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J. Bonet; Susanne Dierolf; J. Wengenroth

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STRONG DUALS OF PROJECTIVE LIMITS OF (LB)-SPACES

J. Bonet, Valencia, S. Dierolf and J. Wengenroth, Trier

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Abstract. We investigate the problem when the strong dual of a projective limit of (LB)-spaces coincides with the inductive limit of the strong duals. It is well-known that the answer is affirmative for spectra of Banach spaces if the projective limit is a quasinormable Fréchet space. In that case, the spectrum satisfies a certain condition which is called "strong P-type". We provide an example which shows that strong P-type in general does not imply that the strong dual of the projective limit is the inductive limit of the strong duals, but on the other hand we show that this is indeed true if one deals with projective spectra of retractive (LB)-spaces. Finally, we apply our results to a question of Grothendieck about biduals of (LF)-spaces.

Keywords: derived projective limit functor, Retakh's condition, weakly acyclic (LF)-spaces

MSC 2000: 46A13, 46M15

1. Introduction

The duality between countable projective limits and countable inductive limits has played a significant role in the theory of locally convex spaces and its applications. We refer for example to Section 1 of Komatsu's article [10]. The naive idea that the strong dual of a Fréchet space (i.e. a reduced countable projective limit of Banach spaces) must be the countable inductive limit of the strong duals is false in general, as was shown by Grothendieck and Köthe, cf. [11, §27]. The Fréchet spaces for which this idea really works are called distinguished. They had been studied by Dieudonné, Schwartz and Grothendieck, but they were thoroughly investigated in the 80's; we refer e.g. to the survey article [3]. Investigations of operators defined on spaces with a more complicated structure, like the spaces of ultradifferentiable functions of Roumieu type, motivated important advances in the theory of projective spectra of (LB)-spaces and its applications to surjectivity problems. In fact recent progress in the theory of projective spectra of (LB)-spaces [6, 8, 16, 17, 19, 20] has led to many applications in analysis to surjectivity problems e.g. in [5], existence of right inverses [8,15], or vector-valued real-analytic functions [4]. In this article we analyze the following natural question: when is the strong dual of a projective limit of inductive limits of Banach spaces the inductive limit of the projective limits of the strong duals of the Banach spaces? Applications, examples and related results for weighted (LF)-spaces of holomorphic functions and infinite dimensional holomorphy can be found in [1].

We use standard notation for locally convex spaces like in [11, 14]. For a locally convex space X we denote by $\mathcal{U}_0(X)$ and $\mathcal{B}(X)$ the systems of absolutely convex 0-neighbourhoods and bounded sets, respectively.

By a projective spectrum $\mathcal{X} = (X_n, \varrho_m^n)$ we mean a sequence $(X_n)_{n\in\mathbb{N}}$ of linear spaces (over the same field of real or complex numbers) and linear spectral maps $\varrho_m^n\colon X_m \to X_n$, $n \leqslant m$, satisfying

$$\varrho_m^n \circ \varrho_k^m = \varrho_k^n \quad \text{and} \quad \varrho_n^n = \mathrm{id}_{X_n} \quad \text{for } n \leqslant m \leqslant k.$$

The projective limit is defined as

$$\operatorname{Proj}\mathcal{X} = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n\colon\ \varrho_m^n(x_m) = x_n \quad \text{for all } n \leqslant m \right\},$$

and it is endowed with the topology induced by the product if the X_n are locally convex spaces and the spectral maps are continuous; moreover, $\varrho^n\colon \operatorname{Proj}\mathcal{X} \to X_n$ denotes the canonical projection onto the n$^{\text{th}}$ component. We always set

$$\Psi\colon \prod_{\mathbb{N}} X_n \to \prod_{\mathbb{N}} X_n, \quad \Psi((x_n)_{n\in\mathbb{N}}) = \big(x_n - \varrho_{n+1}^n(x_{n+1})\big)_{n\in\mathbb{N}}$$

and $\operatorname{Proj}^1\mathcal{X} = \prod_{\mathbb{N}} X_n / \operatorname{Im}(\Psi)$.

A projective spectrum $\mathcal{X} = (X_n, \varrho_m^n)$ consisting of locally convex spaces is said to be of strong P-type if

$$\forall\, n \in \mathbb{N}\ \exists\, B_n \in \mathcal{B}(X_n), m \geqslant n\ \forall\, k \geqslant m, D \in \mathcal{B}(X_m)\ \exists\, K \in \mathcal{B}(X_k)$$
$$\varrho_m^n(D) \subseteq \varrho_k^n(K) + B_n.$$

In [8] it is shown that in this case the sequence B_n can be chosen to satisfy, in addition, $\varrho_{n+1}^n(B_{n+1}) \subseteq B_n$, and then a result of Vogt [16, Theorem 4.9] implies that Ψ lifts bounded sets if $\mathcal{X}$ consists of regular (LB)-spaces; in particular $\operatorname{Proj}^1\mathcal{X} = 0$. If a projective spectrum of strong P-type consists of Banach spaces, its projective limit is a quasinormable Fréchet space, hence its strong dual coincides with the inductive limit of the strong duals. A locally convex space is called distinguished if its strong dual is barrelled. A metrizable locally convex space is distinguished if and only if its strong dual is bornological. Every quasinormable Fréchet space is distinguished; cf. [11].

It could have been hoped that also in the general case the strong P-type condition would be sufficient to deduce that the strong dual of $\operatorname{Proj}\mathcal{X}$ equals the inductive limit of the strong duals. We present in Section 3 an example which destroys this hope. On the other hand, we show in Theorem 1 that the desired result holds for projective limits of retractive (LB)-spaces. This has consequences for a problem of Grothendieck [9, Question non résolue 8] whether the bidual of a strict (LF)-space is again an (LF)-space.

2. Projective spectra of retractive (LB)-spaces

An inductive limit $E = \operatorname{ind}_n E_n$ is called (sequentially) retractive if every null sequence in E is a null sequence in some step. For (LF)-spaces this sequential retractivity is equivalent to many other regularity conditions (like bounded retractivity or acyclicity, see [19]) and, in particular, an (LF)-space is retractive if and only if it is regular (i.e. every bounded set in E is bounded in some E_n) and satisfies the strict Mackey condition. We recall that according to Grothendieck [9], a locally convex space X satisfies the strict Mackey condition (sMc) if each bounded set A of X is contained in an absolutely convex bounded set B whose Minkowski functional induces the original topology on A.

Theorem 1. *Let $\mathcal{X} = (X_n, \varrho^n_m)$ be a projective spectrum of retractive (LB)-spaces which is of strong P-type. Then the strong dual of $\operatorname{Proj}\mathcal{X}$ is bornological and equals the inductive limit $\operatorname{ind}_n(X_n', \beta(X_n', X_n))$.*

Note that the inductive limit of the strong duals need not be injective. However, in most interesting cases, $\varrho^n(\operatorname{Proj}\mathcal{X})$ is dense in X_n. We call such a projective spectrum reduced, and in the reduced case, the inductive limit of the duals is indeed injective.

To prove the theorem we start with a lemma of Meise and Vogt [11, Lemma 26.10]. There, it is only formulated for Fréchet spaces, but the version we need follows with exactly the same proof.

Lemma 1. *Let $0 \to E \overset{i}{\to} F \overset{q}{\to} G \to 0$ be an algebraically exact sequence of locally convex spaces such that i is a topological embedding and q is continuous. If* *the condition*

$$(\star) \qquad \forall\, B \in \mathcal{B}(G)\ \exists\, D \in \mathcal{B}(F)\ \forall\, U \in \mathcal{U}_0(F)\ \exists\, V \in \mathcal{U}_0(G) \quad B \cap V \subseteq q(D \cap U)$$

holds, then $i^t\colon F'_\beta \to E'_\beta$ *is open.*

It is easy to see that $(\star)$ implies that q lifts bounded sets, i.e. every bounded set in G is contained in the image of some bounded subset of F (put $U = F$ in condition $(\star)$).

Our next simple result gives a partial converse. It also yields a different proof of [11, Lemma 26.11].

Lemma 2. *Let* $0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$ *be as in Lemma* 1 *and assume that* q *lifts bounded sets and* G *satisfies the strict Mackey condition. Then* $i^t\colon F'_\beta \to E'_\beta$ *is open.*

Proof. Given $B \in \mathcal{B}(G)$ there is $K \in \mathcal{B}(G)$ with $B \subseteq K$ such that the Minkowski functional p_K induces the same topology on B as the topology of G. Choose $D \in \mathcal{B}(F)$ with $K \subseteq q(D)$. Given $U \in \mathcal{U}_0(F)$ we find $\varepsilon \in (0,1)$ with $\varepsilon D \subseteq U$. Using the coincidence of the topologies above, we find $V \in \mathcal{U}_0(G)$ with $B \cap V \subseteq \varepsilon K$. We obtain

$$B \cap V \subseteq \varepsilon K \subseteq \varepsilon q(D) = q(\varepsilon D \cap U) \subseteq q(D \cap U).$$

Now, Lemma 1 gives the conclusion. □

If q lifts bounded sets, then clearly $q^t\colon G'_\beta \to F'_\beta$ is a homomorphism (it would be sufficient that q lifts bounded sets with closure). Therefore, we have the following consequence:

Corollary 2. *Let* $0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$ *be a topologically exact sequence of locally convex spaces such that* G *satisfies the strict Mackey condition and* q *lifts bounded sets. Then the dual sequence* $0 \to G'_\beta \xrightarrow{q^t} F'_\beta \xrightarrow{i^t} E'_\beta \to 0$ *is again topologically exact.*

Proof of Theorem 1. Let $E = \operatorname{Proj} \mathcal{X}$, $F = G = \prod_{\mathbb{N}} X_n$, let i be the canonical embedding and $q = \Psi$. Clearly, E is a topological subspace and q is continuous. Since $\mathcal{X}$ is of strong P-type, q lifts bounded sets and, in particular, $0 \to E \xrightarrow{i} F \xrightarrow{q} G \to 0$ is algebraically exact.

Each X_n is a retractive (LB)-space and therefore satisfies (sMc). Since (sMc) is stable with respect to countable products (see e.g. [14, Proposition 5.1.31]), G satisfies (sMc). Hence, Lemma 2 implies that $i^t\colon \left(\prod_{\mathbb{N}} X_n\right)'_\beta \to (\operatorname{Proj} \mathcal{X})'_\beta$ is open and clearly continuous. This proves that $(\operatorname{Proj}\mathcal{X})'_\beta$ is a quotient of the (LF)-space $\left(\prod_{\mathbb{N}} X_n\right)'_\beta = \bigoplus_{\mathbb{N}} X'_{n,\beta}$ and therefore itself an (LF)-space. This gives the conclusion.

□

In Section 3 we show that Theorem 1 is false without the assumption that each step is a retractive (LB)-space. What can be said in the general case is contained in our next result which is proved by a classical Mittag-Leffler argument, the proof of which is inspired by [17].

Proposition 3. *Let $\mathcal{X} = (X_n, \varrho^n_m)$ be a reduced projective spectrum of (LB)-spaces which is of strong P-type. Then there are $U_n \in \mathcal{U}_0(X'_{n,\beta})$ with $U_n \subseteq U_{n+1}$ and for each $n \in \mathbb{N}$ there is $m \geqslant n$ such that $\beta(X'_m, X_m)$ and $\beta((\operatorname{Proj}\mathcal{X})', \operatorname{Proj}\mathcal{X})$ coincide on U_n.*

P r o o f. By what we have said above, there are $B_n \in \mathcal{B}(X_n)$ with $\varrho^n_{n+1}(B_{n+1}) \subseteq B_n$ such that

$$\forall\, n \in \mathbb{N}\ \exists\, m \geqslant n\ \forall\, k \geqslant m, D \in \mathcal{B}(X_m)\ \exists\, K \in \mathcal{B}(X_k)$$
$$\varrho^n_m(D) \subseteq \varrho^n_k(K) + B_n.$$

By passing to a subsequence of the projective spectrum—which does not change the projective limit—we may reach that $m = n+1$ satisfies this condition. Replacing D by $\varepsilon^{-1}D$ we obtain for every positive ε

$$\forall\, k \geqslant n+1, D \in \mathcal{B}(X_{n+1})\ \exists\, K \in \mathcal{B}(X_k) \quad \varrho^n_{n+1}(D) \subseteq \varrho^n_k(K) + \varepsilon B_n.$$

We will show that for each $D \in \mathcal{B}(X_{n+1})$ there is $A \in \mathcal{B}(\operatorname{Proj}\mathcal{X})$ such that

$$(+) \qquad \varrho^n_{n+1}(D) \subseteq \varrho^n(A) + B_n$$

holds. Then, the assertion of the proposition will follow by taking polars. Without loss of generality let $n = 1$ and let us fix $D \in \mathcal{B}(X_2)$. Then there is $A_3 \in \mathcal{B}(X_3)$ with

$$\varrho^1_2(D) \subseteq \varrho^1_3(A_3) + 2^{-1}B_1.$$

Proceeding by induction, we find bounded sets $A_k \subseteq X_k$, $k \geqslant 3$ such that

$$\varrho^k_{k+1}(A_{k+1}) \subseteq \varrho^k_{k+2}(A_{k+2}) + 2^{-k}B_k.$$

Now, fix $x \in \varrho^1_2(D)$ and find inductively $a_k \in A_k$, $k \geqslant 3$ with

$$x - \varrho^1_3(a_3) \in 2^{-1}B_1 \quad \text{and} \quad \varrho^k_{k+1}(a_{k+1}) - \varrho^k_{k+2}(a_{k+2}) \in 2^{-k}B_k.$$

For $\nu \in \mathbb{N}$ define $\xi_\nu = \varrho^\nu_{\nu+2}(a_{\nu+2}) + \sum\limits_{k \geqslant \nu+2} \varrho^\nu_{k+1}(a_{k+1}) - \varrho^\nu_k(a_k)$. Since

$$\begin{aligned}\varrho^\nu_{k+1}(a_{k+1}) - \varrho^\nu_k(a_k) &= \varrho^\nu_{k-1}\big(\varrho^{k-1}_{k+1}(a_{k+1}) - \varrho^{k-1}_k(a_k)\big) \\ &\in \varrho^\nu_{k-1}(2^{-k+1}B_{k-1}) \subseteq 2^{-k+1}B_\nu,\end{aligned}$$

the series above converges with respect to the complete topology $\mathcal{R}_\nu$ induced by the Minkowski functional of B_ν on X_ν, the limit ξ_ν satisfies $\xi_\nu = \lim\limits_{k\to\infty} \varrho^\nu_k(a_k)$ and belongs to $\varrho^\nu_{\nu+2}(A_{\nu+2}) + B_\nu$. The continuity of $\varrho^\nu_{\nu+1}$ with respect to the new topologies $\mathcal{R}_\nu$ (here we use $\varrho^\nu_{\nu+1}(B_{\nu+1}) \subseteq B_\nu$) implies $\xi = (\xi_\nu)_{\nu\in\mathbb{N}} \in \operatorname{Proj}\mathcal{X}$, hence

$$\xi \in \bigcap_{\nu\in\mathbb{N}} (\varrho^\nu)^{-1}\big(\varrho^\nu_{\nu+2}(A_{\nu+2}) + B_\nu\big) =: A,$$

which is a bounded set in $\operatorname{Proj}\mathcal{X}$. Moreover, $x - \varrho^1(\xi) = x - \xi_1 = x - \varrho^1_3(a_3) + \sum\limits_{k\geqslant 3} \varrho^1_{k+1}(a_{k+1}) - \varrho^1_k(a_k) \in 2^{-1}B_1 + 2^{-1}B_1 = B_1$. We have shown (+) and this gives the conclusion. □

Corollary 4. *Let $\mathcal{X} = (X_n, \varrho^n_m)$ be a reduced projective spectrum of (LB)-spaces which is of strong P-type. Then*

1. $\operatorname{Proj}\mathcal{X}$ *is quasinormable,*
2. $(\operatorname{Proj}\mathcal{X})'_\beta = \operatorname{ind}_n X'_{n,\beta}$ *if and only if* $(\operatorname{Proj}\mathcal{X})'_\beta$ *is* $\aleph_0$*-quasibarrelled.*

P r o o f. By [17, Theorem 3.4], $\operatorname{Proj}\mathcal{X}$ is barrelled, hence it is quasinormable if and only if its strong dual satisfies (sMc). $(\operatorname{Proj}\mathcal{X})'_\beta$ and $\operatorname{ind}_n X'_{n,\beta}$ have the same bounded sets and the latter space is a retractive (LF)-space, hence it satisfies (sMc) and it is regular. Now, Proposition 3 implies that the strong topology and the inductive topology coincide on the bounded sets of $(\operatorname{Proj}\mathcal{X})'$ and therefore $(\operatorname{Proj}\mathcal{X})'_\beta$ also satisfies (sMc).

The second part follows from Proposition 3 by the theory of generalized inductive limits, see [14, Chapteers 8.1, 8.2] and in particular Proposition 8.2.4 there. □

3. An example

We will now construct a class of examples which show that Theorem 1 is not true without the assumption that the steps of the projective spectrum are retractive. This will be done in the frame of the so-called projective limits of Moscatelli type which had been used for many counterexamples in Fréchet space theory and also in [2, 7] to clarify the duality between (LF)-spaces and projective limits of (LB)-spaces.

Let E, F be locally convex spaces and $f\colon E \to F$ a continuous linear map with dense range. For $n \in \mathbb{N}$ we define

$$X_n = \prod_{k<n} E \times \bigoplus_{k\geqslant n} F, \quad \varrho^n_{n+1} = \prod_{k<n} \mathrm{id}_E \times f \times \prod_{k>n} \mathrm{id}_F$$

and $\varrho^n_m = \varrho^n_{n+1} \circ \dots \varrho^{m-1}_m$. We set $\mathcal{X}(E \xrightarrow{f} F) = (X_n, \varrho^n_m)$ and $X = X(E \xrightarrow{f} F) = \mathrm{Proj}(X_n, \varrho^n_m)$. It is easy to check that we have a canonical isomorphism

$$X(E \xrightarrow{f} F) \cong \left\{ (y_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}\colon\ (f(y_n))_{n\in\mathbb{N}} \in \bigoplus_{\mathbb{N}} F \right\}$$

if we endow the second space with the initial topology with respect to the inclusion $X \hookrightarrow E^{\mathbb{N}}$ and $f^{\mathbb{N}}\big|_X\colon X \to \bigoplus_{\mathbb{N}} F$. This isomorphic space has a 0-neighbourhood basis

$$\left\{ \prod_{k<n} V \times \bigoplus_{k\geqslant n} f^{-1}(U_k)\colon\ n \in \mathbb{N},\ \ V \in \mathcal{U}_0(E),\ \ U_k \in \mathcal{U}_0(F) \right\},$$

and a fundamental system of bounded sets is given by

$$\left\{ \prod_{k<n} B_k \times \prod_{k\geqslant n} (B_k \cap \ker(f))\colon\ n \in \mathbb{N},\ \ B_k \in \mathcal{B}(E) \right\}.$$

The strong dual of X_n is $\prod_{k<n} E'_\beta \times \prod_{k\geqslant n} F'_\beta$, and for the inductive limit of the strong duals (with respect to $(\varrho^n_{n+1})^t$) we have again a canonical isomorphism

$$X'_{\mathrm{ind}} := \operatorname*{ind}_n X'_{n,\beta} \cong \bigoplus_{\mathbb{N}} E' + f^t(F')^{\mathbb{N}} \subseteq (E')^{\mathbb{N}} \quad \text{where the canonical map}$$

$$\bigoplus_{\mathbb{N}} E'_\beta \times \prod_{\mathbb{N}} F'_\beta \to X'_{\mathrm{ind}}$$

$$\big((\varphi_n)_{n\in\mathbb{N}}, (\psi_n)_{n\in\mathbb{N}}\big) \mapsto \big(\varphi_n + f^t(\psi_n)\big)_{n\in\mathbb{N}}$$

is continuous and open. It is shown in [7] that the inductive spectrum $X'_{n,\beta}$ is strict if and only if f lifts bounded sets with closure, whereas $\mathrm{Proj}^1\, \mathcal{X}(E \xrightarrow{f} F) = 0$ if and only if f is surjective. The following proposition complements this result. Its proof is very similar to [7, Proposition 1] and therefore omitted.

Proposition 5. *$\mathcal{X}(E \xrightarrow{f} F)$ is of strong P-type if and only if f lifts bounded sets.*

Proposition 6. *For $X = X(E \xrightarrow{f} F)$ we have $X'_\beta = X'_{\text{ind}}$ if and only if for each $B \in \mathcal{B}(E)$ there is $A \in \mathcal{B}(E)$ with*

$$\overline{f^t(F') + A^\circ}^{\sigma(E',E)} \subseteq f^t(F') + B^\circ.$$

P r o o f. We use the identifications explained above. The duality between $X \cong \Big\{(y_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}} \colon\ (f(y_n))_{n\in\mathbb{N}} \in \bigoplus_{\mathbb{N}} F\Big\}$ and $X' \cong \bigoplus_{\mathbb{N}} E' + f^t(F')^{\mathbb{N}}$ is given by

$$\langle (y_n)_{n\in\mathbb{N}}, (\varphi_n + f^t(\psi_n))_{n\in\mathbb{N}}\rangle = \sum_{n\in\mathbb{N}} \varphi_n(y_n) + \psi_n(f(y_n)).$$

Let now $X'_\beta = X'_{\text{ind}}$ and $B \in \mathcal{B}(E)$ be given. Then $U := \bigoplus_{\mathbb{N}} B^\circ + \prod_{\mathbb{N}} f^t(F')$ is a 0-neighbourhood in X'_{ind}. Hence there is $D \in \mathcal{B}(X)$ with $D^\circ \subseteq U$. There are $n \in \mathbb{N}$ and $A_k \in \mathcal{B}(E)$, $k \in \mathbb{N}$, such that

$$D \subseteq \prod_{k<n} A_k \times \prod_{k\geqslant n} (A_k \cap \ker(f)) =: A.$$

Denote by pr_n and π_n the projections onto the n-th component on $E^{\mathbb{N}}$ and $(E')^{\mathbb{N}}$, respectively. Then we have $\mathrm{pr}_n(A) = A_n \cap \ker(f)$ and hence

$$\prod_{k<n} \{0\} \times (A_n \cap \ker(f))^\circ \times \prod_{k\geqslant n} \{0\} \subseteq A^\circ.$$

Therefore

$$\pi_n(A^\circ) \supseteq (A_n \cap \ker(f))^\circ = \overline{\Gamma\big(A_n^\circ \cup (\ker(f))^\circ\big)}^{\sigma(E',E)} \supseteq \overline{f^t(F') + (2A_n)^\circ}^{\sigma(E',E)}.$$

But, $B^\circ + f^t(F') = \pi_n(U) \supseteq \pi_n(A^\circ)$, and this implies the only if part.

To show the if part let U be a 0-neighbourhood in $X'_{\text{ind}} \cong \bigoplus_{\mathbb{N}} E' + f^t(F')^{\mathbb{N}}$. Then there are $B_k \in \mathcal{B}(E)$ and $n \in \mathbb{N}$ such that

$$U \supseteq \bigoplus_{\mathbb{N}} B_k^\circ + \prod_{k<n} \{0\} \times \prod_{k\geqslant n} F' =: V.$$

Choose $A_k \in \mathcal{B}(E)$ with $B_k \subseteq A_k$ and $\overline{f^t(F') + A_k^\circ}^{\sigma(E',E)} \subseteq f^t(F') + B_k^\circ$, and set

$$A := \prod_{k<n} A_k \times \prod_{k\geqslant n} (A_k \cap \ker(f)).$$

Then A is bounded in X and $A^\circ \subseteq V$, because for $(\psi_k)_{k\in\mathbb{N}} \in A^\circ$ and $k < n$ we have $\psi_k \in A_k^\circ \subseteq B_k^\circ$ and for $k \geqslant n$ we have

$$\psi_k \in (A_k \cap \ker(f))^\circ = \overline{\Gamma(A_k^\circ \cup \overline{f^t(F')})} \subseteq \overline{A_k^\circ + f^t(F')} \subseteq B_k^\circ + f^t(F').$$

This proves that U contains a strong neighbourhood of zero, and since the inductive topology is always finer than the strong topology we obtain $X'_{\mathrm{ind}} = X'_\beta$. □

Next, we investigate the condition of the last proposition in a special situation. Recall that an inductive limit $F = \mathrm{ind}_n F_n$ is weakly acyclic if the spectrum $\mathcal{X} = (F'_n, r^n_m)$, where r^n_m denotes the transpose of the inclusion $F_n \hookrightarrow F_m$, satisfies $\mathrm{Proj}^1 \mathcal{X} = 0$. For more information about such inductive limits we refer to [13, 18].

Lemma 3. *Let $F = \mathrm{ind}_n F_n$ be an (LB)-space, $E = \bigoplus_{\mathbb{N}} F_n$, and $f\colon E \to F$, $(x_n)_{n\in\mathbb{N}} \mapsto \sum_n x_n$. Then the following conditions are equivalent:*

1. *For each $B \in \mathcal{B}(E)$ there is $A \in \mathcal{B}(E)$ with*

$$\overline{f^t(F') + A^\circ}^{\sigma(E',E)} \subseteq f^t(F') + B^\circ.$$

2. *F is weakly acyclic.*

P r o o f. Suppose that 1 is satisfied. According to a classical result of Palamodov [13, Theorem 5.3 and Corollary 5.1], F is weakly acyclic if and only if

$$\forall\, n \in \mathbb{N}\ \exists\, m > n \quad r^n_m F'_m \subseteq r^n F' + B_n^\circ,$$

where B_n is the unit ball of F_n and r^n_m and r^n are the restrictions $F'_m \to F'_n$ and $F' \to F'_n$, respectively, $\varphi \mapsto \varphi\big|_{F_n}$. Of course, we may assume $B_n \subseteq B_{n+1}$. Given $n \in \mathbb{N}$ set $B = \prod_{k\leqslant n} B_k \times \prod_{k>n} \{0\} \in \mathcal{B}(E)$ and choose A according to 1. There is $m > n$ such that $A \subseteq \prod_{k\leqslant m} F_k \times \prod_{k>n} \{0\}$. Hence,

$$\begin{aligned} f^t(F') + A^\circ &\supseteq \{(\varphi\big|_{F_k})_{k\in\mathbb{N}} \colon\ \varphi \in F'\} + \left(\prod_{k\leqslant m} \{0\} \times \prod_{k>m} F'_k\right) \\ &= \{(\varphi\big|_{F_1}, \dots, \varphi\big|_{F_m}) \colon\ \varphi \in F'\} \times \prod_{k>m} F'_k, \end{aligned}$$

and

$$\begin{aligned} \overline{f^t(F') + A^\circ}^{\sigma(E',E)} &\supseteq \overline{\{(\varphi\big|_{F_1}, \dots, \varphi\big|_{F_m}) \colon\ \varphi \in F'\}}^{\prod_{k\leqslant m} \sigma(F'_k, F_k)} \times \prod_{k>m} F'_k \\ &\overset{\star}{=} \{(\varphi\big|_{F_1}, \dots, \varphi\big|_{F_{m-1}}, \varphi) \colon\ \varphi \in F'_m\} \times \prod_{k>m} F'_k =: D, \end{aligned}$$

where in $\star$ the inclusion $\subseteq$ holds because the diagonal $\{(\varphi|_{F_1},\ldots,\varphi)\colon\ \varphi\in F'_m\}$ is $\prod_{k\leqslant m}\sigma(F'_k,F_k)$-closed, and $\supseteq$ is true because the injectivity of $F_m\hookrightarrow F$ implies that r^mF' is weak*-dense in F'_m. Projecting onto the n-th component we obtain

$$r^n_mF'_m=\mathrm{pr}_n(D)\subseteq\mathrm{pr}_n(f^t(F')+B^\circ)\subseteq r^nF'+B^\circ_n.$$

Let now 2 be satisfied and let B be a bounded set in E. Then there is $n\in\mathbb{N}$ with $B\subseteq n\prod_{k\leqslant n}B_k\times\prod_{k>n}\{0\}$. With $\varepsilon:=n^{-2}$ we get

$$f^t(F')+B^\circ\supseteq f^t(F')+\left(\varepsilon\prod_{k\leqslant n}B^\circ_k\times\prod_{k>n}F'_k\right).$$

Choose $m>n$ with $r^n_mF'_m\subseteq r^nF'+\frac{1}{2}\varepsilon B^\circ_n$ and set $A:=2/\varepsilon\prod_{k\leqslant m}B_k\times\prod_{k>m}\{0\}\in\mathcal{B}(E)$. Then

$$\begin{aligned}f^t(F')+A^\circ&\subseteq\{(\varphi|_{F_k})_{k\in\mathbb{N}}\colon\ \varphi\in F'\}+\left(\frac{\varepsilon}{2}\prod_{k\leqslant m}B^\circ_k\times\prod_{k>m}F'_k\right)\\&\subseteq\left(\{(\varphi|_{F_1},\ldots,\varphi|_{F_m})\colon\ \varphi\in F'\}+\frac{\varepsilon}{2}\prod_{k\leqslant m}B^\circ_k\right)\times\prod_{k>m}F'_k.\end{aligned}$$

Using the weak*-compactness of $\prod_{k\leqslant m}B^\circ_k$ we get

$$\begin{aligned}\overline{f^t(F')+A^\circ}&\subseteq\left(\overline{\{(\varphi|_{F_1},\ldots,\varphi|_{F_m})\colon\ \varphi\in F'\}}^{\prod_{k\leqslant m}\sigma(F'_k,F_k)}+\frac{\varepsilon}{2}\prod_{k\leqslant m}B^\circ_k\right)\times\prod_{k>m}F'_k\\&=\left(\{(\varphi|_{F_1},\ldots,\varphi|_{F_{m-1}},\varphi)\colon\ \varphi\in F'_m\}+\frac{\varepsilon}{2}\prod_{k\leqslant m}B^\circ_k\right)\times\prod_{k>m}F'_k\end{aligned}$$

by the same argument as above. Given $\varphi\in F'_m$ there is $\psi\in F'$ with $\varphi|_{F_n}-\psi|_{F_n}\in\frac{1}{2}\varepsilon B^\circ_n$, and since $B_k\subseteq B_n$ for $k\leqslant n$ we even have $\varphi|_{F_k}-\psi|_{F_k}\in\frac{1}{2}\varepsilon B^\circ_k$ for $k\leqslant n$. Therefore,

$$\begin{aligned}\overline{f^t(F')+A^\circ}&\subseteq\left(\{(\varphi|_{F_1},\ldots,\varphi|_{F_n})\colon\ \varphi\in F'\}+\frac{\varepsilon}{2}\prod_{k\leqslant n}B^\circ_k+\frac{\varepsilon}{2}\prod_{k\leqslant n}B^\circ_k\right)\times\prod_{k>n}F'_k\\&=\left(\{(\varphi|_{F_1},\ldots,\varphi|_{F_n})\colon\ \varphi\in F'\}+\varepsilon\prod_{k\leqslant n}B^\circ_k\right)\times\prod_{k>n}F'_k\\&=f^t(F')+\left(\varepsilon\prod_{k\leqslant n}B^\circ_k\times\prod_{k>n}F'_k\right)\\&\subseteq f^t(F')+B^\circ.\end{aligned}$$

$\square$

Now, we can give many examples of projective spectra of (LB)-spaces which are of strong P-type having a non-distinguished projective limit. We formulate this in the following result.

Proposition 7. *Let $F = \operatorname{ind}_n F_n$ be a complete (LB)-space which is not weakly acyclic (e.g. a co-echelon space k_∞ of order ∞ which is not retractive), set $E = \bigoplus_{\mathbb{N}} F_n$ and $f\colon E \to F$, $(x_n)_{n\in\mathbb{N}} \mapsto \sum_n x_n$. Then $\mathcal{X}(E \xrightarrow{f} F)$ is of strong P-type and the strong dual of its projective limit is not $\aleph_0$-quasibarrelled.*

P r o o f. Since F is a regular (LB)-space, f lifts bounded sets and Proposition 5 implies that $\mathcal{X}(E \xrightarrow{f} F)$ is of strong P-type. By Lemma 3 and Proposition 6, the inductive topology on $X(E \xrightarrow{f} E)$ is strictly finer than the strong topology, and Corollary 4 yields that $X(E \xrightarrow{f} F)'_\beta$ is not $\aleph_0$-quasibarrelled. □

4. A question of Grothendieck

In [9, p. 121] A. Grothendieck asked whether the bidual of a strict inductive limit of locally convex spaces (or even Fréchet spaces) equals the inductive limit of the biduals. In [2] it is shown that this need not be the case. However, our Theorem 1 yields an affirmative answer for inductive limits of quasinormable Fréchet spaces. Applications and related results for concrete situations (like weighted (LF)-spaces of holomorphic functions) are given in [1].

Theorem 8. *Let $E = \operatorname{ind}_n E_n$ be a retractive inductive limit of quasinormable Fréchet spaces. Then $E'' = \operatorname{ind}_n E''_n$ holds topologically.*

P r o o f. By [19], E satisfies Retakh's condition (M), i.e. there is an increasing sequence of absolutely convex 0-neighbourhoods of the steps on which almost all topologies of the steps coincide. Taking polars, this yields that the projective spectrum of the strong duals is of strong P-type. Since the steps are quasinormable Fréchet spaces the projective spectrum consists of retractive (LB)-spaces. Moreover, the projective limit of the strong duals is the strong dual of E because E is regular. Now, Theorem 1 gives the assertion. □

Our method even leads to a more general result. According to Palamodov [13] an inductive spectrum $(E_n)_{n\in\mathbb{N}}$ of locally convex spaces is called acyclic if

$$\sigma\colon \bigoplus_{\mathbb{N}} E_n \to \bigoplus_{\mathbb{N}} E_n$$
$$(x_n)_{n\in\mathbb{N}} \mapsto (x_n - x_{n-1})_{n\in\mathbb{N}}, \quad x_0 = 0$$

is an isomorphism onto its range. Palamodov has shown that every strict inductive spectrum is acyclic, and for (LF)-spaces many characterizations can be found in [19].

Theorem 9. *Let $E = \operatorname{ind}_n E_n$ be the regular inductive limit of an acyclic spectrum of quasibarrelled and quasinormable spaces. Then $E'' = \operatorname{ind}_n E_n''$ holds topologically.*

P r o o f. Since E is regular we have $E'_\beta = \operatorname{Proj} E'_{n,\beta}$. The transpose of σ defined above is the map Ψ used for the definition of Proj^1. Since σ is an isomorphism onto its range defined on a quasibarrelled space, $\sigma^t = \Psi\colon \prod_{\mathbb{N}} E'_{n,\beta} \to \prod_{\mathbb{N}} E'_{n,\beta}$ lifts bounded sets. Now, the same argument as in the proof of Theorem 1 applies because each $E'_{n,\beta}$ satisfies the strict Mackey condition. □

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Authors' addresses: J. Bonet, Universidad Politécnica de Valencia, Departamento de Matematica Aplicada, E.T.S. Arquitectura, E–46071 Valencia, Spain, e-mail: jbonet@pleiades.upv.es; J. Wengenroth, S. Dierolf, FB IV—Mathematik, Universität Trier, D–54286 Trier, Germany, e-mails: wengen@uni-trier.de, sdierolf@uni-trier.de.