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SEQUENTIAL COMPLETENESS AND REGULARITY OF INDUCTIVE LIMITS OF WEBBED SPACES

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Abstract. Any inductive limit of bornivorously webbed spaces is sequentially complete iff it is regular.

Keywords: localy convex space, webbed space, sequential completeness, regularity of an inductive limit

MSC 2000: 46A13, 46A04

A web in a vector space F is a countable family of balanced subsets of F, arranged in "layers". The first layer of the web consists of a sequence $(A_p\colon\ p=1,2,\ldots)$ whose union absorbs each point of F. For each set A_p of the first layer there is a sequence $(A_{pq}\colon\ q=1,2,\ldots)$ of sets, called the sequence determined by A_p such that

$$A_{pq}+A_{pq}\subset A_p \ \text{ for each } \ q, \ \text{ and}$$
$$\bigcup\{A_{pq}\colon\ q=1,2,\ldots\} \ \text{absorbs each point of } \ A_p.$$

Further layers are made up in a corresponding way so that each set of the k-th layer is indexed by a row of k integers and at each step the two conditions above are satisfied. Suppose that one chooses a set A_p then A_{pq} from the sequence determined by A_p, and so on. The resulting sequence $S=(A_p,A_{pq},A_{pqr},\ldots)$ is called a strand. Whenever we are dealing with only one strand we can simplify the notation by writing $W_1=A_p$, $W_2=A_{pq}$, etc.; thus $S=(W_k)$ is a strand where for each k, W_k is a set in the k-th layer.

A web in a locally convex space is called strict if each of its sets is absolutely convex and for any strand (W_k) there exists a sequence $a_1>a_2>\ldots>0$, called

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completing, such that for any choice of $b_k \in [0, a_k]$, $x_k \in W_k$, $k \in \mathbb{N}$, and any $n \in \mathbb{N}$, the sequence $\sum\limits_{k=n+1}^{\infty} b_k x_k$ converges to some element in W_n. A locally convex space F is strictly webbed if it admits a strict web. It is bornivorously webbed if it is strictly webbed and for every set B bounded in F there exist a strand (W_k) of a strict web and a sequence (λ_k) such that $B \subset \lambda_k W_k$ for every $k \in \mathbb{N}$ ([1], [2], [3]).

Lemma. *Let (F, α) be a strictly webbed space with a locally convex topology α. Take a strand (W_k) of a strict webb in F and denote by γ the topology generated by the subbasis $\{W_k \colon k \in \mathbb{N}\}$. Then:*

(a) *The topology γ is stronger than α.*

(b) *The space (F, γ) is Fréchet.*

P r o o f. (a) Let (a_k) be a completing sequence for the strand (W_k). Assume that the topology γ is not stronger than α. Then there exists a balanced 0-neighbourhood $U \in \alpha$ such that $a_k W_k \setminus U \neq \emptyset$ for any $k \in \mathbb{N}$. Take a 0-neighbourhood V in α such that $V - V \subset U$, choose $y_k \in a_k W_k \setminus U$, and put $x_k = a_k^{-1} y_k$, $k \in \mathbb{N}$. Since the space (F, α) is strictly webbed, the series $\sum\limits_{k=1}^{\infty} a_k x_k$ converges in (F, α) and there exists $k \in \mathbb{N}$ such that for any $m \geqslant k$, we have $\sum\limits_{n=m}^{\infty} a_k x_k \in V$. This implies $y_k = a_k x_k = \sum\limits_{n=k}^{\infty} a_n x_n - \sum\limits_{n=k+1}^{\infty} a_n x_n \in V - V \subset U$, a contradiction.

(b) Since the topology γ has a countable subbasis, it is metrizable. It remains to show that (F, γ) is complete. Take a Cauchy sequence $\{y_k;\ k \in \mathbb{N}\}$ in (F, γ) and a completing sequence (a_k) for the strand (W_k). Without a loss of generality, we may assume that $y_{k+1} - y_k \in a_k W_k$, $k \in \mathbb{N}$. Then $x_k = a_k^{-1}(y_{k+1} - y_k) \in W_k$ for any $k \in \mathbb{N}$ and $\sum\limits_{k=1}^{\infty} a_k x_k = \sum\limits_{k=1}^{\infty} (y_{k+1} - y_k)$ converges to an element $y \in F$. Moreover $\sum\limits_{k=n+1}^{\infty} (y_{k+1} - y_k) = \sum\limits_{k=n+1}^{\infty} a_k x_k \in W_n$, $n \in \mathbb{N}$. This implies that, in the topology γ, the series $\sum\limits_{k=1}^{\infty} (y_{k+1} - y_k)$ converges to an element $y \in F$ and $y_{k+1} = y_1 + \sum\limits_{k=1}^{n} (y_{k+1} - y_k) \longrightarrow y_1 + y \in F$. $\square$

Let $E_1 \subset E_2 \subset \ldots$ be a sequence of locally convex spaces with respective topologies τ_n, $n \in \mathbb{N}$, such that the identity map $\mathrm{id} \colon (E_n, \tau_n) \to (E_{n+1}, \tau_{n+1})$ is continuous for every $n \in \mathbb{N}$. Its locally convex inductive limit is denoted by $\operatorname{ind} E_n$.

We call $\operatorname{ind} E_n$ regular if for each set B, bounded in $\operatorname{ind} E_n$, there exists $n \in \mathbb{N}$ such that B is bounded in (E_n, τ_n).

Theorem. *If each space E_n, $n \in \mathbb{N}$, is bornivorously webbed then* $\operatorname{ind} E_n$ *is sequentially complete if and only if it is regular.*

P r o o f. For brevity, we will write just E instead of $\operatorname{ind} E_n$.

Suppose that the space E is sequentially complete, then it is fast complete. By Theorem 1 in [4], fast complete inductive limit of webbed spaces is regular.

Let E be regular, $\{x_n;\ n \in \mathbb{N}\}$ a Cauchy sequence in E, and $B_n = \operatorname{cl}_E \operatorname{co} \bigcup\{x_m;\ m \geqslant n\}$, $n \in \mathbb{N}$. The set B_1 is bounded in E and, by the regularity of E, it is also bounded in some space E_m. Without a loss of generality, we may assume $m = 1$.

The space (E_1, τ_1) is bornivorously webbed, hence there exists a strand (W_k) in E_1 and a sequence $\{\alpha_k;\ k \in \mathbb{N}\}$ such that $B_1 \subset \alpha_k W_k$ for each $k \in \mathbb{N}$. Denote by γ the topology on E_1 generated by the subbasis $\{W_k;\ k \in \mathbb{N}\}$ and, for brevity, by F the space (E_1, γ).

The set $B_1 \subset E_1$ is closed in E. Hence it is closed in (E_1, τ_1), and by Lemma, it is closed in the locally convex space $F = (E_1, \gamma)$. Since B_1 is convex, it is also weakly closed in F.

By Lemma, F is a Fréchet space. Hence the canonical inbedding $F \to F''$, where F'' is the second dual of F equipped with the strong topology, is a topological isomorphism into F''. Since F is complete, it is closed in F'' and each functional from the strong dual F' of F can be continuously extended to F''. Thus the $\sigma(F, F')$-closed set B_1 is also $\sigma(F'', F')$-closed in F''.

Further, since B_1 is bounded in F'', it is equicontinuous in F'. Hence, by Alaoglu Theorem, the set B_1 is relatively $\sigma(F'', F')$-compact. This, together with the $\sigma(F'', F')$-closedness, implies that B_1 is $\sigma(F'', F')$-compact in F''.

Similarly, all sets B_n, $n \in \mathbb{N}$, are $\sigma(F'', F')$-compact. Every finite intersection $\bigcap\{B_n;\ 1 \leqslant n \leqslant m\} = B_m$, $m \in \mathbb{N}$, is non-empty. Hence there exists $x_0 \in \bigcap\{B_n;\ n \in \mathbb{N}\} \subset B_1 \subset E_1$. This implies the existence of an upper-triangular matrix $\Lambda = (\lambda_{nm})$ with all entries $\lambda_{nm} \geqslant 0$, only finite number of non-zeros in each row, and the sum of all entries in each row equal to 1, such that the sequence $\{y_n = \sum\limits_{m=n}^{\infty} \lambda_{nm} x_m;\ n \in \mathbb{N}\}$ converges to x_0 in the topology γ. Then the continuity of the identity maps: $(E_1, \gamma) \to (E_1, \tau_1) \to \operatorname{ind} E_n$ implies the convergence $y_k \to x_0$ in $\operatorname{ind} E_n$.

Take a balanced, convex, 0-neighbourhood V in E. Then there exist $p, q \in \mathbb{N}$ such that $y_n - x_0 \in V$ for $n \geqslant p$ and $x_m - x_n \in V$ for $m \geqslant n \geqslant q$. Then for $n \geqslant \max(p, q)$, we have $x_0 - x_n = (x_0 - y_n) + (y_n - x_n) = (x_0 - y_n) + \sum\limits_{m=n}^{\infty} \lambda_{nm}(x_m - x_n) \in V + V$. This implies $x_n \to x_0$ in the space E. $\square$

Corollaries.

(a) *Since any LF-space is bornivorously webbed, it is sequentially complete iff it is regular.*

(b) *Theorem* 1 *in* [4] *reads: If all locally convex spaces* (E_n, τ_n), $n \in \mathbb{N}$, *are webbed and their* $\operatorname{ind} E_n$ *is fast complete, then* $\operatorname{ind} E_n$ *is regular. This result, combined with our Theorem, implies: In a regular LF-space, the sequential and fast completenesses are equivalent.*

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