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# ON A THEOREM OF HOLICKÝ AND ZELENÝ CONCERNING BOREL MAPS WITHOUT $\sigma$-COMPACT FIBERS

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*Abstract.* The paper is concerned with a recent very interesting theorem obtained by Holický and Zelený. We provide an alternative proof avoiding games used by Holický and Zelený and give some generalizations to the case of set-valued mappings.

*Keywords*: Borel maps, $\sigma$-compact sections, set-valued maps

*MSC 2000*: 54H05, 26A21, 28A05, 54C10

## 1. Introduction

We shall consider only separable metrizable spaces. Our terminology follows Kuratowski [8] and Kechris [6]. We denote by $2^{\mathbb{N}}$ the Cantor set and $\mathbb{N}^{\mathbb{N}}$ is the space of irrationals.

Holický and Zelený [5] proved recently a very interesting theorem that if $f\colon X \to Y$ is a Borel map between complete spaces with uncountably many non-$\sigma$-compact fibres, then $f$ takes a closed set in $X$ to a non-Borel set in $Y$.

More specifically, Holický and Zelený established that for any such map $f\colon X \to Y$, there is a Cantor set $K$ in $Y$ and a homeomorphism $k\colon K \times \mathbb{N}^{\mathbb{N}} \to P$, $P \subset X$, such that $f \circ k(y,t) = y$ for $(y,t) \in K \times \mathbb{N}^{\mathbb{N}}$, and $f(x) \notin K$ for $x \in \overline{P} \setminus P$.

A key element in the proof in [5] is a parametric version of the Kechris-Louveau-Woodin theorem, cf. Section 2. The proof given by Holický and Zelený involves a closed game introduced by Louveau and Saint Raymond.

In this note we use an approach from [10] to present a proof of a certain parametric version of the Kechris-Louveau-Woodin theorem, based directly on a classical theorem of Hurewicz. We shall also give an extension of the Holický-Zelený theorem to the case of set-valued functions.

We would like to thank the referee for remarks which improved the exposition.

## 2. Some background

Let $X$ be a complete space and let $A$ be an analytic not $F_\sigma$ set in $X$. A classical theorem of Hurewicz asserts that there is a copy $T$ of $2^{\mathbb{N}}$ in $X$ with $T\setminus A$ countable and dense in $T$, cf. [6, 21.18]. Kechris, Louveau and Woodin [7], [6, 21.22] strengthened this result as follows: if $B \subset X \setminus A$ and each $F_\sigma$ set in $X$ containing $A$ hits $B$, then there is a copy $T$ of $2^{\mathbb{N}}$ in $A \cup B$ with $T \cap B$ countable and dense in $T$.

We shall use the following closely related fact.

**2.1. Lemma.** *Let $G$ be a $G_\delta$ set in a complete space $X$, let $H \subset X \setminus G$, and let $R \subset X \times X$ be a closed symmetric relation on $X$. Assume that each $F_\sigma$ set in $X$ containing $G$ intersects $H$, and $G \times H \cap R = \emptyset$. Then there is a copy $T$ of $2^{\mathbb{N}}$ in $G \cup H$ with $T \cap H$ countable and dense in $T$ and $(s,t) \notin R$ for any distinct $s, t \in T$.*

A justification requires only some adaptations of standard proofs of the Hurewicz theorem, cf. [4, p. 333]. To be more specific, first one removes from $X$ all open sets $U$ such that $U \cap G$ is contained in some $F_\sigma$ set disjoint from $H$, and next, one replaces $H$ by its countable dense subset. This allows one to concentrate on the case where both $G$ and $H$ are dense in $X$ and $H$ is countable. In this case the classical Hurewicz's arguments need only a slight modification. One can also get Lemma 2.1 from [1, Proposition 2.1].

Incidentally, the Kechris-Louveau-Woodin theorem can be derived from Lemma 2.1, cf. [9].

We shall need also Jankov-von Neumann selection theorem [6, 29.9]. Let $\mathscr{BA}(X)$ be the $\sigma$-algebra generated by analytic sets in complete space $X$. A mapping $f\colon X \to Y$ is $\mathscr{BA}$-measurable if $f^{-1}[U] \in \mathscr{BA}(X)$ for any open $U$ in $Y$. The Jankov-von Neumann theorem asserts that for any analytic set $E \subset X \times Y$ in the product of complete spaces, with all vertical sections $E_x$ nonempty, there is a $\mathscr{BA}$-measurable mapping $f\colon X \to Y$ such that $f(x) \in E_x$, for $x \in X$.

## 3. A parametric version of the Kechris-Louveau-Woodin theorem

Given an $M \subset S \times T$ we denote respectively the vertical and the horizontal sections of $M$ by

$$M_s = \{t\colon (s,t) \in M\}, \quad M^t = \{s\colon (s,t) \in M\}. \tag{1}$$

Let $Z$ be a complete space. We denote by $\mathscr{H}(2^{\mathbb{N}}, Z)$ the space of embeddings of the Cantor set into $Z$ with the topology of uniform convergence and by $\mathscr{K}(Z)$ the space of compact subsets of $Z$ with the Vietoris topology, cf. [6]. Both spaces are completely metrizable.

Let us recall that $f\colon X \to Z$ is $\mathscr{B}\mathscr{A}$-measurable if $f^{-1}[U]$ is in the $\sigma$-algebra generated by analytic sets in $X$, for any open $U$ in $Z$, cf. Section 2.

**3.1. Theorem.** *Let $A, B \subset X \times Y$ be disjoint analytic sets in the product of complete spaces $X$, $Y$ such that every $F_\sigma$ set in $Y$ containing $A_x$ hits $B_x$, $x \in X$. Let $C, D$ be disjoint countable dense sets in $2^{\mathbb{N}}$. Then there are $\mathscr{B}\mathscr{A}$-measurable mappings $h\colon X \to \mathscr{H}(2^{\mathbb{N}}, Y)$ and $H_n\colon X \to \mathscr{K}(2^{\mathbb{N}})$ such that, with $G(x) = 2^{\mathbb{N}} \setminus \bigcup_{n=1}^{\infty} H_n(x)$,*

(2) $$C \subset G(x), \quad h(x)[G(x)] \subset A_x, \quad \text{for } x \in X,$$

*and*

(3) $$h(x)[D] \subset B_x, \qquad \text{for } x \in X.$$

Proof. Let $\pi\colon X \times Y \times 2^{\mathbb{N}} \to X \times Y$, $p\colon Y \times 2^{\mathbb{N}} \to Y$ be the projections. Let $G \subset X \times Y \times 2^{\mathbb{N}}$ be a $G_\delta$ set such that

(4) $$A = \pi[G] \quad \text{and} \quad H = \pi^{-1}[B].$$

Let

(5) $$E = \{(x, f) \in X \times \mathscr{H}(2^{\mathbb{N}}, Y \times 2^{\mathbb{N}})\colon\ f[C] \subset G_x,\ f[D] \subset H_x,\ p \circ f \in \mathscr{H}(2^{\mathbb{N}}, Y)\}.$$

We shall check that

(6) $$E \text{ is analytic and } E_x \neq \emptyset \text{ for } x \in X.$$

For any $t \in 2^{\mathbb{N}}$ let us consider the continuous mapping $e_t\colon X \times \mathscr{H}(2^{\mathbb{N}}, Y \times 2^{\mathbb{N}}) \to X \times Y \times 2^{\mathbb{N}}$ defined by $e_t(x, f) = (x, f(t))$. Then the set

$$E' = \bigcap_{c \in C} e_c^{-1}[G] \cap \bigcap_{d \in D} e_d^{-1}[H]$$

is analytic, $C, D$ being countable. The set $W = \{f \in \mathscr{H}(2^{\mathbb{N}}, Y \times 2^{\mathbb{N}})\colon p \circ f \in \mathscr{H}(2^{\mathbb{N}}, Y)\}$ is of type $G_\delta$. We conclude that $E = E' \cap (X \times W)$ is analytic.

To check the second part of (6), let us fix $x \in X$. Since $p[G_x] = A_x$, $H_x = p^{-1}(B_x)$ and the projection parallel to the compact axis takes closed sets to closed sets, every $F_\sigma$ set in $Y \times 2^{\mathbb{N}}$ containing $G_x$ hits $H_x$. Let $R = \{(u, v) \in (Y \times 2^{\mathbb{N}})^2 \colon p(u) = p(v)\}$. Then Lemma 2.1 can be applied to the triple $G_x, H_x, R$, providing a copy $T \subset G_x \cup H_x$ of $2^{\mathbb{N}}$ with $T \cap H_x$ countable and dense in $T$ and $p$ injective on $T$. Let $f \colon 2^{\mathbb{N}} \to T$ be a homeomorphism with $f[D] = T \cap H_x$, cf. [3, 4.3.H(e)]. Then $f[C] \subset G_x$ and $p \circ f \in \mathscr{H}(2^{\mathbb{N}}, Y)$. It follows that $f \in E_x$.

Having checked (6), one can apply the Jankov-von Neumann theorem, cf. Section 2, to get a $\mathscr{B}\mathscr{A}$-measurable mapping

$$k \colon X \to \mathscr{H}(2^{\mathbb{N}}, Y \times 2^{\mathbb{N}}), \quad k(x) \in E_x. \tag{7}$$

Let $X \times Y \times 2^{\mathbb{N}} \setminus G = \bigcup_{n=1}^{\infty} F_n$, where $F_n$ are closed. We set

$$h(x) = p \circ k(x), \quad H_n(x) = k(x)^{-1}[(F_n)_x]. \tag{8}$$

Then $h \colon X \to \mathscr{H}(2^{\mathbb{N}}, Y)$ and $H_n \colon X \to \mathscr{K}(2^{\mathbb{N}})$ are $\mathscr{B}\mathscr{A}$-measurable mappings. This is transparent for $h$. To check $\mathscr{B}\mathscr{A}$-measurability of $H_n$ let us notice that $H_n = \varphi \circ \psi$, where $\varphi(f, K) = f^{-1}[K]$ ($f \in \mathscr{H}(2^{\mathbb{N}}, Y \times 2^{\mathbb{N}})$, $K \in \mathscr{K}(Y \times 2^{\mathbb{N}})$) and $\psi(x) = (k(x), (F_n)_x)$. Since $\varphi$ is Borel and $\psi$ is $\mathscr{B}\mathscr{A}$-measurable, $x \mapsto (F_n)_x$ being Borel, cf. [6, 11.4 ii)], the composition $\varphi \circ \psi$ is $\mathscr{B}\mathscr{A}$-measurable. Therefore, $h$ and $H_n$ satisfy the assertion of the theorem, cf. (4), (5) and (7), (8). □

Let us comment on Theorem 3.1. Since the mappings $h$ and $H_n$ are $\mathscr{B}\mathscr{A}$-measurable, there is a dense $G_\delta$ set $P$ in $X$ such that the restrictions of $h$ and $H_n$ to $P$ are continuous, cf. [6, 29.5]. Let $K$ be any compact set in $P$. The continuity of $H_n$ on $K$ implies that $G = 2^{\mathbb{N}} \setminus \bigcup_{n=1}^{\infty} \bigcup\{H_n(x) \colon x \in K\}$ is a $G_\delta$ set. By (2) and (3), $C \subset G$, $h(x)[G] \subset A_x$, and $h(x)[D] \subset B_x$, for any $x \in K$. Since $G$ and $D$ are dense in $2^{\mathbb{N}}$, Lemma 2.1 (with $R$ being the diagonal) provides a Cantor set $T \subset G \cup D$ with $\overline{T \cap D} = T$. The map $\Phi \colon K \times T \to K \times Y$ defined by $\Phi(x, t) = (x, h(x)(t))$ is an embedding which sends $K \times (T \setminus D)$ to $A$ and $K \times (T \cap D)$ to $B$.

This is a parametric version of the Kechris-Louveau-Woodin theorem, established by Holický and Zelený [5, Lemma 1].

We shall close this section with a lemma containing some observations which will be useful in the next section. The expression "for almost every compact set in $X$" refers to the Baire category in $\mathscr{K}(X)$. Let us recall that in a complete space $X$ without isolated points almost every nonempty compact set is a Cantor set.

**3.2. Lemma.** *Let $A, B, C, D$ and $h$ be as in Theorem 3.1. Assume in addition that $X$ has no isolated points and*

$$(9) \qquad A^y \text{ is meager in } X \text{ for any } y \in Y.$$

*Then for almost every Cantor set $K$ in $X$, there is a Cantor set $T$ in $2^{\mathbb{N}}$ such that*

$$(10) \qquad \overline{T \cap D} = T,$$

*and*

$$(11) \qquad h(x_1)[T \setminus D] \cap h(x_2)[T] = \emptyset, \quad \text{for } x_1 \neq x_2,\ x_1, x_2 \in K.$$

P r o o f. Let $P$ be a dense $G_\delta$ set in $X$ such that the $\mathscr{BA}$-measurable mapping $h$ restricted to $P$ is continuous. Let us fix $c \in C$, $u \in C \cup D$, $c \neq u$, and let $\mathscr{G}_{c,u}$ be the collection of compact sets $K$ in $P$ such that

$$(12) \qquad h(x_1)(c) \neq h(x_2)(c) \neq h(x_1)(u), \quad \text{for } x_1 \neq x_2,\ x_1, x_2 \in K.$$

Then $\mathscr{G}_{c,u}$ is a $G_\delta$ set in $\mathscr{K}(X)$. Let us check that $\mathscr{G}_{c,u}$ is dense, and hence comeager, in $\mathscr{K}(X)$. To this end let us consider nonempty open sets $V_1, \ldots, V_n$ in $X$. We have to find $K \subset \bigcup_{i=1}^{n} V_i$ intersecting all $V_i$ and satisfying (12). Let us set

$$(13) \qquad k(x) = h(x)(c), \quad l(x) = h(x)(u), \quad x \in P.$$

The functions $k\colon P \to Y$, $l\colon P \to Y$ are continuous and $k^{-1}(y) \subset A^y$, cf. (2). By (9), the fibers of $k$ are meager in $P$. If $l^{-1}(y)$ is nonmeager, then being closed in $P$, it contains the intersection of a nonempty open set with $P$. It follows that the set $J$ of points $y$ with $l^{-1}(y)$ nonmeager in $P$ is at most countable. Therefore, one can choose inductively $a_j \in V_j \cap P \setminus k^{-1}[J]$ such that $a_j \notin \bigcup_{i<j} (k^{-1}[k(a_i)] \cup k^{-1}[l(a_i)] \cup l^{-1}[k(a_i)])$. Then $K = \{a_1, \ldots, a_n\}$ satisfies (12), cf. (13).

We have demonstrated that each $\mathscr{G}_{c,u}$ is comeager, and in effect almost every Cantor set $K$ in $P$ satisfies (12) simultaneously for all pairs $c \neq u$ with $c \in C$, $u \in C \cup D$. Let us fix any such $K$. We shall find a Cantor set $T$ in $2^{\mathbb{N}}$ satisfying (10) and (11). Let

$$(14) \qquad G = \{t \in 2^{\mathbb{N}} : h(x_1)(t) \neq h(x_2)(t) \text{ for any } x_1 \neq x_2,\ x_1, x_2 \in K,$$
$$\text{and } h(x_1)(t) \neq h(x_2)(d) \text{ for any } d \in D \text{ and } x_1, x_2 \in K\}.$$

Using the continuity of the mapping $(x,t) \mapsto h(x)(t)$ on the product $K \times 2^{\mathbb{N}}$, one easily verifies that $G$ is a $G_\delta$ set. It is transparent that $G \cap D = \emptyset$ and, by (12), $C \subset G$. Let $R$ be the closed symmetric set in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ consisting of pairs $(s,t)$ such that $h(x_1)(s) = h(x_2)(t)$ for some $x_1, x_2 \in K$. Then $G \times D \cap R = \emptyset$, cf. (14). Therefore, Lemma 2.1 can be applied to the triple $G, D, R$, providing a Cantor set $T \subset G \cup D$ with $\overline{T \cap D} = T$ and $(s,t) \notin R$ for any distinct $s, t \in T$. One readily checks that $T$ satisfies also (11). □

## 4. Set-valued Borel functions

Let us recall that $\mathscr{K}(E)$ is the space of compact subsets of $E$ with the Vietoris topology and the phrase "almost all" refers to the Baire category.

The following fact provides an extension of the Holický-Zelený theorem.

**4.1. Theorem.** *Let $S, E$ be complete spaces without isolated points and let $F\colon S \to \mathscr{K}(E)$ be a Borel mapping whose values are boundary in $E$. Then the following conditions are equivalent:*

(i) *for almost all $x \in E$, the set $\{s \in S\colon\ x \in F(s)\}$ is not $\sigma$-compact,*

(ii) *for almost all Cantor sets $K$ in $E$ there is a homeomorphism $k\colon\ K \times \mathbb{N}^{\mathbb{N}} \to P$, $P \subset S$, such that $x \in F(k(x,t))$ and $F(s) \cap K = \emptyset$ for $s \in \overline{P} \setminus P$.*

We shall first establish a counterpart to Lemma 2 in [5].

**4.2. Lemma.** *Let $X$ be a dense $G_\delta$ subset of the complete space without isolated points $E$ and let $H\colon\ X \to \mathscr{K}(E)$ be a Borel mapping such that $x \notin H(x)$ and the interior of $H(x)$ is empty for all $x \in X$. Then for almost all $K \in \mathscr{K}(E)$, $K \cap \bigcup\{H(x)\colon\ x \in K\} = \emptyset$.*

P r o o f. Let $P$ be a dense $G_\delta$ subset of $X$ such that $H$ restricted to $P$ is continuous. Since $\mathscr{K}(P) \subset \mathscr{K}(E)$ is comeager and the set $\mathscr{L} = \{K \in \mathscr{K}(P)\colon$ $K \cap H(x) = \emptyset$ for $x \in K\}$ is open in $\mathscr{K}(P)$, it is sufficient to prove the density of $\mathscr{L}$. Let $V_1, \dots, V_n$ be nonempty open subsets of $P$. Since $\{(x,t)\colon\ x \in P,\ t \in H(x)\}$ is Borel with all vertical sections meager, by the Kuratowski-Ulam theorem the set $Z = \{t \in E\colon\ \{x \in P\colon\ t \in H(x)\}$ is nonmeager$\}$ is meager. Therefore we can successively choose $t_i \in P \cap V_i \setminus \Bigl[\bigcup_{j=1}^{i-1} H(t_j) \cup \bigcup_{j=1}^{i-1} \{x \in P\colon\ t_j \in H(x)\} \cup Z\Bigr]$ for $i = 1, \dots, n$. Then $\{t_1, \dots, t_n\}$ belongs to $\mathscr{L}$ and hits every $V_i$. □

Before passing to the proof of Theorem 4.1, let us make a simple observation. If $\mathscr{L}$ is a comeager family of compact subsets of a complete space $X$, then $\bigcup \mathscr{L}$ is comeager in $X$. Indeed, let $\mathscr{L}' \subset \mathscr{L}$ be dense $G_\delta$ in $\mathscr{K}(X)$. Then $\bigcup \mathscr{L}' \subset X$ is analytic. Thus $\bigcup \mathscr{L}' = (U \setminus M_1) \cup M_2$, where $U$ is open and $M_i$ are meager in $X$, cf. [6, 8.21, 29.5]. If $\overline{U \setminus M_1} = X$, $\bigcup \mathscr{L}'$ is comeager in $X$. Otherwise let us take the nonempty open $V = X \setminus \overline{U \setminus M_1}$. Then $\mathscr{K}(V \setminus M_2)$ is comeager in $\mathscr{K}(V)$ and disjoint from $\mathscr{L}'$, a contradiction. □

Proof of Theorem 4.1. (i)⇒(ii). Let $X \subset E$ be a dense $G_\delta$ set such that $\{s \in S\colon\ x \in F(s)\}$ is not $\sigma$-compact for all $x \in X$. Let us set

$$A = \{(x, s) \in X \times S\colon\ x \in F(s)\}, \quad B = X \times S \setminus A. \tag{15}$$

Every $\sigma$-compact set containing $A_x$ hits $B_x$, $x \in X$. Therefore, the assumptions of Theorem 3.1 are satisfied and let $h$ be the map described in this theorem. By Lemma 3.2 for almost all Cantor sets $K$ in $X$, and hence in $E$, there is a Cantor set $T$ in $2^{\mathbb{N}}$ satisfying (10) and (11). Moreover, one can also demand that $(x, h(x)(t)) \in A$ for $x \in K$, $t \in T \setminus D$, using a simple argument that follows the proof of Theorem 3.1. The map $h$ is continuous on a dense $G_\delta$ subset $X'$ of $X$. To simplify the notation we shall assume that $X' = X$. Let $d \in D$ and let us define $H\colon\ X \to \mathscr{K}(E)$ by $H(x) = F(h(x)(d))$. By (15) and condition (3) in Theorem 3.1, $x \notin H(x)$. Therefore, by Lemma 4.2, we can assume in addition that

$$K \cap F(h(x)(d)) = \emptyset \quad \text{for all } x \in X \text{ and } d \in D. \tag{16}$$

We shall verify that each such $K$ satisfies (ii). Let $g\colon\ \mathbb{N}^{\mathbb{N}} \to T \setminus D$ be any homeomorphism. Let us define $k\colon\ K \times \mathbb{N}^{\mathbb{N}} \to S$ by

$$k(x, t) = h(x)(g(t)), \quad P = k[K \times \mathbb{N}^{\mathbb{N}}]. \tag{17}$$

Using (11), one easily checks that $k$ is an embedding. By (17), $x \in F(k(x, t))$ for $(x, t) \in K \times \mathbb{N}^{\mathbb{N}}$. Let us note that $\overline{P}$ is included in the compact set $\bigcup_{x \in K} h(x)[T]$. Thus, if $s \in \overline{P} \setminus P$ then $s = h(x)(d)$ for some $d \in D$. Therefore, by (16), $F(s) \cap K = \emptyset$ for $s \in \overline{P} \setminus P$.

(ii)⇒(i) Let us note that if $K$ is a Cantor set described in (ii), then for every $x \in K$ the set $\{s \in S\colon\ x \in F(s)\}$ is not $\sigma$-compact. Therefore, (i) follows from the remark preceding the proof. □

**Corollary 4.3.** *Let $S, E$ be complete spaces and let $F\colon\ S \to \mathscr{K}(E)$ be a Borel mapping. Suppose that the set $\{x \in S\colon\ y \in F(x)\}$ is not $\sigma$-compact for uncountably many $y \in E$. Then there is a closed subset $\tilde{S}$ of $S$ and a Borel function $f\colon\ S \to E$ with $f(x) \in F(x)$ whenever $F(x) \neq \emptyset$, such that $f[\tilde{S}]$ is not Borel.*

P r o o f. Let us first assume that $E$ is the ternary Cantor set $\mathscr{C} \subset [0,1]$, all values of $F$ are nonempty and the set $\{x \in S\colon\ y \in F(x)\}$ is not $\sigma$-compact for all but countably many $y \in \mathscr{C}$. Let us set

$$A = \{(y,t)\colon\ t \in S,\ \ y \in F(t)\}, \quad B = \mathscr{C} \times S \setminus A.$$

For each open interval $J_k$ with rational endpoints let us put $A(J_k) = \{s\colon\ J_k \subset F(s)\}$. Since every horizontal section of the Borel set $(J_k \times S) \cap B$ is $\sigma$-compact, the set $S \setminus A(J_k) = \pi_S[(J_k \times S) \cap B]$ is Borel, $\pi_S$ being the projection. Let us consider two cases.

**A**. The set $A(J_k)$ is not $\sigma$-compact for some $k \in \mathbb{N}$. Then by the Hurewicz theorem there is a Cantor set $T \subset S$ such that $T \setminus A(J_k)$ is countable and dense in $T$. Let $g_1\colon\ T \cap A(J_k) \to J_k$ be a continuous function with $g_1[T \cap A(J_k)]$ not Borel, and let $g_2\colon\ S \to \mathscr{C}$ be any Borel selection of $F$. Then $\tilde{S} = T$ and the function $f\colon\ S \to \mathscr{C}$ that agrees with $g_1$ on $\tilde{S} \cap A(J_k)$ and with $g_2$ on $S \setminus (A(J_k) \cap \tilde{S})$ has the required properties.

**B**. $A(J_k)$ is $\sigma$-compact for all $k \in \mathbb{N}$. Let us put $A' = A \setminus \bigcup\limits_{k=1}^{\infty} J_k \times A(J_k)$. Then every horizontal section of the Borel set $A'$ is compact, boundary and nonempty. For any $x$, the section $(A')_x$ is the difference of $A_x$ and its $\sigma$-compact subset, and hence it is not $\sigma$-compact. Therefore we can assume without loss of generality that the set $F(x)$ is boundary for all $x \in S$. Let $K$ and $k$ be respectively a Cantor set and a homeomorphism, such as in Theorem 4.1. Let $M \subset K \times \mathbb{N}^{\mathbb{N}}$ be a closed set such that $\pi_K[M]$ is not Borel. Let us put $N = k[M]$, $\tilde{S} = \overline{k[M]}$ and let $f\colon\ S \to \mathscr{C}$ coincides on $N$ with $\pi_K \circ k^{-1}(x)$ and with any Borel selection for $F$ on $S \setminus N$. Since $K \cap f[\tilde{S}] = \pi_K[M]$, the set $f[\tilde{S}]$ is not Borel.

Let us consider now the general case. Because the set of all $y \in E$ such that $\{t \in S\colon\ y \in F(t)\}$ is not $\sigma$-compact is analytic (see [2, Remarque (b), p. 255]), it contains a Cantor set $\mathscr{C}$. Let $X = \{x \in S\colon\ F(x) \cap \mathscr{C} \neq \emptyset\}$. We shall apply the reasoning from the first part of the proof to the Borel function $F'$ on $S$ which sends $x \in X$ to $F(x) \cap \mathscr{C}$ and associates to $x \in S \setminus X$ a fixed singleton $\{c_0\}$, $c_0 \in \mathscr{C}$. In effect, we obtain a closed set $\tilde{S}$ in $S$ and a Borel map $f\colon\ S \to \mathscr{C}$, $f(x) \in F'(x)$, with $f[\tilde{S}]$ non-Borel. It suffices to replace the function $f$ on the set $\{x \in S \setminus X\colon\ F(x) \neq \emptyset\}$ by any Borel selection for $F$. The proof is completed. □

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