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AN ITERATION PROCESS FOR NONLINEAR MAPPINGS IN UNIFORMLY CONVEX LINEAR METRIC SPACES

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Abstract. We obtain necessary conditions for convergence of the Cauchy Picard sequence of iterations for Tricomi mappings defined on a uniformly convex linear complete metric space.

Keywords: linear metric space, fixed point, uniformly convex

MSC 2000: 47H10, 54H25

1. Introduction and preliminaries

An interesting class of nonexpansive mappings for which the Cauchy Picard sequence of iterations converges was discovered by Moreau [11]. Subsequently Beauzamy [1] extended Moreau's results from Hilbert spaces to uniformly convex Banach spaces. Kirk [10] and Ding [7] also constructed an iteration process for nonexpansive mappings in metric spaces. Recently Shimizu and Takahashi [15] and Beg [2, 3] have started the study of the fixed points of nonexpansive mappings on uniformly convex complete metric spaces. The aim of this paper is to study convergence of the Cauchy Picard type sequence of iterations for a more general class of mappings, namely the Tricomi mappings (first introduced and studied by Tricomi [16]) defined on a linear metric space. Our result generalizes the known results of Moreau [11], Beauzamy [1] and several other authors.

Definition 1.1. Let (X, d) be a metric space and $\mathbb{R}$ the set of real numbers. A mapping $W\colon X \times X \times \mathbb{R} \longrightarrow X$ is said to be a *linear structure* on X if for each

$(x, y, \lambda) \in X \times X \times \mathbb{R}$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leqslant |\lambda| d(u, x) + |1 - \lambda| d(u, y).$$

A metric space X together with the linear structure W is called a *linear metric space*. Obviously, $W(x, x, \lambda) = x$. The convex metric space of Takahashi [15] is a special case of our Definition 1.1. Takahashi [15] has used the unit interval $[0, 1]$ instead of $\mathbb{R}$. All normed spaces are convex metric spaces. However, there are many examples of convex metric spaces which are not embedded in any normed space (see Takahashi [15]). Guay, Singh and Whitfield [9], Naimpally, Singh and Whitfield [12], Beg et al [2, 3, 4, 5], Shimizu and Takahashi [14], Ciric [6], Gajic and Stojakavic [8] and many other authors have studied fixed point theorems on convex metric spaces.

Definition 1.2. A linear metric space X is said to have *property* (B), if it satisfies
(i) $W(x_1, x_2, t) = W(x_2, x_1, 1 - t)$,
(ii) $d(W(x_1, x_2, t_1), W(x_1, x_2, t_2)) = |t_1 - t_2| d(x_1, x_2)$

and
(iii) $d\big(v, W\big(x_1, x_2, \frac{t}{2}\big)\big) = d\big(W\big(W(v, x_1, 1 + t), W(v, x_2, 1 - t), \frac{1}{2}\big), x_2\big)$.
Each normed space has property (B), if we define $W(x_1, x_2, t) = tx_1 + (1 - t)x_2$.

Definition 1.3 (Beg [3])**.** A linear complete metric space X is said to be *uniformly convex* if for all $x, y, a \in X$,

$$\Big[d\Big(a, W\Big(x, y, \frac{1}{2}\Big)\Big)\Big]^2 \leqslant \frac{1}{2}\Big(1 - \alpha\Big(\frac{d(x, y)}{\max\{d(a, x), d(a, y)\}}\Big)\Big)([d(a, x)]^2 + [d(a, y)]^2),$$

where the function α is an increasing function on the set of strictly positive numbers and $\alpha(0) = 0$.

Uniformly convex Banach spaces are uniformly convex linear metric spaces.

Definition 1.4. Let X be a metric space. A mapping $T\colon\ X \longrightarrow X$ is said to be *nonexpansive* if $d(T(x), T(y)) \leqslant d(x, y)$ for every $x, y \in X$. A point $x \in X$ is called *a fixed point* of T if $T(x) = x$.

Definition 1.5. A continuous mapping T defined on a subset of a linear complete metric space X is called *a Tricomi mapping* if:
(i) $F(T)$, the set of fixed points of T, is the nonempty set;
(ii) for any x in the set on which T is defined and any $u \in F(T)$, $d(f(x), u) \leqslant d(x, u)$.

For examples and other details about the Tricomi mappings we refer to Tricomi [16] and Moreau [11]. From the definition of Tricomi mappings it is clear that this class of mappings is larger than the class of nonexpansive mappings.

Remark 1.6. Let $x, y, z \in X$, then

$$d(x,z) \leqslant d(x,y) + d(y,z).$$

This implies that

$$[d(x,z)]^2 \leqslant [d(x,y)]^2 + [d(y,z)]^2 + 2d(x,y)d(y,z) \leqslant 2[d(x,y)]^2 + 2[d(y,z)]^2.$$

It further implies that

$$\frac{1}{2}[d(x,z)]^2 \leqslant [d(x,y)]^2 + [d(y,z)]^2. \tag{1}$$

2. Convergence of iterates of Tricomi mappings

Let X be a uniformly convex linear complete metric space having property (B). Let C be a subset of X and $T\colon C \to C$ a Tricomi mapping. Suppose that $F(T)$ contains a nonempty open set. For any $x \in C$, define $m = d(v,x)$ where v is the centre of an open sphere $S(v,r)$ which is contained in $F(T)$. Consider another element p defined by $d(v,p) = \inf\limits_{t\in\mathbb{R}} d(v, W(x, T(x), t))$, i.e., p is the best approximation of v on the set $Y = \{y\colon\ y = W(x,T(x),t) \text{ and } t \in \mathbb{R}\}$. Clearly, for any $z \in Y$,

$$[d(v,p)]^2 \leqslant \Big[d\Big(v, W\Big(z,p,\frac{1}{2}\Big)\Big)\Big]^2. \tag{2}$$

Theorem 2.1. *Let $z \in Y$ be such that $d(z,v) < 2m$. Then*

$$[d(v,p)]^2 + \frac{1}{2}[d(p,z)]^2\alpha\Big(\frac{d(p,z)}{2m}\Big) \leqslant [d(v,z)]^2.$$

P r o o f. Since X is a uniformly convex metric space, using inequality (2) we have

$$\begin{aligned}[d(v,p)]^2 &\leqslant \Big[d\Big(v, W\Big(z,p,\frac{1}{2}\Big)\Big)\Big]^2 \\ &\leqslant \frac{1}{2}\Big(1 - \alpha\Big(\frac{d(z,p)}{\max\{d(v,z), d(v,p)\}}\Big)\Big)([d(v,z)]^2 + [d(v,p)]^2).\end{aligned}$$

It implies that

$$\frac{1}{2}[d(v,p)]^2 + \frac{1}{2}\alpha\Big(\frac{d(z,p)}{2m}\Big)([d(v,z)]^2 + [d(v,p)]^2\Big) \leqslant \frac{1}{2}[d(v,z)]^2.$$

Inequality (1) further implies

$$\frac{1}{2}[d(v,p)]^2 + \frac{1}{2}\alpha\Big(\frac{d(z,p)}{2m}\Big)\frac{1}{2}[d(p,z)]^2 \leqslant \frac{1}{2}[d(v,z)]^2.$$

Hence,

$$[d(v,p)]^2 + \frac{1}{2}[d(p,z)]^2\alpha\Big(\frac{d(p,z)}{2m}\Big) \leqslant [d(v,z)]^2.$$

□

Remark 2.2. Let $x \in C$ be fixed, then the function f defined by

$$f(t) = [d(v, W(x, T(x), t))]^2$$

is a convex function.

Remark 2.3. Let t_0 be a real number such that $p = W(x, T(x), t_0)$.

Lemma 2.4. *Let t_1 and t_2 be two points such that $t_0 \leqslant t_1 \leqslant t_2$. If $f(t_1) \leqslant 4m^2$ and $f(t_2) \leqslant 4m^2$ then*

$$\frac{1}{2}(t_2 - t_1)^2[d(x, T(x))]^2\alpha\Big(\frac{(t_2 - t_1)d(x, T(x))}{2m}\Big) \leqslant f(t_2) - f(t_1).$$

Proof. Since f is a convex function, therefore

$$f(t_2) - f(t_1) \geqslant f(t_0 + t_2 - t_1) - f(t_0)$$

and

$$f(t_0 + (t_2 - t_1)) \leqslant 4m^2.$$

Using Theorem 2.1 we have

$$\begin{aligned}
f(t_2) - f(t_1) &\geqslant f(t_0 + t_2 - t_1) - f(t_0) \\
&= [d(v, W(x, T(x), t_0 + t_2 - t_1))]^2 - [d(v, W(x, T(x), t_0)]^2 \\
&\geqslant \frac{1}{2}[d(W(x, T(x), t_0 + t_2 - t_1), W(x, T(x), t_0))]^2 \\
&\quad\ \times \alpha\Big(\frac{d(W(x, T(x), t_0 + t_2 - t_1), W(x, T(x), t_0))}{2m}\Big)
\end{aligned}$$

(using property (B))

$$= \frac{1}{2}(t_2 - t_1)^2[d(x, T(x))]^2\alpha\Big(\frac{(t_2 - t_1)d(x, T(x))}{2m}\Big).$$

□

Similarly, we can prove the following lemma.

Lemma 2.5. *Let t_1 and t_2 be two points such that $t_2 \leqslant t_1 \leqslant t_0$. If $f(t_1) \leqslant 4m^2$ and $f(t_2) \leqslant 4m^2$ then*

$$\frac{1}{2}(t_1 - t_2)^2[d(x,T(x))]^2\alpha\Big(\frac{(t_1-t_2)d(x,T(x))}{2m}\Big) \leqslant f(t_1) - f(t_2).$$

Lemma 2.4 and Lemma 2.5 imply the following theorem.

Theorem 2.6. *Let t_1 and t_2 be two points such that $t_0 \leqslant t_1, t_2$ or $t_2, t_1 \leqslant t_0$. If $f(t_1) \leqslant 4m^2$ and $f(t_2) \leqslant 4m^2$ then*

$$\frac{1}{2}(t_1 - t_2)^2[d(x,T(x))]^2\alpha\Big(\frac{|t_1-t_2|d(x,T(x))}{2m}\Big) \leqslant |f(t_1) - f(t_2)|.$$

Theorem 2.7. *Let $T\colon C \to C$ be a Tricomi mapping, then*

$$\frac{r^2}{8m^2}[d(x,T(x))]^2\alpha\Big(\frac{r}{4m^2}d(x,T(x))\Big) \leqslant [d(v,x)]^2 - [d(v,T(x))]^2 = f(1) - f(0),$$

where r is the radius of an open sphere $S(v,r) \subset F(T)$.

P r o o f. Case I ($[t_0 \leqslant 0]$). Choose $t_1 = 0$, $t_2 = 1$. Since T is a Tricomi mapping, we obtain

$$f(0) = [d(v, W(x,T(x),0)]^2 \leqslant [d(v,T(x))]^2 \leqslant [d(v,x)]^2 = m^2 \leqslant 4m^2$$

and

$$f(1) = [d(v, W(x,T(x),1)]^2 \leqslant [d(v,x)]^2 = m^2 \leqslant 4m^2.$$

Thus Lemma 2.4 implies that

$$\frac{1}{2}[d(x,T(x))]^2\alpha\Big(\frac{d(x,T(x))}{2m}\Big) \leqslant f(1) - f(0). \tag{3}$$

Case II ($[t_0 \in (0,1)]$). Let

$$w = W\Big(W\Big(v,x,1-\frac{r}{m}\Big), W\Big(v,T(x),1+\frac{r}{m}\Big),\frac{1}{2}\Big),$$

then

$$d(v,w) \leqslant \frac{1}{2}\Big[d\Big(v, W\Big(v, x, 1-\frac{r}{m}\Big)\Big) + d\Big(v, W\Big(v, T(x), 1+\frac{r}{m}\Big)\Big)\Big]$$
$$\leqslant \frac{r}{2m}[d(v,x) + d(v,T(x))]$$

(T is a Tricomi mapping)

$$\leqslant \frac{r}{2m}[d(v,x) + d(v,x)] = r.$$

Therefore $w \in S(v,r) \subset F(T)$. This further implies that

$$\begin{aligned} d\Big(v, W\Big(x, T(x), -\frac{r}{2m}\Big)\Big) &= d(w, T(x)) \quad \text{(by property B(iii))} \\ &\leqslant d(w,x) \quad \text{(since } T \text{ is a Tricomi mapping)} \\ &= d\Big(v, W\Big(x, T(x), 1-\frac{r}{2m}\Big)\Big) \quad \text{(by property B(iii))}. \end{aligned}$$

It further gives

$$f\Big(-\frac{r}{2m}\Big) \leqslant f\Big(1-\frac{r}{2m}\Big).$$

Since the minimum of the convex function f is attained at t_0 (see the definition of t_0), $t_0 \in (0,1)$, therefore,

$$f\Big(-\frac{r}{2m}\Big) \geqslant f(0) \geqslant f(t_0)$$

and

$$f(1) \geqslant f\Big(1-\frac{r}{2m}\Big).$$

However, $1 - r/(2m) \geqslant 0$. Thus $t_0 \leqslant 1 - r/(2m)$. Choosing $t_1 = 1 - r/(2m)$ and $t_2 = 1$, Theorem 2.6 implies

$$f(1) - f(0) \geqslant f(1) - f\Big(1-\frac{r}{2m}\Big) \geqslant \frac{1}{2}\Big(\frac{r}{2m}\Big)^2 [d(x,T(x))]^2 \alpha\Big(\frac{r}{4m^2} d(x,T(x))\Big).$$

Since the estimate in inequality (3) is optimal, we have

$$\frac{r^2}{8m^2}[d(x,T(x))]^2 \alpha\Big(\frac{r}{4m^2} d(x,T(x))\Big) \leqslant [d(v,x)]^2 - [d(v,T(x))]^2 = f(1) - f(0).$$

$\square$

Theorem 2.8. *Let X be a uniformly convex linear complete metric space having property* (B) *and let C be a subset of X. Let $T\colon C \to C$ be a Tricomi mapping and let $F(T)$ contain a nonempty open set. Then for any $x \in C$, the sequence $\{T^n(x)\}$ converges to a fixed point of T.*

Proof. Let v be a fixed point of T and x an arbitrary point in C, then $d(v, T^{n+1}(x)) \leqslant d(v, T^n(x))$. Thus the sequence $\{d(v, T^n(x))\}$ is a monotonic decreasing sequence of positive numbers. Therefore $\{d(v, T^n(x))\}$ is a Cauchy sequence. Theorem 2.7 further implies that $\{T^n(x)\}$ is a Cauchy sequence in X. Hence it is convergent to a fixed point of T. □

Remark 2.9. Theorem 2.8 generalizes the results of Moreau [11] and Beauzamy [1].

Remark 2.10. It is of great interest to observe that if T is a Tricomi mapping and $F(T)$ reduces to a singleton, then $\{T^n(x)\}$ need not be convergent. E.g., let $X = \mathbb{R}^2$ with the usual metric $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and $C = \{(x, y)\colon \sqrt{x^2 + y^2} \leqslant 1\}$. Let $T\colon C \to C$ be given by $T(x_1, x_2) = \bigl(-x_1, -\frac{x_2}{2}\bigr)$. Clearly T is a Tricomi mapping and $(0, 0)$ is the unique fixed point of T. For any $(x_1, x_2) \in C$ we have $d(T^n(x_1, x_2), T^{n+1}(x_1, x_2)) = (2x_1)^2 + \bigl(3x_2/2^{n+1}\bigr)^2$, which does not converge to zero unless $x_1 = 0$.

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