Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 53 (2003), No. 2, 437–447

Persistent URL: `http://dml.cz/dmlcz/127812`

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Czechoslovak Mathematical Journal, 53 (128) (2003), 437–447

A CANTOR-BERNSTEIN THEOREM FOR σ-COMPLETE MV-ALGEBRAS

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(Received June 14, 2000)

Abstract. The Cantor-Bernstein theorem was extended to σ-complete boolean algebras by Sikorski and Tarski. Chang's MV-algebras are a nontrivial generalization of boolean algebras: they stand to the infinite-valued calculus of Łukasiewicz as boolean algebras stand to the classical two-valued calculus. In this paper we further generalize the Cantor-Bernstein theorem to σ-complete MV-algebras, and compare it to a related result proved by Jakubík for certain complete MV-algebras.

Keywords: Cantor-Bernstein theorem, MV-algebra, boolean element of an MV-algebra, partition of unity, direct product decomposition, σ-complete MV-algebra

MSC 2000: 06D35, 06D30, 06C15, 03G20

1. Introduction

The Cantor-Bernstein theorem states that, if a set X can be embedded into a set Y, and vice versa, then there is a one-one map of X onto Y. The theorem was proved by Dedekind in 1887, conjectured by Cantor in 1895, and again proved by Bernstein in 1898, [6, p. 85].

For any boolean algebra A, let $[0,a]$ denote the boolean algebra of all $x \in A$ such that $0 \leqslant x \leqslant a$, equipped with the restriction of the join and meet of A, where the complement of $y \in [0,a]$ is the meet of a with the complement $\neg y$ of y in A. (Note that Sikorski [8, p. 29] writes $A|a$ instead of $[0,a]$.)

Research supported by COST ACTION 15 on Many-valued logics for computer science applications, Project on Logic of the Italian MURST, Project VS 96049 of the Czech Ministry of Education and Grant 201/97/0437 of the Grant Agency of the Czech Republic.

Sikorski [9] and Tarski [10] proved the following generalization of the (Dedekind)-Cantor-Bernstein theorem: For any two σ-complete boolean algebras A and B and elements $a \in A$ and $b \in B$, if B is isomorphic to $[0, a]$ and A is isomorphic to $[0, b]$, then A and B are isomorphic. To obtain the Cantor-Bernstein theorem it suffices to assume that A and B are the powersets of X and Y, respectively, with the natural set-theoretic boolean operations.

Our aim in this paper is to further generalize the Cantor-Bernstein theorem to MV-algebras—the latter being an interesting "non-commutative" extension of boolean algebras (see [7] for a precise formulation of this). Since, as proved in [3] and [5], the σ-completeness assumption is indispensable already in the boolean algebraic setup, our Cantor-Bernstein theorem shall be proved for σ-complete MV-algebras.

2. MV-ALGEBRAS

An *MV-algebra* $A = (A, 0, \oplus, \neg)$ is an algebra where the operation $\oplus\colon A \times A \to A$ is associative and commutative with 0 as the neutral element, the operation $\neg\colon A \to A$ satisfies the identities $\neg\neg x = x$ and $x \oplus \neg 0 = \neg 0$, and, in addition,

$$y \oplus \neg(y \oplus \neg x) = x \oplus \neg(x \oplus \neg y). \tag{1}$$

Example 2.1. The real unit interval $[0, 1]$ equipped with the operations $x \oplus y = \min(1, x + y)$ and $\neg x = 1 - x$ is an MV-algebra.

Following common usage, for any elements x, y of an MV-algebra we will use the abbreviations $1 = \neg 0$, $x \odot y = \neg(\neg x \oplus \neg y)$ and $x \ominus y = x \odot \neg y$. We will denote by $(A, \vee, \wedge)$ the underlying distributive lattice of A, where $x \vee y = x \oplus \neg(x \oplus \neg y)$ and $x \wedge y = x \odot \neg(x \odot \neg y)$. With reference to the underlying order of A, for any element $a \in A$ we let the interval $[0, a]$ be defined by

$$[0, a] = \{x \in A \mid 0 \leqslant x \leqslant a\}.$$

An MV-algebra A is *σ-complete* (*complete*) iff every sequence (every family, respectively) of elements of A has supremum in A with respect to the underlying order of A.

As shown by Chang, boolean algebras coincide with MV-algebras satisfying the equation $x \oplus x = x$. In this case the operation $\oplus$ coincides with $\vee$, and the operation $\odot$ coincides with $\wedge$.

An element a in an MV-algebra A is called *boolean* iff $a \oplus a = a$. We let $\mathbf{B}(A)$ denote the set of boolean elements of A. It is not hard to see that the operations of A make $\mathbf{B}(A)$ a boolean algebra. As shown in Corollary 3.3 below, if A is a σ-complete

MV-algebra, then $\mathbf{B}(A)$ is a σ-complete boolean algebra, and the σ-infinitary operations of $\mathbf{B}(A)$ agree with the restrictions of the corresponding operations of A.

A *homomorphism* between two MV-algebras is a map that sends zero to zero, and preserves the operations $\oplus$ and $\neg$. A one-one surjective homomorphism is called an *isomorphism.*

For further information on MV-algebras we refer to [1], [2] and [7].

Definition 2.2. Let A be an MV-algebra and z a fixed, but otherwise arbitrary, element of A. Let the map $h_z\colon A \to [0,z]$ be defined by

$$h_z(x) = x \wedge z. \tag{2}$$

Further, we define the operation $\neg_z\colon [0,z] \to [0,z]$ by

$$\neg_z x = z \odot \neg x = z \ominus x, \tag{3}$$

and the operation $\oplus_z\colon [0,z] \times [0,z] \to [0,z]$ by

$$x \oplus_z y = (x \oplus y) \wedge z. \tag{4}$$

A moment's reflection shows that the ranges of both operations $\neg_z$ and $\oplus_z$ coincide with $[0,z]$.

Proposition 2.3. *Let A be an MV-algebra and $b \in A$. We then have*

(i) *for each element $b \in A$, the structure $([0,b], \oplus_b, \neg_b, 0)$ is an MV-algebra.*

If, in addition, b is a boolean element of A then

(ii) $\neg_b x = b \wedge \neg x$ *for all* $x \in [0,b]$;

(iii) *the interval $[0,b]$ (as well as the interval $[0,\neg b]$) is an ideal of A;*

(iv) *The map h_b defined in (2) is a homomorphism of A onto $[0,b]$ whose kernel coincides with $[0,\neg b]$;*

(v) *The MV-algebra $[0,b]$ is isomorphic to the quotient MV-algebra $A/[0,\neg b]$;*

(vi) *$[0,b]$ is a subalgebra of A iff $b = 1$ iff $[0,b] = A$.*

Proof. (i) For every $x \in [0,b]$ we have

$$\neg_b\neg_b x = b \odot \neg(b \odot \neg x) = b \wedge x = x$$

and

$$x \oplus_b \neg_b 0 = x \oplus_b b = (x \oplus b) \wedge b = b = \neg_b 0.$$

Associativity of $\oplus_b$ follows from the identities

$$\begin{aligned}(x \oplus_b y) \oplus_b z &= (((x \oplus y) \wedge b) \oplus z) \wedge b \\ &= ((x \oplus y \oplus z) \wedge (b \oplus z)) \wedge b \\ &= (x \oplus y \oplus z) \wedge b = \ldots = x \oplus_b (y \oplus_b z).\end{aligned}$$

With reference to (1) we shall now prove the identity

$$y \oplus_b \neg_b(y \oplus_b \neg_b x) = x \oplus_b \neg_b(x \oplus_b \neg_b y) \quad \text{for all } x, y \in [0, b]. \tag{5}$$

First, using distributivity of $\odot$ over $\vee$, we transform a part of the expression on the left-hand side of (5) as follows:

$$\begin{aligned}\neg_b(y \oplus_b \neg_b x) &= b \odot \neg((y \oplus (b \odot \neg x)) \wedge b) \\ &= b \odot ((\neg y \odot \neg(b \odot \neg x)) \vee \neg b) = b \odot \neg y \odot \neg(b \odot \neg x) \\ &= \neg y \odot (b \wedge x) = \neg y \odot x = \neg(y \oplus \neg x).\end{aligned}$$

We can now simplify the left-hand term in (5) as follows:

$$y \oplus_b \neg_b(y \oplus_b \neg_b x) = (y \oplus \neg(y \oplus \neg x)) \wedge b = (y \vee x) \wedge b = y \vee x,$$

which settles (5). The remaining verifications needed to show that $[0, b]$ is an MV-algebra are all trivial.

Following now the proof of [2, Proposition 6.4.1], let us assume that $b \in \mathbf{B}(A)$. Then condition (ii) is an immediate consequence of the definition of $\neg_b$ and of the fact that $\odot$ coincides with $\wedge$ whenever one of its arguments is boolean, [2, Theorem 1.5.3]. Similarly, (iii) follows from the definition of a boolean element, [2, Corollary 1.5.6], and we also see that $\oplus_b$ coincides with the restriction of $\oplus$ to $[0, b]$. To prove (iv), for all $x, y \in A$ we can write $(x \wedge b) \oplus (y \wedge b) = ((x \wedge b) \oplus y) \wedge ((x \wedge b) \oplus b)$. From $(x \wedge b) \oplus b = (x \wedge b) \vee b = b$ we get $(x \wedge b) \oplus (y \wedge b) = (x \oplus y) \wedge (b \oplus y) \wedge b = (x \oplus y) \wedge b$. We conclude that $h_b(x \oplus y) = h_b(x) \oplus h_b(y) = h_b(x) \oplus_b h_b(y)$. The rest is trivial. The proof of (v) and (vi) is the same as in [2, Proposition 6.4.3]. □

Remarks. As shown by (ii) above, whenever b is a boolean element of A, there is no discrepancy between our present definition of $\neg_b$ and the definition in [2, (6.4)].

If in a boolean algebra B we denote by $\mathcal{I}$ the principal ideal generated by $\neg b$, then $\mathcal{I} = [0, \neg b]$ and the algebra $[0, b]$ is isomorphic to $B/\mathcal{I}$ via the map $x \in [0, b] \mapsto x/\mathcal{I} \in B/\mathcal{I}$. Condition (v) is a generalization of this fact to MV-algebras.

If a is not a boolean element of A, then $[0, a]$ need not be a homomorphic image of A. For an example, let $A = \{0, 1/2, 1\}$ be a subalgebra of the MV-algebra $[0, 1]$

of Example 2.1. Then $[0,1/2] = \{0,1/2\}$ is not a homomorphic image of A, because A has no other ideals than $\{0\}$. One more example is given in 5.2 below.

On the other hand, the existence of a homomorphism of A onto $[0,a]$ need not imply that a is a boolean element of A. As a matter of fact, in the MV-algebra $[0,1]$ of Example 2.1, multiplication by $1/2$ is a homomorphism of $[0,1]$ onto the interval MV-algebra $[0,1/2]$, but the element $1/2$ is not boolean in $[0,1]$.

The proof of the following result is immediate.

Lemma 2.4. *Let A and B be MV-algebras and let $\alpha\colon A \to B$ be an isomorphism of A onto B. For any $a \in A$, the restriction of the map α to the interval $[0,a]$ of A is an isomorphism of the MV-algebra $[0,a]$ onto the interval $[0,\alpha(a)]$ of B, once these two intervals are equipped with the MV-algebraic operations of Definition* 2.2 *and Proposition* 2.3 (i).

Corollary 2.5. *For each $a \in \mathbf{B}(A)$, the mapping $x \mapsto (x \wedge a, x \wedge \neg a)$ is an isomorphism of A onto the product MV-algebra $[0,a] \times [0,\neg a]$.*

Proof. The same as for [2, Lemma 6.4.5]. □

3. Partitions of unity and decompositions

In Lemma 3.4 below we will give an infinitary generalization of Corollary 2.5. To this purpose, we prepare

Notation. We set $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

Lemma 3.1. *Let A be a σ-complete MV-algebra. Let $x_1, x_2, \ldots \in A$. Then the following countably infinitary de Morgan identities hold:*

$$\bigwedge_{n\in\mathbb{N}} x_n = \neg \bigvee_{n\in\mathbb{N}} \neg x_n \tag{6}$$

and

$$\bigvee_{n\in\mathbb{N}} x_n = \neg \bigwedge_{n\in\mathbb{N}} \neg x_n. \tag{7}$$

Proof. Let $a = \bigwedge\limits_{n\in\mathbb{N}} x_n$ and $b = \bigvee\limits_{n\in\mathbb{N}} \neg x_n$. For all $n \in \mathbb{N}$ we have $a \leqslant x_n$, whence $\neg a \geqslant \neg x_n$. Therefore $\neg a \geqslant \bigvee\limits_{n\in\mathbb{N}} \neg x_n = b$. Similarly, for all $n \in \mathbb{N}$ we have $b \geqslant \neg x_n$, $\neg b \leqslant x_n$, whence $\neg b \leqslant \bigwedge\limits_{n\in\mathbb{N}} x_n = a$ and $b \geqslant \neg a$, which settles (6). The proof of (7) is similar. □

As an immediate consequence we get the following infinitary distributive laws:

Lemma 3.2. *Let A be a σ-complete MV-algebra. Let $x_1, x_2, \ldots \in A$. Then for each $x \in A$ we have*

$$x \wedge \bigvee_{n\in\mathbb{N}} x_n = \bigvee_{n\in\mathbb{N}} (x \wedge x_n) \tag{8}$$

and

$$x \vee \bigwedge_{n\in\mathbb{N}} x_n = \bigwedge_{n\in\mathbb{N}} (x \vee x_n). \tag{9}$$

Proof. This is an easy adaptation of the proof of [2, Lemma 6.6.4]. □

Corollary 3.3. *Let A be a σ-complete MV-algebra. Then*

(i) $\mathbf{B}(A)$ *is a σ-complete boolean algebra. As a matter of fact, for any sequence $b_1, b_2, \ldots \in \mathbf{B}(A)$ we have*

$$\bigvee_{n\in\mathbb{N}} b_n \in \mathbf{B}(A) \tag{10}$$

and

$$\bigwedge_{n\in\mathbb{N}} b_n \in \mathbf{B}(A). \tag{11}$$

(ii) *For every $b \in \mathbf{B}(A)$, letting $h_b\colon A \to [0, b]$ be as in (2), it follows that $[0, b]$ is a σ-complete MV-algebra, and h_b preserves all existing infima and suprema. Therefore, for all $x_1, x_2, \ldots \in A$ we can write*

$$h_b\Big(\bigvee_{n\in\mathbb{N}} x_n\Big) = \bigvee_{n\in\mathbb{N}} h_b(x_n) \tag{12}$$

and

$$h_b\Big(\bigwedge_{n\in\mathbb{N}} x_n\Big) = \bigwedge_{n\in\mathbb{N}} h_b(x_n). \tag{13}$$

Proof. An easy adaptation of the proof of [2, Corollary 6.6.5]. □

From Lemma 3.2 we obtain the desired generalization of Corollary 2.5, given by the following variant of [2, Lemma 6.6.6].

Lemma 3.4. *Let A be a σ-complete MV-algebra. Suppose that a sequence $b_1, b_2, \ldots \in \mathbf{B}(A)$ satisfies the following conditions:*

$$\text{(Partition of unity)} \qquad \bigvee_{n\in\mathbb{N}} b_n = 1 \quad \textit{and} \quad b_j \wedge b_k = 0 \quad \textit{for all } j \neq k.$$

Then the map $x \mapsto (x \wedge b_1, x \wedge b_2, \ldots) = (x \wedge b_n)_{n\in\mathbb{N}}$ is an isomorphism of A onto the product MV-algebra $\prod\limits_{n\in\mathbb{N}} [0, b_n]$.

4. MV-ALGEBRAIC CANTOR-BERNSTEIN THEOREM

In this section we prove the following MV-algebraic generalization of the Cantor-Bernstein theorem.

Theorem 4.1. *Let A and B be σ-complete MV-algebras. Let $a \in \mathbf{B}(A)$, $b \in \mathbf{B}(B)$, and assume α to be an isomorphism of A onto the interval algebra $[0, b]$ of B, and β an isomorphism of B onto the interval algebra $[0, a]$ of A. Then A and B are isomorphic.*

Proof. Skipping all trivialities, we may safely assume $0 < a < 1$ and $0 < b < 1$. Also, A and B can be safely assumed disjoint. We can now define sequences $a_0, a_1, a_2, \ldots \in A$ and $b_0, b_1, b_2, \ldots \in B$ by the following inductive stipulation:

$$\begin{aligned} a_0 &= 1, & b_0 &= 1, \\ a_{n+1} &= \beta(b_n), & b_{n+1} &= \alpha(a_n). \end{aligned}$$

For each $n = 0, 1, 2, \ldots$, both elements a_n and b_n are boolean. From the assumed injectivity of α and β we obtain

$$a_0 > a_1 > a_2 > \ldots \quad \text{and} \quad b_0 > b_1 > b_2 > \ldots \tag{14}$$

Let $a_\infty \in A$ and $b_\infty \in B$ be given by $a_\infty = \bigwedge\limits_{n\in\mathbb{N}_0} a_n$ and $b_\infty = \bigwedge\limits_{n\in\mathbb{N}_0} b_n$. The existence of a_∞ and b_∞ is ensured by the assumed σ-completeness of A and B. By Corollary 3.3 (i) both a_∞ and b_∞ are boolean elements. For all $n \in \mathbb{N}_0$ we have the identities $a_{n+2} = (\beta \circ \alpha)(a_n)$ and $b_{n+2} = (\alpha \circ \beta)(b_n)$. Since the mapping $\beta \circ \alpha$ is an isomorphism of A onto $[0, a_2]$, it preserves countable infima and suprema. Since for each $n = 0, 1, 2, \dots$ the underlying orders of the interval MV-algebras $[0, a_n]$ and $[0, a_{n+1}]$ agree, we have

$$(\beta \circ \alpha)(a_\infty) = (\beta \circ \alpha)\Big(\bigwedge_{n \in \mathbb{N}} a_n\Big) = \bigwedge_{n \in \mathbb{N}} (\beta \circ \alpha)(a_n) = \bigwedge_{n \in \mathbb{N}} a_{n+2} = a_\infty.$$

Similarly, $b_\infty = (\alpha \circ \beta)(b_\infty)$. One similarly obtains

$$\alpha(a_\infty) = b_\infty \quad \text{and} \quad \beta(b_\infty) = a_\infty. \tag{15}$$

In particular, $a_\infty = 0$ iff $b_\infty = 0$. For each $n = 0, 1, 2, \dots$ let us define $d_n = a_n \ominus a_{n+1} = a_n \wedge \neg a_{n+1}$ and $e_n = b_n \ominus b_{n+1} = b_n \wedge \neg b_{n+1}$. Then for each $n = 0, 1, 2, \dots$ we have

$$\alpha(d_{2n}) = e_{2n+1} \quad \text{and} \quad \beta(e_{2n}) = d_{2n+1}. \tag{16}$$

A straightforward computation shows that, for any two distinct $m, n \in \mathbb{N}_0$, $d_m \wedge d_n = 0 = e_m \wedge e_n$.

Lemma 3.1 together with (14) yields

$$\bigvee_{n \in \mathbb{N}} d_{n-1} = \bigvee_{n \in \mathbb{N}} \bigvee_{k=1}^{n} d_{k-1} = \bigvee_{n \in \mathbb{N}} (1 \ominus a_n) = \bigvee_{n \in \mathbb{N}} \neg a_n = \neg \bigwedge_{n \in \mathbb{N}} a_n = \neg a_\infty.$$

It follows that the sequence $(a_\infty, d_0, d_1, d_2, \dots)$ is a partition of unity in $\mathbf{B}(A)$. Analogously, the sequence $(b_\infty, e_0, e_1, e_2, \dots)$ is a partition of unity in $\mathbf{B}(B)$. By Lemma 3.4, the map

$$x \mapsto (x \wedge a_\infty, x \wedge d_0, x \wedge d_1, x \wedge d_2, \dots)$$

is an isomorphism of A onto the product MV-algebra $[0, a_\infty] \times [0, d_0] \times [0, d_1] \times [0, d_2] \times \dots$ Similarly, the map

$$y \mapsto (y \wedge b_\infty, y \wedge e_0, y \wedge e_1, y \wedge e_2, \dots)$$

is an isomorphism of B onto $[0, b_\infty] \times [0, e_0] \times [0, e_1] \times [0, e_2] \times \dots$ By Lemma 2.4 and (15), the restriction of α to $[0, a_\infty]$ is an isomorphism of $[0, a_\infty]$ onto $[0, b_\infty]$, in symbols (and with a slight abuse of notation),

$$\alpha \colon [0, a_\infty] \cong [0, b_\infty].$$

Another application of Lemma 2.4 together with (16) yields, for each $n = 0, 1, 2, \ldots$, an isomorphism

$$\alpha\colon\ [0, d_{2n}] \cong [0, e_{2n+1}].$$

Similarly, from the isomorphism $\beta\colon\ [0, e_{2n}] \cong [0, d_{2n+1}]$ one obtains an isomorphism

$$\beta^{-1}\colon\ [0, d_{2n+1}] \cong [0, e_{2n}]$$

for each $n = 0, 1, 2, \ldots$ It is now easy to obtain an isomorphism of $[0, a_\infty] \times [0, d_0] \times [0, d_1] \times [0, d_2] \times \ldots$ onto $[0, b_\infty] \times [0, e_0] \times [0, e_1] \times [0, e_2] \times \ldots$, whence one has the desired isomorphism of A onto B. $\square$

If A happens to be a boolean algebra, the above theorem reduces to the boolean-algebraic Cantor-Bernstein theorem stated in the introduction, and proved by Sikorski and Tarski.

5. A RELATED RESULT BY JAKUBÍK

In his paper [4], Jakubík proved a different form of Cantor-Bernstein theorem for MV-algebras. In this section we shall compare Jakubík's result with our Theorem 4.1.

A *lattice isomorphism* between two MV-algebras A and B is a one-one map of A onto B that preserves the underlying lattice structures of A and B. We say that A and B are *lattice isomorphic* iff there is a lattice isomorphism between A and B.

Let $\mathcal{D} \subseteq [0, 1]$ be the MV-algebra consisting of all rational numbers in $[0, 1]$ whose denominator is $1, 2, 4, 8, 16, \ldots$ Let $\mathcal{Q}$ be the subalgebra of $[0, 1]$ consisting of all rational numbers in $[0, 1]$. Then $\mathcal{D}$ and $\mathcal{Q}$ are lattice isomorphic (as denumerable, densely ordered chains with two endpoints) but they are not isomorphic MV-algebras. As a matter of fact, the equation $x \oplus x = \neg x$ has a solution in $\mathcal{Q}$, but does not have any solution in $\mathcal{D}$. Thus, the existence of a lattice isomorphism between two MV-algebras need not imply that the two MV-algebras are isomorphic. Trivially, if two MV-algebras are isomorphic then their underlying lattices are isomorphic.

For any MV-algebra A let us consider the following property:

(∗) If $a \in A$ and $[0, a]$ is a boolean algebra, then $a \in \mathbf{B}(A)$.

Jakubík proved

Theorem 5.1 [4]. *Let A and B be complete MV-algebras satisfying condition (∗). Suppose that for some $a \in A$, $b \in B$, A is lattice isomorphic to $[0, b]$ and B is lattice isomorphic to $[0, a]$. Then A and B are isomorphic as MV-algebras.*

The rest of this section is devoted to a comparison between Jakubík's theorem and our Theorem 4.1. To this aim, we present an example that simultaneously shows the necessity of condition $(*)$ in Jakubík's Theorem 5.1 and the necessity of the assumption that a and b are boolean in our Theorem 4.1.

Example. Let $\mathcal{K} = \{0, 1/2, 1\}$ be the uniquely determined three-element subalgebra of the MV-algebra $[0, 1]$ from Example 2.1. Denote by A the product of denumerably many copies of $\mathcal{K}$,

$$A = \mathcal{K} \times \mathcal{K} \times \mathcal{K} \times \ldots$$

With pointwise defined operations, A is a complete MV-algebra. Let elements $a, b \in A$ be defined by

$$\begin{aligned} a &= (1/2, 1, 1, 1, \ldots), \\ b &= (0, 1, 1, 1, \ldots). \end{aligned}$$

Then $B = [0, a]$ equipped with the operations from Definition 2.2 is a complete MV-algebra which is (isomorphic to) an interval of A. On the other hand, A is isomorphic to $[0, b]$ via the isomorphism $\alpha\colon A \to [0, b]$ defined by

$$\alpha((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, x_3, \ldots).$$

A fortiori, B is lattice isomorphic to an interval of A, and A is lattice isomorphic to the interval $[0, b]$ of B. Nevertheless, A and B are not isomorphic MV-algebras. Indeed, the element $c = (1/2, 0, 0, \ldots)$ is an atom of B (minimal nonzero element) and it also belongs to the boolean algebra $\mathbf{B}(B)$, while no atom of A is boolean.

Trivially, the interval $[0, c] = \{0, c\}$ is a boolean algebra, but the atom c is not boolean in A, and condition $(*)$ is not satisfied. On the other hand, all the other assumptions of Theorem 5.1 are satisfied. This shows the necessity of assumption $(*)$ in Jakubík's Theorem 5.1. The present example also shows that our Theorem 4.1 would no longer hold without assuming the elements a and b therein to be boolean.

Note that Theorem 4.1 also holds for MV-algebras not satisfying condition $(*)$. We can, for example, apply it to the MV-algebras A and B of the above example. On the other hand, the assumption that a and b are boolean is not needed in Theorem 5.1.

We finally remark that Theorem 5.1 is stated for complete MV-algebras, while our result here is valid for a larger class of σ-complete MV-algebras.

Altogether, Theorems 5.1 and 4.1 are incomparable.

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