space.*
(ii) $\lim\limits_n \sum\limits_{i=1}^{\infty} \left| \frac{1}{n} \sum\limits_{j=1}^{n} a_{ij} \right| = 0.$

Proof. This is just Theorem 5.1, (i) $\Leftrightarrow$ (iii), and Theorem 5.5 with $E = \ell$. $\square$

Proposition 5.7. *If A is a sum preserving ℓ-ℓ method then ℓ_A is not a C_1-wedge space.*

Proof. Recall that A is a sum preserving ℓ-ℓ method if and only if

$$\text{(a)}\ \sup_k \sum_{n=1}^{\infty} |a_{nk}| < \infty, \qquad \text{(b)}\ \sum_{n=1}^{\infty} a_{nk} = 1 \quad (k = 1, 2, \ldots),$$

see [7]. Using (b) we get

$$\sum_{i=1}^{\infty} \left| \frac{1}{n} \sum_{j=1}^{n} a_{ij} \right| \geqslant \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{\infty} a_{ij} = 1.$$

Thus $\sum\limits_{i=1}^{\infty} \left| \frac{1}{n} \sum\limits_{j=1}^{n} a_{ij} \right| \nrightarrow 0$ $(n \to \infty)$. By Corollary 5.6, ℓ_A is not a C_1-wedge space. $\square$

Corollary 5.8. *The following conditions are equivalent for any matrix A:*
(i) *bv_A is a (weak) C_1-wedge space.*
(ii) $\lim\limits_{n\to\infty} \left\{ \sum\limits_{i=1}^{\infty} \left| \frac{1}{n} \sum\limits_{j=1}^{n} (a_{ij} - a_{i+1,j} \right| + \lim\limits_i \left| \frac{1}{n} \sum\limits_{j=1}^{n} a_{ij} \right| \right\} = 0.$

Proof. This follows from Theorem 5.1, (i) $\Leftrightarrow$ (iii), and Theorem 5.5 with $E = bv$. $\square$

Acknowlegment. The author is indebted to Prof. G. Bennett of Indiana University (USA) for suggesting the subject, and to Prof. J. Boos of Fernuniversität Hagen (Germany) for his assistance.

References

[1] *G. Bennett*: The Glinding Humps technique for FK-spaces. Trans. Amer. Math. Soc. *166* (1972), 285–292.

[2] *G. Bennett*: A new class of sequence spaces with applications in summability theory. J. Reine Angew. Math. *266* (1974), 49–75.

[3] *N. Dunford and J. T. Schwartz*: Linear Operators. Interscience Publishers, New York, 1958.

[4] *G. Goes and S. Goes*: Sequences of bounded variation and sequences of Fourier coefficients. I. Math. Z. *118* (1970), 93–102.

[5] *G. Goes*: Sequences of bounded variation and sequences of Fourier coefficients. II. J. Math. Anal. Appl. *39* (1972), 477–494.

[6] *P. K. Kamthan and M. Gupta*: Sequence Spaces and Series. Marcel Dekker, New York, Basel, 1981.

[7] *K. Knopp and G. G. Lorentz*: Beiträge zur absoluten Limitierung. Arch. Math. *2* (1949), 10–16.

[8] *G. Köthe*: Topological Vector Spaces I. Springer-Verlag, New York, 1969.

[9] *A. P. Robertson and W. J. Robertson*: Topological Vector Spaces. University Press, Cambridge, 1964.

[10] *A. K. Snyder*: An embedding property of sequence spaces related to Meyer-König and Zeller type theorems. Indiana Univ. Math. J. *35* (1986), 669–679.

[11] *A. K. Snyder and A. Wilansky*: Inclusion theorems and semiconservative FK-spaces. Rocky Mountain J. Math. *2* (1972), 595–603.

[12] *A. Wilansky*: Functional Analysis. Blaisdell Press, New York-Toronto-London, 1964.

[13] *A. Wilansky*: Summability Through Functional Analysis. North Holland, Amsterdam-New York-Oxford, 1984.

[14] *K. Zeller*: Allgemeine Eigenschaften von Limitierungsverfahren. Math. Z. *53* (1951), 463–487.

[15] *K. Zeller*: Theorie der Limitierungsverfahren. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958.

Author's address: Ankara University, Faculty of Science, Department of Mathematics, 06100 Tandogan, Ankara, Turkey, e-mail: `ince@science.ankara.edu.tr`.