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ACHROMATIC NUMBER OF $K_5 \times K_n$ FOR SMALL n

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Abstract. The achromatic number of a graph G is the maximum number of colours in a proper vertex colouring of G such that for any two distinct colours there is an edge of G incident with vertices of those two colours. We determine the achromatic number of the Cartesian product of K_5 and K_n for all $n \leqslant 24$.

Keywords: complete vertex colouring, achromatic number, Cartesian product, complete graph

MSC 2000: 05C15

1. Introduction

Consider a simple finite graph G and its vertex k-colouring f mapping $V(G)$ into $\{1, 2, \ldots, k\}$. As usual, f is proper if $f(u) \neq f(v)$ whenever $uv \in E(G)$. Let $\mathrm{chr}(G)$ denote the chromatic number of G, the minimum k such that there is a proper vertex k-colouring of G. It is easy to see that any proper vertex $\mathrm{chr}(G)$-colouring of G is *complete*: for every $i, j \in \{1, 2, \ldots, \mathrm{chr}(G)\}$, $i \neq j$, there is an edge uv in G with $f(u) = i$ and $f(v) = j$. In other words, $\mathrm{chr}(G)$ is the minimum k admitting a complete proper vertex k-colouring of G. It is natural to ask also for the *maximum* l admitting a complete proper vertex l-colouring of G, i.e., for the *achromatic number* of G, in symbol $\mathrm{achr}(G)$. This graph invariant was introduced by Harary, Hedetniemi and Prins in [5], where the authors proved among other things also the following interpolation theorem:

Theorem 1. *If G is a graph and k an integer with $\mathrm{chr}(G) \leqslant k \leqslant \mathrm{achr}(G)$, then there exists a complete proper vertex k-colouring of G.*

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It is known, see Yannakakis and Gavril [8], that, given a graph G and a positive integer k, to decide whether $\operatorname{achr}(G) \geqslant k$ is an NP-complete problem. Note that classes of graphs with exactly determined achromatic number are quite rare. A reader can find a survey of results on the achromatic number in Edwards [4].

Cartesian products of complete graphs form a class of graphs with structure simple enough to evaluate (at least for some subclasses) the achromatic number. The Cartesian product of complete graphs K_m and K_n is the graph $K_m \times K_n$ with $V(K_m \times K_n) = \{(i,j)\colon i \in \{1,2,\ldots,n\}\}$, in which (i_1, j_1) is adjacent to (i_2, j_2) if and only if the pairs (i_1, j_1), (i_2, j_2) have exactly one common co-ordinate. Since the graphs $K_m \times K_n$ and $K_n \times K_m$ are isomorphic, when analyzing $\operatorname{achr}(K_m \times K_n)$ we may suppose that $m \leqslant n$. The achromatic number of $K_m \times K_n$ is completely determined for $m = 1,2,3,4$: It is known that $\operatorname{achr}(K_1 \times K_n) = \operatorname{achr}(K_n) = n$ (trivially), $\operatorname{achr}(K_2 \times K_n) = n+1$ (easily), $\operatorname{achr}(K_3 \times K_3) = 5$ and $\operatorname{achr}(K_3 \times K_n) = \lfloor \frac{3}{2}n \rfloor$ for $n \geqslant 4$ (proved independently by Horňák and Puntigán [7] and Chiang and Fu [2]), $\operatorname{achr}(K_4 \times K_n) = 2n$ if $4 \leqslant n \leqslant 12$, $\operatorname{achr}(K_4 \times K_{13}) = 24$, $\operatorname{achr}(K_4 \times K_n) = \lfloor \frac{4}{3}n \rfloor$ if $14 \leqslant n \leqslant 24$ and $\operatorname{achr}(K_4 \times K_n) = \lfloor \frac{5}{3}n \rfloor$ for $n \geqslant 25$, see [7]. Bouchet [1] found that $\operatorname{achr}(K_6 \times K_6) = 18$. Chiang and Fu [3] generalized his result in an important way by showing that $\operatorname{achr}(K_m \times K_m) = \frac{1}{2}p^{2r}(p^r+1)$ holds for an odd prime p, a positive integer r and $m = \frac{1}{2}p^r(p^r+1)$. We succeeded in establishing values of $\operatorname{achr}(K_5 \times K_n)$ in [6] for $n \geqslant 25$; they are resumed in Theorem 4. The aim of the present paper is to complete the results of [6] for $n \leqslant 24$.

For integers p, q, we denote by $[p,q]$ the set of all integers z with $p \leqslant z \leqslant q$. Using the structure of $K_m \times K_n$, we can transform the problem of determining $\operatorname{achr}(K_m \times K_n)$ as follows: For a positive integer p, let $M^p_{m,n}$ be the set of all $m \times n$ matrices A with entries from $[1,p]$ (an entry in the row i and the column j is the colour of the vertex (i,j)) such that the entries in any *line* (a row or a column) of A are distinct (the corresponding p-colouring of $K_m \times K_n$ is proper) and for every $i, j \in [1,p]$, $i \neq j$, there is a line of A containing both i and j (the colouring is complete). Evidently, $\operatorname{achr}(K_m \times K_n)$ is the maximum p with $M^p_{m,n} \neq \emptyset$. If we permute rows and/or columns of a matrix in $M^p_{m,n}$, what results is again a matrix in $M^p_{m,n}$. This trivial (but important) fact will be frequently used throughout the paper. A colour (an entry) of a matrix $A \in M^p_{m,n}$ is a *k-colour* if it appears in A exactly k times.

2. Constructions

In this section we present some $5 \times n$ matrices which will turn out to be optimal for the achromatic number of $K_5 \times K_n$ in Section 3. We define $I_3 := \{1, 6\}$, $I_2 := \{2, 4, 5, 7, 8, 10\}$, $I_1 := \{3, 9\} \cup [11, 14]$, $I_0 := [15, 24]$ and $c(n) := 2n + a$ for $n \in I_a$, $a = 0, 1, 2, 3$.

Theorem 2. *If* $n \in [1, 24]$, *then* $\operatorname{achr}(K_5 \times K_n) \geqslant c(n)$.

P r o o f. For $n \leqslant 4$ we simply use the results of [7]. In what follows, we restrict ourselves to $n \in [5, 24]$.

For $n \in [5, 10]$ we present a matrix belonging to $M_{5,n}^{c(n)}$ in which $\bar{k}$ stands for $k+10$ and $\bar{\bar{l}}$ for $l + 20$:

$$\begin{pmatrix} 1&2&3&4&5 \\ 6&1&2&3&7 \\ 8&9&\bar{0}&7&4 \\ 5&\bar{1}&9&\bar{2}&6 \\ \bar{0}&\bar{2}&8&\bar{1}&9 \end{pmatrix} \quad \begin{pmatrix} 1&2&3&4&5&6 \\ 2&1&7&8&9&\bar{0} \\ \bar{1}&\bar{2}&4&3&7&\bar{3} \\ 5&\bar{4}&\bar{5}&\bar{0}&\bar{2}&8 \\ \bar{3}&\bar{5}&\bar{4}&9&6&\bar{1} \end{pmatrix} \quad \begin{pmatrix} 1&2&3&4&5&6&7 \\ 2&1&8&9&\bar{0}&\bar{1}&\bar{2} \\ \bar{3}&\bar{4}&4&3&\bar{5}&8&\bar{1} \\ \bar{1}&7&\bar{6}&\bar{0}&9&\bar{3}&8 \\ \bar{6}&\bar{5}&\bar{2}&6&\bar{4}&5&\bar{3} \end{pmatrix}$$

$$\begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 2&1&9&\bar{0}&\bar{1}&\bar{2}&\bar{3}&\bar{4} \\ \bar{5}&\bar{6}&4&3&\bar{3}&\bar{7}&\bar{1}&\bar{8} \\ \bar{8}&\bar{5}&\bar{4}&\bar{6}&6&5&\bar{7}&9 \\ \bar{7}&\bar{8}&\bar{5}&\bar{2}&\bar{6}&\bar{0}&8&7 \end{pmatrix} \quad \begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ \bar{3}&\bar{4}&\bar{5}&7&4&5&6&\bar{1}&\bar{2} \\ 3&\bar{0}&\bar{5}&\bar{6}&\bar{7}&\bar{8}&\bar{9}&\bar{2}&\bar{1} \\ \bar{5}&\bar{3}&\bar{4}&\bar{0}&9&\bar{6}&\bar{7}&1&8 \\ \bar{4}&\bar{5}&\bar{3}&\bar{8}&\bar{9}&\bar{0}&8&9&1 \end{pmatrix} \quad \begin{pmatrix} 1&2&3&4&5&6&7&8&9&\bar{0} \\ \bar{1}&\bar{2}&\bar{3}&\bar{4}&\bar{5}&\bar{6}&\bar{0}&7&8&9 \\ 2&\bar{7}&\bar{6}&\bar{8}&1&\bar{3}&\bar{9}&\bar{\bar{0}}&\bar{\bar{1}}&\bar{\bar{2}} \\ 3&\bar{5}&4&\bar{\bar{1}}&\bar{2}&\bar{\bar{2}}&\bar{7}&\bar{8}&\bar{9}&\bar{\bar{0}} \\ \bar{4}&\bar{9}&5&\bar{1}&6&\bar{\bar{0}}&\bar{\bar{1}}&\bar{\bar{2}}&\bar{7}&\bar{8} \end{pmatrix}$$

For $n \in [11, 14]$, consider the following matrices B_{n-8} and C_8:

$$B_3 = \begin{pmatrix} \bar{\bar{\bar{2}}}&1&2 \\ 2&\bar{\bar{\bar{3}}}&1 \\ 3&4&5 \\ 5&3&4 \\ 4&5&3 \end{pmatrix} \quad B_4 = \begin{pmatrix} \bar{\bar{\bar{4}}}&1&2&3 \\ 2&3&\bar{\bar{\bar{5}}}&1 \\ 4&5&6&7 \\ 7&4&5&6 \\ 6&7&4&5 \end{pmatrix} \quad B_5 = \begin{pmatrix} \bar{\bar{\bar{6}}}&1&2&3&4 \\ 3&4&\bar{\bar{\bar{7}}}&1&2 \\ 5&6&7&8&9 \\ 9&5&6&7&8 \\ 8&9&5&6&7 \end{pmatrix} \quad B_6 = \begin{pmatrix} \bar{\bar{\bar{8}}}&1&2&3&4&5 \\ 3&4&5&\bar{\bar{\bar{9}}}&1&2 \\ 6&7&8&9&\bar{0}&\bar{1} \\ \bar{1}&6&7&8&9&\bar{0} \\ \bar{0}&\bar{1}&6&7&8&9 \end{pmatrix}$$

$$C_8 = \begin{pmatrix} -16 & -15 & -14 & -13 & -12 & -11 & -10 & -9 \\ -8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \\ -13 & -16 & -15 & -14 & -2 & -1 & -4 & -3 \\ +1 & -9 & -8 & -7 & -11 & -12 & 0 & -10 \\ -6 & -5 & +1 & -10 & -9 & 0 & -11 & -12 \end{pmatrix}$$

Let $C_{8,2n}$ be the matrix obtained from C_8 by increasing all its entries by $2n$. The block matrix $M_n = (B_{n-8} C_{8,2n})$ has the following colour structure: colours of $[1, n-9]$ are 2-colours appearing in both rows 1, 2 of B_{n-8}, colours of $[n-8, 2n-$

17] are 3-colours appearing in all three rows 3, 4, 5 of B_{n-8}, colours of $[2n-16, 2n-13] \cup [2n-8, 2n-1]$ are 2-colours appearing in exactly one of the rows 1, 2 and in exactly one of the rows 3, 4, 5 of $C_{8,2n}$, colours of $[2n-12, 2n-9]$ are 3-colours appearing in all three rows 1, 4, 5 of $C_{8,2n}$, and colours of $[2n, 2n+1]$ are 3-colours appearing in exactly one of the rows 1, 2 of B_{n-8} and in both rows 4, 5 of $C_{8,2n}$.

All connections between 2-colours of B_{n-8} and 3-colours of B_{n-8} are realized in columns of B_{n-8}: any 3-colour of B_{n-8} covers three consecutive (modulo $n-8$) columns of B_{n-8}, and a maximum "column gap" between two exemplars of any 2-colour of B_{n-8} consists of $\lceil \frac{1}{2}(n-10) \rceil \leqslant 2$ columns. All other colour connections involving 2-colours of B_{n-8} are realized in one of the rows 1, 2 of M_n and all colour connections between 3-colours of B_{n-8} and 2-colours of $C_{8,2n}$ are realized in one of the rows 3, 4, 5 of M_n. It is easy to check that all colour connections between 2-colours of $C_{8,2n}$ and colours appearing not only in B_{n-8} are present in M_n. Clearly, because of the Pigeonhole Principle (PP), it is unnecessary to look for colour connections involving two 3-colours. Finally, as all rows of M_n contain n distinct colours and all columns of M_n contain five distinct colours, we have $M_n \in \mathcal{M}_{5,n}^{2n+1}$.

To conclude the proof, it is sufficient to use Proposition 1 of [6], showing that $\operatorname{achr}(K_5 \times K_n) \geqslant 2n$ for $n \in [12, 24]$. □

3. Optimality

Theorem 3. *If* $n \in [1, 24]$, *then* $\operatorname{achr}(K_5 \times K_n) = c(n)$.

Proof. Again we omit the case $n \in [1, 4]$. Let $n \in I_a$, so that $c(n) = 2n + a$. Because of Theorem 2, it suffices to show that $\operatorname{achr}(K_5 \times K_n) \leqslant 2n + a$. Proceeding by the way of contradiction, we assume that $\operatorname{achr}(K_5 \times K_n) \geqslant 2n + a + 1$. Then, by Theorem 1, we know that there is a matrix $A \in \mathcal{M}_{5,n}^{2n+a+1}$.

For a positive integer i, let C_i be the set of i-colours of A; put $c_i := |C_i|$, $c_{3+} := c_3 + c_4 + c_5$, $c_{4+} := c_4 + c_5$.

Claim 1. *If* $c_i > 0$, *then* $i \in [2, 5]$.

Proof of Claim 1. Clearly, $c_i = 0$ for $i \geqslant 6$ (PP). If some colour appears only once in A, all colours of A must be present in the corresponding row or in the corresponding column of A, so their number is at most $n + 4$. However, $2n + a + 1 \geqslant 2n + 1 \geqslant n + 5 + 1 > n + 4$, a contradiction. □

By Claim 1, we have $2n + a + 1 \leqslant \lfloor \frac{5}{2} n \rfloor$, which yields immediately a contradiction if $n \in [5, 6]$. Thus, from now on we suppose that $n \in [7, 24]$.

Claim 2. $c_2 \geqslant c_{4+} + n + 3a + 3$ *and* $c_{3+} \leqslant n - 2a - 2$.

Proof of Claim 2. Claim 1 implies $2n+a+1 = c_2+c_3+c_{4+}$ and $5n = \sum_{i=2}^{5} ic_i \geqslant 2c_2 + 3c_3 + 4c_{4+} = 2(2n + a + 1) + c_3 + 2c_{4+}$, so that $c_{3+} \leqslant c_3 + 2c_{4+} \leqslant n - 2a - 2$ and $c_2 - c_{4+} = (2n + a + 1 - c_3 - c_{4+}) - c_{4+} \geqslant 2n + a + 1 - (n - 2a - 2)$. □

Claim 3. $c_2 \geqslant 15$.

Proof of Claim 3. As a consequence of Claim 2, we obtain the following inequalities for $a = 0, 1$ and 2, respectively: $c_2 \geqslant n + 3 \geqslant 18$, $c_2 \geqslant n + 6 \geqslant 15$ and $c_2 \geqslant n + 9 \geqslant 16$. □

For sets $S_1 \subseteq [1, 5]$ and $S_2 \subseteq [1, n]$, an S_1-*row* is a row whose number is in S_1 and an S_2-*column* is a column whose number is in S_2. Instead of $\{s_1\}$-rows and $\{s_2\}$-columns we speak simply about s_1-rows and s_2-columns. For $i, j \in [1, 5]$, $i \neq j$, let $R_{i,j}$ denote the set of 2-colours occurring in both $\{i, j\}$-rows, $S_{i,j}$ the set of numbers of columns covered by the colours of $R_{i,j}$ and, for $l \in [1, 2]$, let $S_{i,j}^{(l)}$ be the set of numbers of $S_{i,j}$-columns containing l colours of $R_{i,j}$. For a colour α, we denote by S_α the set of numbers of columns covered by α. Put $r_{i,j} := |R_{i,j}|$, $s_{i,j} := |S_{i,j}|$, $s_{i,j}^{(l)} := |S_{i,j}^{(l)}|$, and let $t_{i,j}$ be the *total* number of colours appearing in both $\{i, j\}$-rows. Sets $R_{i,j,k}$ (of 3-colours) and numbers $r_{i,j,k}$ are defined analogously.

We associate with the matrix A an edge-labelled graph $K_5(A)$ as the graph K_5 with $V(K_5) = [1, 5]$, in which an edge $\{i, j\}$ is labelled with $r_{i,j}$.

Claim 4. *If* $i, j \in [1, 5], i \neq j$ *and* $r_{i,j} > 0$, *then* $t_{i,j} \leqslant 5 - a$. *Consequently, the graph* $K_5(A)$ *is labelled with numbers from* $[0, 5 - a]$.

Proof of Claim 4. Consider a 2-colour $\alpha \in R_{i,j}$. Because of connections with α, all colours missing in both $\{i, j\}$-rows must be present in one of the two S_α-columns, and the total number of colours in A is $2n + a + 1 \leqslant (2n - t_{i,j}) + 6$, so that $r_{i,j} \leqslant t_{i,j} \leqslant 5 - a$. □

The *weight* $w(G)$ of a subgraph G of the graph $K_5(A)$ is the sum of labels of all edges of G. Thus, $w(K_5(A)) = c_2$. By $\overline{w}(G)$ we denote the weight of $\overline{G}$, the complement of G.

Claim 5. *Any subgraph* $K_{1,4}$ *of* $K_5(A)$ *is of weight at least* $n - c_{3+} \geqslant 2a + 2$.

Proof of Claim 5. Since, by Claim 2, $c_{3+} \leqslant n - 2a - 2$, the claim follows from the fact that the number of 2-colours in any row of A is at least $n - c_{3+}$. □

Claim 6. *The graph $K_5(A)$ has a subgraph $K_2 \cup K_3$ of weight at least $\lceil \frac{2}{5} c_2 \rceil \geqslant \lceil \frac{2}{5}(n+3a+3) \rceil$.*

P r o o f of Claim 6. The graph $K_5(A)$ has ten subgraphs $K_2 \cup K_3$ and each of its edges appears in four such subgraphs: once in a K_2-component and three times in a K_3-component. So, by Claim 2, the sum of weights of those ten subgraphs is $4c_2 \geqslant 4(n+3a+3)$, and the maximum weight is at least $\lceil \frac{4}{10} c_2 \rceil$. □

Denote by $K(i,j)$ the subgraph $K_2 \cup K_3$ of $K_5(A)$ with $V(K_2) = \{i,j\}$ and by $K(i)$ the subgraph $K_{1,4}$ of $K_5(A)$ with parts $\{i\}$ and $[1,5] - \{i\}$. We may suppose without loss of generality that the subgraph $K(1,2)$ is of the maximum weight $w = r_{1,2} + (r_{3,4} + r_{3,5} + r_{4,5})$, and that $r_{3,4} \geqslant r_{3,5} \geqslant r_{4,5}$. We assume also that $r_{1,2}$ is the maximum weight of a K_2-component among all subgraphs $K_2 \cup K_3$ of $K_5(A)$ of weight w. Put $R := R_{3,4} \cup R_{3,5} \cup R_{4,5}$, $r := |R|$, $R_i := R_{1,i} \cup R_{2,i}$, $r_i := |R_i|$, $i \in [3,5]$, $\tilde{R} := R_3 \cup R_4 \cup R_5$ and $\tilde{r} := |\tilde{R}|$. Thus, r is the weight of the K_3-component of $K(1,2)$ and $c_2 = w + \tilde{r}$.

Claim 7. *If $\{i,j,k\} = [3,5]$, then $r_i \leqslant r_{j,i} + r_{k,i}$. If, moreover, $r_{j,k} > r_{1,2}$, then $r_i < r_{j,i} + r_{k,i}$.*

P r o o f of Claim 7. As $r_{j,k} + (r_{1,2} + r_{1,i} + r_{2,i}) = w(K(j,k)) \leqslant w(K(1,2)) = r_{1,2} + (r_{j,i} + r_{k,i} + r_{j,k})$, the first part of the claim is proved. The second issues from the assumption on $r_{1,2}$. □

Claim 8. $r_{1,2} + 3r \geqslant c_2 \geqslant n + 3a + 3$.

P r o o f od Claim 8. By Claim 7 we have $r_3 + r_4 + r_5 \leqslant 2r$, hence it follows from Claim 2 that $n + 3a + 3 \leqslant c_2 = r_{1,2} + r + r_3 + r_4 + r_5 \leqslant r_{1,2} + 3r$. □

Claim 9. $w \geqslant 7$.

P r o o f of Claim 9. If $n \neq 9$, it suffices to apply Claim 6. For $n = 9$ the same claim yields $r_{1,2} + r \geqslant 6$. So, suppose that $r_{1,2} + r = 6$. Returning to the proofs of Claims 6, 7 and 8 we see that then $c_2 = 15$, all ten subgraphs $K_2 \cup K_3$ of $K_5(A)$ are of weight 6, and $r_{1,2} + 3r = 15$. This, however, leads to $2r = 9$, a contradiction. □

Claim 10. $r_{1,2} \leqslant 2$.

P r o o f of Claim 10. By Claims 4 and 9 we know that $r_{1,2} \leqslant 5$ and $r_{1,2} + r \geqslant 7$. However, $r_{1,2} = 5$ is impossible: in such a case any 2-colour missing in both $[1,2]$-rows (and there are at least $7 - 5 = 2$ such colours in R) has at most $2 \cdot 2 = 4$ connections with (colours of) $R_{1,2}$, a contradiction.

So, suppose that $r_{1,2} \in [3,4]$. Since any exemplar of a colour $\alpha \in R$ realizes in its column at most two connections with $R_{1,2}$, we have $S_\alpha \subseteq S_{1,2}$, $S_\alpha \cap S_{1,2}^{(2)} \neq \emptyset$ and, if $r_{1,2} = 4$, even $S_\alpha \subseteq S_{1,2}^{(2)}$.

Assume first that $r_{4,5} > 0$. Any colour of R_i, $i \in [3,5]$, must have at least one of its exemplars in an $S_{1,2}$-column, otherwise its connections with $R_{j,k}$, where $\{j,k\} = [3,5] - \{i\}$, would be missing. Thus, for the number p of places in the $S_{1,2}$-columns filled in with 2-colours, we obtain $2(r_{1,2}+r) + (c_2 - (r_{1,2}+r)) \leqslant p \leqslant 5s_{1,2}$, hence, by Claims 3 and 9, $7 + 15 \leqslant (r_{1,2}+r) + c_2 \leqslant 5s_{1,2}$ and $s_{1,2} \geqslant 5$. Similarly, for $r_{1,2} = 4$, we obtain $22 \leqslant 5s_{1,2}^{(2)}$ and $s_{1,2}^{(2)} \geqslant 5$ in contradiction with the immediate bound $s_{1,2}^{(2)} \leqslant 4$. Clearly, we have $s_{1,2}^{(1)} + s_{1,2}^{(2)} = s_{1,2}$, $s_{1,2}^{(1)} + 2s_{1,2}^{(2)} = 2r_{1,2}$ and, consequently, $s_{1,2} + s_{1,2}^{(2)} = 2r_{1,2}$. Thus, $r_{1,2} = 3$ yields $s_{1,2}^{(2)} = 6 - s_{1,2} \leqslant 6 - 5 = 1$, and then $r \leqslant 3$ in contradiction with Claim 9.

From now on we suppose that $r_{4,5} = 0$. We cannot have $s_{1,2} = s_{1,2}^{(2)} = 3$, because in such a case $r_{1,2} = 3$, $r_{3,4} + r_{3,5} \leqslant 3$ (any colour of $R = R_{3,4} \cup R_{3,5}$ has its 3-row exemplar in $\{3\} \times S_{1,2}$) and $r_{1,2} + r \leqslant 3 + 3$. So, $s_{1,2} \geqslant 4$ and it is easy to see that there are colours $\alpha, \beta \in R_{1,2}$ sharing no column. Then 3-row exemplars of colours of R must appear in $\{3\} \times (S_\alpha \cup S_\beta)$, $r = r_{3,4} + r_{3,5} \leqslant 4$, $r_{1,2} + 3r \leqslant 16$, and Claim 8 yields $n \in \{7,9\}$. Since $r_{3,5} \leqslant 2$, it follows from Claim 7 that $w(K(5)) = r_5 + r_{3,5} + r_{4,5} \leqslant 2 + 2 + 0 = 4$.

Hence, by Claim 5, the only remaining possibility is $n = 9$. If $r_{3,5} \leqslant 1$, Claim 7 yields $w(K(5)) \leqslant 2(1+0)$ in contradiction with Claim 5. Thus, we must have $r_{3,4} = r_{3,5} = 2$. Claims 5 and 7 imply $r_4 = r_5 = 2$.

If $i \in [4,5]$, then each colour of R_i must have an exemplar in one of the $S_{1,2}$-columns: it needs connections with $R_{j,k}$, where $\{j,k\} = [3,5] - \{i\}$. Since $r_4 + r_5 = 4$, we cannot have $s_{1,2} = 3$ (at least fourteen places in the $S_{1,2}$-columns are occupied by colours of $R_{1,2} \cup R$). From $s_{1,2} \geqslant 4$ we obtain, as above, that there are two colours $\alpha, \beta \in R_{1,2}$ with $S_\alpha \cap S_\beta = \emptyset$. We may suppose without loss of generality that $S_\alpha = [1,2]$ and $S_\beta = [3,4]$. Every colour of R has both its exemplars in the $[1,4]$-columns and, as $r > 3$, any colour of $R_{1,2}$ must also have both its exemplars in the $[1,4]$-columns. Thus, in the rectangle $[1,2] \times [1,4]$ (in the intersection of the set of the $[1,2]$-rows and the set of the $[1,4]$-columns) of the matrix A there are at most two positions for colours of the set $R_4 \cup R_5$ and at least two positions for colours of $R_4 \cup R_5$ must be in the rectangle $[4,5] \times [1,4]$ (note that in $\{3\} \times [1,4]$ there are all four colours of R).

A colour missing in both $[1,2]$-rows has at least two its exemplars in $[3,5] \times [1,4]$ (connections with $R_{1,2}$); the number of such colours is therefore at most $\lfloor \frac{1}{2}(12-2) \rfloor = 5$. As the $[1,2]$-rows contain at most $18 - r_{1,2}$ colours, the total number of colours in A is $20 \leqslant 23 - r_{1,2}$, so that $r_{1,2} = 3$, there are five colours missing in both $[1,2]$-rows (four of R and the fifth of $R_{3,4,5}$), any colour of $R_4 \cup R_5$ has exactly one exemplar in $[1,5] \times [1,4]$ and the distribution of $R_4 \cup R_5$ in the rectangles $[1,2] \times [1,4]$ and $[3,5] \times [1,4]$ is $2+2$. Let γ, δ be colours of $R_4 \cup R_5$ occurring in $[1,2] \times [1,4]$. Because of the distribution of $R_{1,2}$ in $[1,2] \times [1,4]$, it is clear that a connection γ/δ can only be provided by γ_2 and δ_2. (For a 2-colour μ we denote its two exemplars by μ_1 and μ_2, and we assume that μ_1 is the exemplar entering into our considerations as the first.)

The mentioned colour of $R_{3,4,5}$ occupies two positions in $[4,5] \times [1,4]$, hence one position in that rectangle is occupied by a colour of R_4 and one by a colour of R_5. That is why, if $\gamma \in R_{l,i}$, $l \in [1,2]$, $i \in [4,5]$, then (because of $r_4 = r_5 = 2$) $\delta \in R_{3-l,9-i}$. Thus, a connection γ/δ is realized in a column. However, that column must contain also all colours of R_3, because the colour $\gamma \in R_{l,i}$ needs connections with $R_{3,9-i}$ (its second exemplar cannot help, as all exemplars of R_3 are in $[1,5] \times [5,9]$) and, analogously, the colour $\delta \in R_{3-l,9-i}$ needs connections with $R_{3,i}$. This leads to a contradiction since $r_3 = c_2 - w - (r_4 + r_5) \geqslant 15 - 7 - 4 = 4$. □

Claim 11. *If* $\{i,j,k,l,m\} = [1,5]$, $r_{i,j} = 5$, *then* $r_{k,l} = r_{k,m} = r_{l,m} = 0$, $s_{i,j} = r_{k,l,m} = 6$ *and all positions in* $\{k,l,m\} \times S_{i,j}$ *are filled in with colours of* $R_{k,l,m}$.

P r o o f of Claim 11. From Claim 4 we obtain $a = 0$. The number of colours missing in both $\{i,j\}$-rows is then $(2n+1) - (2n-5) = 6$, and each exemplar of such a colour provides at most two connections with $R_{i,j}$. Hence, $r_{k,l} = r_{k,m} = r_{l,m} = 0$ and $r_{k,l,m} = 6$.

Any colour of $R_{k,l,m}$ occupies three positions in $\{k,l,m\} \times S_{i,j}$ and at least two positions in $\{k,l,m\} \times S^{(2)}_{i,j}$, that is why $18 = 3r_{k,l,m} \leqslant 3s_{i,j}$ and $12 = 2r_{k,l,m} \leqslant 3s^{(2)}_{i,j}$. Moreover, $s^{(1)}_{i,j} + s^{(2)}_{i,j} = s_{i,j}$, $s^{(1)}_{i,j} + 2s^{(2)}_{i,j} = 2r_{i,j} = 10$, consequently $s_{i,j} = 10 - s^{(2)}_{i,j}$, $6 \leqslant 10 - s^{(2)}_{i,j} \leqslant 10 - 4 = 6$, $s^{(2)}_{i,j} = 4$, $s_{i,j} = 6$, and the proof follows. □

Claim 12. *If* $\{i,j,k,l,m\} = [1,5]$ *and* $r_{i,j} \in [3,4]$, *then* $r_{k,l} + r_{k,m} \leqslant 4$.

P r o o f of Claim 12. Suppose first that there are colours $\alpha, \beta \in R_{i,j}$ with $S_\alpha \cap S_\beta = \emptyset$. Evidently, any colour of $R_{k,l} \cup R_{k,m}$ must have its k-row exemplar in an $(S_\alpha \cup S_\beta)$-column, and so $r_{k,l} + r_{k,m} = |R_{k,l} \cup R_{k,m}| \leqslant |\{k\} \times (S_\alpha \cup S_\beta)| = 4$.

If the above assumption is not fulfilled, then $s_{i,j} = 3$ and any colour of $R_{k,l} \cup R_{k,m}$ must have its k-row exemplar in an $S_{i,j}$-column, hence $r_{k,l} + r_{k,m} \leqslant |\{k\} \times S_{i,j}| = 3$. □

Claim 13. *If* $\{i,j,k,l,m\} = [1,5]$ *and* $r_{i,j} \geqslant 1$, *then* $r_{k,l} + r_{k,m} + r_{l,m} + r_{k,l,m} \leqslant 6$.

P r o o f of Claim 13. If $\alpha \in R_{i,j}$, then any colour of $R_{k,l} \cup R_{k,m} \cup R_{l,m} \cup R_{k,l,m}$ must be present in $\{k,l,m\} \times S_\alpha$. □

Claim 14. *If* $\{i,j,k,l,m\} = [1,5]$ *and* $r_{i,j} \geqslant 1$, *then* $r_{i,j} + r_{k,l} + r_{k,m} \leqslant 8$. *Moreover, the equality can apply only if* $r_{i,j} \in \{2,4\}$.

Proof of Claim 14. The claim is a direct consequence of Claims 11, 12 and 13. □

Claim 15. *If* $r_{1,2} \in [1,2]$, *then* $(r_{3,4}, r_{3,5}, r_{4,5}) \in \{(2,2,1),(2,2,2)\}$.

Proof of Claim 15. By Claim 13, we have $r \in [5,6]$ and so $w \in [7,8]$. If $r = 5$ (and $r_{1,2} = 2$), then, by Claims 6 and 5, $n \leqslant 11$ and $w(K(5)) \geqslant 4$. The assumption $r_{3,4} = 2$ leads to $r_{3,5} = 2$ and $r_{4,5} = 1$. On the other hand, if $r_{3,4} \geqslant 3$, using Claim 7 we obtain $4 \leqslant w(K(5)) < 2(r_{3,5} + r_{4,5}) = 2(5 - r_{3,4})$ and $r_{3,4} < 3$, a contradiction.

So, suppose that $r = 6$. If $r_{3,4} \geqslant 4$, Claim 7 implies $w(K(5)) < 2(6 - r_{3,4}) \leqslant 4$, hence, by Claim 5, $n \geqslant 15$. By Claim 2, we have $c_2 \geqslant 18$, $\tilde{r} = \sum\limits_{l=1}^{2}(r_{l,3} + r_{l,4} + r_{l,5}) \geqslant 18 - w$ and, as $w(K(1,5)) + w(K(2,5)) = \tilde{r} + 2r_{3,4}$, there exists $l \in [1,2]$ with $w(K(l,5)) \geqslant r_{3,4} + \lceil \frac{1}{2}(18 - w) \rceil \geqslant \frac{1}{2}(26 - w) > w$, a contradiction.

Henceforth we assume that $r_{3,4} = 3$ (otherwise we are done). If $n \geqslant 15$, then, by Claim 2, $c_2 \geqslant n + 3 \geqslant 18$ and $\tilde{r} = c_2 - w \geqslant 18 - 8 = 10$. Moreover, $16 \geqslant w(K(1,5)) + w(K(2,5)) = 2r_{3,4} + \tilde{r} \geqslant 16$, so that $w(K(1,5)) = w(K(2,5)) = 8$, $\tilde{r} = 10$, $c_2 = 18$, $n = 15$, $w = 8$, $r_{1,2} = 2$, $c_3 = c_{3+} = 13$. Claim 7 yields $r_3 + r_4 \leqslant r_{3,4} + r = 9$ and $r_5 \leqslant 2$, hence $r_5 = \tilde{r} - (r_3 + r_4) \geqslant 10 - 9 = 1$. If $l \in [1,2]$, then $w(K(l,5)) = 8$ by virtue of Claim 13 implies $r_{l,5} \neq 1$, therefore there is $l \in [1,2]$ with $r_{l,5} = 2$, $r_{3-l,3} + r_{3-l,4} = 3$, $r_{3-l,5} = 0$ and $r_{l,3} + r_{l,4} = 5$. Since $r_{3,5} \geqslant 2$, from Claim 11 we know that $r_{l,4} \leqslant 4$ and $r_{l,3} \geqslant 1$. If $r_{l,3} = 5$ and $r_{l,4} = 0$, then $w(K(3-l,4)) \geqslant r_{l,3} + r_{l,5} + r_{3,5} \geqslant 5 + 2 + 2 = 9$, a contradiction.

Thus, $r_{l,3}r_{l,4} > 0$ and, by Claim 13, $(r_{3-l,4} + r_{3-l,5} + r_{4,5} + r_{3-l,4,5}) + (r_{3-l,3} + r_{3-l,5} + r_{3,5} + r_{3-l,3,5}) = 6 + r_{3-l,3,5} + r_{3-l,4,5} \leqslant 12$ and $r_{3-l,3,5} + r_{3-l,4,5} \leqslant 6$. Consider a colour $\alpha \in R_{1,2}$. Clearly, all positions in $[3,5] \times S_\alpha$ are occupied by six distinct colours of R. At least one colour of $R_{l,5}$, say β, is out of S_α, therefore $s_{3,4}^{(2)} = 2$, $s_{3,4} = 4$ and $S_{3,4} = S_\alpha \cup S_\beta$. Because of connections $R_{l,5}/(R_{3-l,3} \cup R_{3-l,4})$, in $\{3-l,3,4\} \times S_\beta$ there are all three colours of $R_{3-l,3} \cup R_{3-l,4}$ (together with all three colours of $R_{3,4}$). We have $S_{l,5} \subseteq S_{3,4}$, and so connections $R_{l,5}/(R_{3-l,3} \cup R_{3-l,4})$ imply $S_{l,5} = S_\beta$. Consequently, $S_{1,2} = S_\alpha$ and $r_{1,2,5} (= r_{3-l,l,5}) = 0$, since all places in $\{1,2,5\} \times S_{3,4}$ are filled in exclusively with colours of $R_{1,2} \cup R_{l,5} \cup R_{3,5} \cup R_{4,5} \cup R_{3-l,3} \cup R_{3-l,4}$. From $r_{3-l,l} + (r_{3-l,3} + r_{3-l,4}) + r_{3-l,5} = 2 + 3 + 0 = 5$ and $r_{l,5} + r_{3-l,5} + (r_{3,5} + r_{4,5}) = 2 + 0 + 3 = 5$ we see that in both $\{3-l,5\}$-rows there are ten 3-colours. Since $c_3 = 13$, at least seven 3-colours are in both $\{3-l,5\}$-rows, i.e. $r_{3-l,l,5} + r_{3-l,3,5} + r_{3-l,4,5} = 0 + r_{3-l,3,5} + r_{3-l,4,5} \geqslant 7$ in contradiction with $r_{3-l,3,5} + r_{3-l,4,5} \leqslant 6$.

If $n \leqslant 14$, then, by Claims 5 and 7, $1 \leqslant r_5 \leqslant 2$. Let us find a lower bound for the number $\hat{c}$ of colours of $R_3 \cup R_4$ needing a column connection with (at least one of) colours of R_5: If $r_{m,5} = 0$ for some $m \in [1,2]$, then $r_{3-m,5} \in [1,2]$ and, by Claim 5, $\hat{c} = r_{m,3} + r_{m,4} \geqslant 2$; on the other hand, if $r_{1,5} = r_{2,5} = 1$, then $\hat{c} = r_3 + r_4 = c_2 - w - r_5 \geqslant 15 - 8 - 1 - 1 = 5$. The number of colours missing in both $[3,4]$-rows is $r_{1,2} + r_{1,5} + r_{2,5} + r_{1,2,5} = 2n + a + 1 - (2n - t_{3,4}) \geqslant r_{3,4} + a + 1 = a + 4 \geqslant 5$. Since $r_{3,4} = 3$, all colours of $\dot{R} := R_{1,2} \cup R_{1,5} \cup R_{2,5} \cup R_{1,2,5}$ must have at least two exemplars in $\{1,2,5\} \times S_{3,4}$. Consider a colour $\alpha \in R_{1,2}$; clearly, all positions in $[3,5] \times S_\alpha$ are filled in with colours of R, and so $s_{3,4} \in [4,5]$ (three positions outside of $[3,5] \times S_\alpha$ are occupied by colours of $R_{3,4}$).

If $s_{3,4} = 4$, then in $[1,5] \times S_{3,4}$ there are at least $2|\dot{R}| \geqslant 10$ places occupied by colours of $\dot{R}$ and at least $r + r_{3,4} = 9$ places occupied by colours of R, hence at most one position can be occupied there by a colour of $R_3 \cup R_4$ in contradiction with $\hat{c} \geqslant 2$ (note that any colour of R_5 has both its exemplars in $\{1,2,5\} \times S_{3,4}$).

If $s_{3,4} = 5$, then $s^{(2)}_{3,4} = 1$, $S^{(2)}_{3,4} \subseteq S_\alpha$ and $r_{1,2} + r_{1,5} + r_{2,5} \leqslant 2$, because any colour of $R_{1,2} \cup R_{1,5} \cup R_{2,5}$ must be present in $[1,2] \times S^{(2)}_{3,4}$; thus we have $r_{1,2} = r_{3-m,5} = 1$, $r_{m,5} = 0$ and $r_{1,2,5} \geqslant 3$. Consequently, $14 \geqslant w(K(1,5)) + w(K(2,5)) = 2r_{3,4} + \tilde{r} = 6 + (c_2 - w) \geqslant 6 + 15 - 7 = 14$ and $w(K(3-m,5)) = 7$, $\hat{c} = r_{m,3} + r_{m,4} = 3$. Evidently, an exemplar of a colour of $R_{3-m,5}$ in an $S^{(2)}_{3,4}$-column does not provide connections with $R_{m,3} \cup R_{m,4}$ (in that column there are only colours of $R_{1,2} \cup R_{3-m,5} \cup R$) and all three connections are realized in the unique remaining $S_{3-m,5}$-column (that is not an S_α-column); however, this is impossible, as colours of $R_{1,2} \cup R_{3-m,5} \cup R_{1,2,5}$ occupy in $\{1,2,5\} \times S_{3,4} - (\{5\} \times S_\alpha)$ at least $2 \cdot 2 + 3 \cdot 3$ (and so all) positions. □

Claim 16. *If $r_{1,2} \in [1,2]$, $\alpha \in R_{1,2}$, $i \in [3,5]$, $\beta, \gamma \in R_i$ and $S_\alpha \cap (S_\beta \cup S_\gamma) = \emptyset$, then $S_\beta \cap S_\gamma \neq \emptyset$.*

Proof of Claim 16. Let $\{j,k\} = [3,5] - \{i\}$ and consider a colour $\delta \in R_{j,k} \neq \emptyset$ (Claim 15). Because of connections with β and γ, we have $S_\delta \neq S_\alpha$ and an $(S_\delta - S_\alpha)$-column contains both β and γ. □

Claim 17. *If $r_{1,2} = 2$, then $s_{1,2} = 2$.*

Proof of Claim 17. If $R_{1,2} = \{\alpha, \beta\}$, we may suppose without loss of generality that α is in (1,1) and (2,2). Put $S := S_{3,4} \cup S_{3,5} \cup S_{4,5}$.

If $S_\alpha \cap S_\beta = \emptyset$ (or, equivalently, $s_{1,2} = 4$), it follows from $r \geqslant 5$ that all colours of R must have one exemplar in an S_α-column and the other in an S_β-column and, consequently, $S \subseteq S_\alpha \cup S_\beta$. Any colour of $C_2 - R_{1,2} - R$ has one exemplar in one of the $[1,2]$-rows and another one in an i-row, $i \in [3,5]$; if $\{i,j,k\} = [3,5]$, this colour needs connections with the set $R_{j,k} \neq \emptyset$ (Claim 15), and therefore must have at least one exemplar in an $S_{j,k}$-column, and hence in an S-column. Colours of $R_{1,2} \cup R$ have both their exemplars in the S-columns, and so, with help of Claims 3 and 9, $15 + 7 \leqslant c_2 + w = 2(r_{1,2} + r) + (c_2 - r_{1,2} - r) \leqslant 5|S| = 20$, a contradiction.

If $s_{1,2} = 3$, we may assume without loss of generality that β occupies the positions (1,3) and (2,1). Clearly, all colours of R that are not in the 1-column must share both $[2,3]$-columns.

If three colours of R share the $[2,3]$-columns, it is easily seen that, for any $i \in [3,5]$ and $j \in [3,5] - \{i\}$, there is a colour $\mu \in R_{i,j}$ with $S_\mu = [2,3]$; if $\{i,j,k\} = [3,5]$, then, because of a connection with μ, any colour of R_k must have an exemplar in $\{(1,2),(2,3)\}$. Therefore, $\tilde{r} = r_3 + r_4 + r_5 \leqslant 2$ and $c_2 = r_{1,2} + r + \tilde{r} \leqslant 2 + 6 + 2$ in contradiction with Claim 3.

Thus, we see that exactly two colours of R share the $[2,3]$-columns, $r = 5$ and $r_{4,5} = 1$. If the colours in the $[2,3]$-columns are not both from $R_{3,4}$ or $R_{3,5}$, then there are $i,j,k \in [3,5]$ such that $\{i,j,k\} = [3,5]$ and the $[2,3]$-columns share exactly one colour of $R_{i,j}$ and exactly one colour of $R_{i,k}$. Because of connections with $R_{i,j}$ (with $R_{i,k}$), any colour of R_k (of R_j) must occur in the $[2,3]$-columns, and so $r_j + r_k \leqslant 4$. For a colour $\gamma \in R_{j,k}$ (by Claim 15, $r_{j,k} \geqslant 1$) we have $S_\gamma = \{1,l\}$, $l \in [2,n]$. Any colour of R_i must be in $\{1,2,i\} \times \{l\}$ (it needs a connection with γ), and so $r_i \leqslant 3$. As a consequence, $c_2 = r_{1,2} + r + \tilde{r} \leqslant 2 + 5 + (4+3) = 14$ in contradiction with Claim 3.

What remains is the following possibility: the $[2,3]$-columns share both colours of $R_{3,i}$ with $i \in [4,5]$ and the 1-column is filled in with colours of $R_{1,2} \cup R_{3,9-i} \cup R_{4,5}$. By Claim 7, $\max\{r_4, r_5\} \leqslant 3$. Moreover, because of a connection with the unique colour of $R_{4,5}$, all colours of R_3 must appear in a unique $(S_{4,5} - \{1\})$-column so that $r_3 \leqslant 3$, too. Claim 3 yields $\tilde{r} = r_3 + r_4 + r_5 = c_2 - w \geqslant 15 - 7 = 8$, hence $\min\{r_j :\ j = 3,4,5\} \geqslant 2$ and at most one of the numbers r_3, r_4 ,r_5 is 2. Furthermore, $c_2 = w + r_3 + r_4 + r_5 \leqslant 7 + 3 + 3 + 3 = 16$, and so $n \in \{7,9\}$ (Claim 2) and $a \geqslant 1$.

We have $S_{3,9-i} \cap S_{4,5} = \{1\}$: if an l-column, $l \in [2,n]$, contains a colour of $R_{3,9-i}$ and a colour of $R_{4,5}$, it contains all colours of R_3, R_i and $R_{4,5}$, altogether at least $(r_3 + r_i) + r_{4,5} + 1 \geqslant 5 + 1 + 1 = 7$ colours, a contradiction. Thus, we may suppose without loss of generality that $S_{3,9-i} = \{1\} \cup [4, s_{3,9-i} + 2]$ and $S_{4,5} = \{1, s_{3,9-i} + 3\}$ (note that the "rectangle" $\{9-i\} \times [2,3]$ is free of colours of $R_{3,9-i} \cup R_{4,5}$, since $\min\{r_3, r_i\} \geqslant 2$).

If $s_{3,9-i} = 3$, then, since all connections of a colour $\gamma \in R_i$ with $R_{3,9-i}$ are realized out of the 1-column, we have $S_{1,i} \cup S_{2,i} = [4,5]$, and so $r_i = 2$, $r_3 = r_{9-i} = 3$, $c_2 = 15$ and $n = 9$. Because of connections with $R_{4,5}$, all three colours of R_3 are in $[1,3] \times \{6\}$. At least one of colours of R_3 in $[1,2] \times \{6\}$, say δ in $(l,6)$, $l \in [1,2]$, is out of $\{3\} \times [4,5]$ (one position in $\{3\} \times [4,5]$ is occupied by a colour of $R_{3,9-i}$). Because of connections δ / R_i we have $R_i = R_{l,i}$. Clearly, $S_\delta \subseteq [6,9]$ and $S_\delta \cap S_{l,i} = \emptyset$. As $r_{9-i} = 3$, we have

$r_{3-l,9-i} \geqslant 1$. For a colour $\varepsilon \in R_{3-l,9-i}$, ε_1 situated in $\{3-l, 9-i\} \times [2,3]$ provides no connections with $\{\delta\} \cup R_{l,i}$; however, $S_\delta \cap S_{l,i} = \emptyset$ means that ε_2 cannot provide all connections with $\{\delta\} \cup R_{l,i}$.

If $s_{3,9-i} = 2$, then $S_{3,9-i} = \{1,4\}$ and $S_{4,5} = \{1,5\}$. If a colour $\mu \in \tilde{R}$ appears in $[1,2] \times [6,n]$, all its connections with R are realized by μ_2. Therefore, μ_2 must occupy one of the positions in the set $\tilde{S} := \{(9-i,2), (9-i,3), (i,4), (3,5)\}$. Let $\tilde{C}$ be the set of colours of $\tilde{R}$ appearing in $[1,2] \times [6,n]$. Since $\tilde{r} \geqslant 8$, we have $|\tilde{C}| \geqslant 2$.

Suppose first that there is a 3-element set $\tilde{C}' \subseteq \tilde{C}$ such that its colours occupy three positions in $\tilde{S}$ forming an independent set of vertices in the graph $K_5 \times K_n$ corresponding to A. Then, clearly, all connections between the colours of $\tilde{C}'$ are provided by exemplars of $\tilde{C}'$ in $[1,2] \times [6,n]$, and this is possible only if those exemplars share an m-row, $m \in [1,2]$. By Claim 5, $w(K(3-m)) \geqslant 4$ and, since in $\{3-m\} \times [6,n]$ there are no 2-colours (such a 2-colour would miss at least one connection with $\tilde{C}'$), in $\{3-m\} \times [2,5]$ there are at least two colours of $\tilde{R}$; hence some of them, say γ, is such that γ_2 does not occupy a position in $\tilde{S}$. Then γ_2 does not provide all connections γ/R so that, if $\gamma \in R_j$, $j \in [3,5]$ and $\{k,l\} = [3,5] - \{j\}$, γ_1 must be in a column containing (all) colours of $R_{k,l}$. There are altogether at most three connections $\gamma/\tilde{C}'$ (one row connection and at most two column connections); however, two of them are connections with the unique colour of $\tilde{C}' \cap R_j$, and so at least one connection $\gamma/\tilde{C}'$ is missing.

So we see that $|\tilde{C}| \leqslant 3$ and, if $|\tilde{C}| = 3$, then two colours of $\tilde{C}$, say γ and δ, occupy positions $(9-i,2)$ and $(9-i,3)$, respectively; a third colour $\varepsilon \in \tilde{C}$ occupies a position of $\tilde{S}$ in one of the $[4,5]$-columns. First, let $|\tilde{C}| = 3$. If γ_2, δ_2 and ε_2 share an m-row, $m \in [1,2]$, consider two colours $\zeta, \eta \in \tilde{R}$ occurring in $\{3-m\} \times [1,5]$ (they do exist by Claim 5, since $a \geqslant 1$ and in $\{3-m\} \times [6,n]$ there is no colour of $\tilde{R}$). Because of connections $\{\zeta,\eta\}/(\{\gamma,\delta\} \cup R)$, ζ_2 and η_2 appear in $\{9-i\} \times [6,n]$. This, however, is in contradiction with Claim 16 (possibly, if $m = 2$, with β in the role of a colour of $R_{1,2}$).

Now, suppose that δ_2 and ε_2 share an m-row, $m \in [1,2]$, and γ_2 in the $(3-m)$-row shares a column with ε_2. Since $\tilde{r} \geqslant 8$, at least three colours of $\tilde{R}$ are present in the square $[1,2] \times [4,5]$. Consider colours $\zeta, \eta \in \tilde{R}$, occupying diagonal positions in $[1,2] \times [4,5]$. Evidently, because of connections $\{\gamma,\delta\}/\{\zeta,\eta\}$, ζ_2 and η_2 must appear in the columns of γ_2 and δ_2 (in an appropriate way), and we have again obtained a contradiction with Claim 16.

The only remaining possibility (with respect to connections γ/ε and δ/ε) is that γ_2 and ε_2 share an m-row, $m \in [1,2]$, and δ_2 in the $(3-m)$-row shares a column with ε_2; this is solved analogously as the preceding case.

Assume, finally, that $|\tilde{C}| = 2$. Then in $[1,2] \times [2,5]$ there are six colours of $\tilde{R}$, $\tilde{r} = 8$, $c_2 = 15$, $n = 9$ and $c_3 = c_{3+} = 5$. As five colours of C_3 occupy $8-2 = 6$

positions in $[1,2]\times[6,9]$, at least one of them, say γ, appears twice in that rectangle. Because of connections γ/R, γ_3 (the third exemplar of γ) must be in $\tilde{S}$.

Let $\tilde{F}$ be the set of six colours of $\tilde{R}$ appearing in $[1,2]\times[2,5]$ and let an $\tilde{F}$-*pair* be a pair of colours $\{\mu,\nu\}\subseteq\tilde{F}$ such that the positions of μ_1 and ν_1 correspond to nonadjacent vertices of $K_5\times K_n$. The number of $\tilde{F}$-pairs is $3\cdot 3-2=7$. Note that if $\{\mu,\nu\}$ is an $\tilde{F}$-pair, then, by Claim 16 (possibly with β in the role of α) there is a column connection μ/ν. Let $\tilde{F}_1$ be the set of those $\mu\in\tilde{F}$ that μ_2 is in $[3,5]\times[2,5]$; clearly, $|\tilde{F}_1|\leqslant 2$.

Consider an l-column, $l\in[2,5]$, containing p colours of $\tilde{F}_1$, $p\in[1,2]$. If $p=1$, the number of column connections corresponding to an $\tilde{F}$-pair that are realized in the considered column is at most 1. If $p=2$, that number is at most 3. On the other hand, if an m-column, $m\in[6,9]$, contains q colours of $\tilde{F}$, in that column at most $\binom{q}{2}$ column connections corresponding to an $\tilde{F}$-pair are realized.

Therefore, if $|\tilde{F}_1|=2$, the total number of column connections corresponding to an $\tilde{F}$-pair is at most $3+\binom{3}{2}+\binom{1}{2}=6$, which is insufficient, as seven such connections should be present. If $|\tilde{F}_1|=1$, that number is at most $1+\binom{3}{2}+\binom{2}{2}=5<7$. Finally, for $|\tilde{F}_1|=0$ we have an upper bound $2\cdot\binom{3}{2}=6<7$. □

Consider a colour $\alpha\in R_{1,2}$. A 3-element set $\{\beta,\gamma,\delta\}$ of colours of R_i, $i\in[3,5]$, is said to be an α-*appropriate triple*, if $S_\beta\cap S_\gamma\cap S_\delta\neq\emptyset$ (i.e., the colours β,γ,δ share a column) and $S_\alpha\cap(S_\beta\cup S_\gamma\cup S_\delta)=\emptyset$ (i.e., there are no column connections $\alpha/\{\beta,\gamma,\delta\}$).

Claim 18. *If $r_{1,2}\in[1,2]$ and $\alpha\in R_{1,2}$, then there is an α-appropriate triple.*

P r o o f of Claim 18. We may suppose without loss of generality that α is in $(1,1)$ and $(2,2)$. If $r_{1,2}=2$, then, by Claim 17, the square $[1,2]\times[1,2]$ is filled in with colours of $R_{1,2}$. Claim 3 yields $15\leqslant c_2=2+r+\tilde{r}$, hence $\tilde{r}=r_3+r_4+r_5\geqslant 13-r$. By Claims 9 and 13, we have $r\in[5,6]$.

If $r=6$, there is $i\in[3,5]$ with $r_i=3$. Let $\{j,k\}=[3,5]-\{i\}$; since the $[1,2]$-columns are filled in with colours of $R_{1,2}$ and R, all connections $R_i/R_{j,k}$ are realized in the $[3,n]$-columns. Therefore, an l-column, $l\in[3,n]$, containing a colour of the (non-empty) set $R_{j,k}$, contains also colours of R_i. Thus, R_i is an α-appropriate triple.

Now, suppose that $r=5$ (and $\tilde{r}\geqslant 8$). If there is $i\in[3,5]$ with $r_i\geqslant 4$, there is a 3-element subset of R_i representing an α-appropriate triple, since at most one colour of R_i is present in an S_α-column. On the other hand, if there are $i,j\in[3,5]$, $i\neq j$, with $r_i=r_j=3$, then at least one of the sets R_i and R_j is an α-appropriate triple.

If $r_{1,2}=1$ (and $r=6$), we have $\tilde{r}\geqslant 15-1-6=8$. By Claim 15, $r_{3,4}=r_{3,5}=r_{4,5}=2$, hence Claim 7 yields $r_i\leqslant(2+2)-1=3$, $i=3,4,5$. Thus, there are

$i, j, k \in [3,5]$ such that $\{i,j,k\} = [3,5]$, $r_i = r_j = 3$ and $r_k \in [2,3]$. There are only two positions that can prevent a 3-element set R_l, $l \in [3,5]$, from being an α-appropriate triple (by carrying a colour of R_l), namely $(1,2)$ and $(2,1)$ (because of connections R_l/R).

Therefore, it is sufficient to deal with the case when $r_k = 2$ (implying $c_2 = 15$, $n = 9$ and $c_3 = c_{3+} = 5$), the position $(1,2)$ is occupied by a colour $\beta \in R_i$ and the position $(2,1)$ by a colour $\gamma \in R_j$. Clearly, β_2 and γ_2 must share a column (a connection β/γ), without loss of generality the 3-column. Because of connections with β and γ, both colours $\delta, \varepsilon \in R_k$ are in $\{1,2,k\} \times \{3\}$. In the 3-column there are no colours of $R_{i,j}$, and so connections $\{\delta,\varepsilon\}/R_{i,j}$ are realized by δ_2 and ε_2 in a column, without loss of generality in the 4-column. If $R_i = \{\beta,\zeta,\eta\}$ and $R_j = \{\gamma,\vartheta,\iota\}$, then, because of connections $\{\delta,\varepsilon\}/\{\zeta,\eta,\vartheta,\iota\}$ (that can be realized only by exemplars of ζ, η, ϑ, ι in the $[1,2]$-rows), it is clear that δ and ε must share an l-row, $l \in [1,2]$ (otherwise, if δ and ε occupy diagonal positions in $[1,2]\times[3,4]$, only the remaining two positions in that square provide both connections with δ and ε). We may assume without loss of generality that that δ_1 is in $(l,3)$ and ε_2 in $(l,4)$. By Claim 5, $w(K(3-l)) \geqslant 4$ and so at least two of the colours ζ, η, ϑ, ι must be present in the $(3-l)$-row. Therefore, using Claim 16, we see that the "rectangle" $\{3-l\} \times [3,4]$ is filled in with one colour of $\{\zeta,\eta\}$, say ζ, and one colour of $\{\vartheta,\iota\}$, say ϑ. Then, evidently, all connections $\zeta/R_{j,k}$ are realized by ζ_2 (without loss of generality in $(i,5)$), and all connections $\vartheta/R_{i,k}$ by ϑ_2 (without loss of generality in $(j,6)$). So, with an additional use of Claim 16, the 5-column contains all four colours of $\{\zeta,\eta\} \cup R_{j,k}$, and the 6-column all four colours of $\{\vartheta,\iota\} \cup R_{i,k}$. Thus, all six positions in $[1,2]\times[7,9]$ are occupied by 3-colours, and at least one of them, say κ, has two its exemplars in that rectangle. Since κ_3 is in $[3,5] \times [7,9]$, two of connections κ/R are missing. $\square$

Claim 19. $r_{1,2} = 0$ *and, consequently,* $r_{3,4} \geqslant 3$.

Proof of Claim 19. If $r_{1,2} \in [1,2]$ and $\alpha \in R_{1,2}$, by Claim 18 there is $i \in [3,5]$ and an α-appropriate triple $\{\beta,\gamma,\delta\} \subseteq R_i$. We may suppose without loss of generality that α is in $(1,1)$, $(2,2)$, β in $(1,3)$, $(i,4)$, γ in $(2,3)$, $(i,5)$ and δ in $(i,3)$ (δ_2 is unimportant for the moment). We suppose also that $\{\beta,\gamma,\delta\}$ maximizes the number of colours of R in the unique common column of its colours among all possible α-appropriate triples.

Consider the set $B := \{j,k\} \times [6,n]$, where $\{j,k\} = [3,5] - \{i\}$. Let b_R be the number of colours of R in B and, for $l \in [1,2]$ and $m \in [2,5]$, let $b_m^{(l)}$ be the number of colours in $C_m - R_{1,2} - R$ that appear l times in B. We have $b_2^{(1)} + b_3^{(2)} \leqslant 2$: to have all connections with $R_{1,2} \cup \{\beta,\gamma\}$, all colours contributing to $b_2^{(1)} + b_3^{(2)}$ must have an exemplar in $(1,5)$ or $(2,4)$. Further, $b_2^{(2)} = 0$ (a connection with α). As a consequence, the number of positions in B is $2(n-5) = b_R + \sum_{l=2}^{5} b_l^{(1)} + 2\sum_{l=3}^{5} b_l^{(2)} \leqslant b_R + (b_2^{(1)} + b_3^{(2)}) + c_3 + 2c_4 + 3c_5 \leqslant b_R + 2 + \sum_{l=2}^{5}(l-2)c_l = b_R + 2 + 5n - 2(2n + a + 1) = b_R + n - 2a$. Thus, we have $b_R \geqslant n + 2a - 10 \geqslant 1$.

For a set $Q \subseteq [3,5] \times [1,n]$, let $q(Q)$ be the number of positions in Q occupied by colours of $\tilde{R} = C_2 - R_{1,2} - R$. Let us show that $q(B) = b_2^{(1)} \leqslant 1$. Suppose that $b_2^{(1)} = 2$ and that colours $\varepsilon, \zeta \in \tilde{R}$ contribute to $b_2^{(1)}$. Then ε_2 and ζ_2 occupy the positions $(1,5)$, $(2,4)$ and ε_1, ζ_1 must be in a common line of A. By Claim 16, this line must be a column, without loss of generality the 6-column. Now, any colour of R realizes its connection with one of the colours β, ε, ζ in a column (those three colours cover all the $[3,5]$-rows), and so $(S_{3,4} \cup S_{3,5} \cup S_{4,5}) - [1,2] \subseteq S_\beta \cup S_\varepsilon \cup S_\zeta = [3,6]$. This inclusion, however, means that $b_R = 0$ (note that in $\{j,k\} \times \{6\} \subseteq B$ there are ε_1 and ζ_1), a contradiction.

Put $q_1 := q([3,5] \times [1,2])$, $q_2 := q(\{j,k\} \times \{3\})$ and $q_3 := q(\{i\} \times [6,n])$. We are going to prove that $q_1 + q_2 + q_3 + q(B) \leqslant 9 - r_{1,2} - r$. First, since all connections of the α-appropriate triple $\{\beta, \gamma, \delta\}$ with any colour of $R_{j,k}$ are realized in the 3-column, we have $q_2 \leqslant 2 - r_{j,k} = 2 + r_{i,j} + r_{i,k} - r \leqslant 2 + 2 + 2 - r = 6 - r$ (Claim 15).

Suppose that $r = 6$ and, consequently, $r_{3,4} = r_{3,5} = r_{4,5} = 2$. A colour contributing to q_3 needs connections with $R_{j,k}$, and they can be realized only in the $[1,2]$-columns (clearly, the 3-column is of no use). However, not more than one of the $[1,2]$-columns contains both colours of $R_{j,k}$, so that $q_3 \leqslant 2 - r_{1,2}$ (for $r_{1,2} = 2$ use Claim 17). Altogether, we obtain $q_1 + q_2 + q_3 + q(B) \leqslant 0 + 0 + (2 - r_{1,2}) + 1 = 9 - r_{1,2} - r$.

If $r = 5$, then $r_{1,2} = 2$ (Claim 9) and $q_3 = 0$ (as above). Since $q_1 + q_2 + q(B) \leqslant 1 + 1 + 1$, to prove our inequality it suffices to find a contradiction if $q_1 = q_2 = q(B) = 1$. So, suppose that q_1, q_2, $q(B)$ are all 1's, and that ε, ζ and η are colours of $\tilde{R}$ contributing to q_1, q_2 and $q(B)$, respectively; we may assume without loss of generality that η_1 is in $(j,6)$ (the only assumption imposed on j,k so far is $\{j,k\} = [3,5] - \{i\}$). Evidently, $q_2 = 1$ means that $r_{j,k} = 1$ and $r_{i,j} = r_{i,k} = 2$.

Suppose first that ε_1 is not in the i-row. Since ε and η need connections both with β and γ, ε_2 and η_2 must occupy positions $(l, 6-l)$ and $(3-l, 3+l)$, respectively, for some $l \in [1,2]$. Therefore, ε_1 and η_1 must share the j-row (a connection ε/η), and ε_1 is in (j,m) for some $m \in [1,2]$. Now, ζ_1 cannot be in $(k,3)$: in such a case ζ_2 is in $(l,6)$ (connections with ε and η), and ζ misses a connection with at least one colour of $R_{i,j}$ (in the 3-column there is no such colour and in $(j,6)$ there is η_1). Thus, ζ_1 is in $(j,3)$, and in $(k,3)$ there is a colour $\vartheta \in R_{j,k}$. So, ϑ_2 is in $(j, 3-m)$, and a colour ι in $(k, 3-m)$ belongs to $R_{i,k}$. Hence, ι_2 is in (i,p) with $p \in [6,n]$, and a connection ε/ι is missing.

Now, assume that ε_1 is in (i,l) for some $l \in [1,2]$. If ζ_1 is in $(j,3)$, then, by Claim 16, $S_\zeta \cap S_\eta \neq \emptyset$. Clearly, there is only one column shared by ζ and η, and that column must contain both colours of $R_{i,k}$; hence, it must be the 6-column. Because of connections $R_j/R_{i,k}$, we have $r_j \leqslant 3$. However, $r_j = 3$ is impossible: in such a case R_j would be an α-appropriate triple with $r_{i,k} = 2$ colours of R in a column shared by colours of R_j in contradiction with the fact that $\{\beta,\gamma,\delta\}$ has only $r_{j,k} = 1$ colour of R in "its" 3-column; so, $r_j \leqslant 2$. Further, $r_k \leqslant 2$, since k-row exemplars of R_k can only be in $\{k\} \times [4,5]$ (recall that $q_2 = 1$ is realized by ζ_1 and $q(B) = 1$ by η_1). Claim 7 yields $r_i \leqslant 4$ so that $r_i = 4$, $r_j = r_k = 2$, $c_2 = 15$ and, by Claim 2, $n = 9$, $c_{4+} = 0$ and $c_3 = c_{3+} = 5$. Moreover, in $(k,4)$ and $(k,5)$ there are colours of R_k, say ϑ and ι, respectively. Also, ζ_2 is in $(p,6)$ for some $p \in [1,2]$ (connections $\{\zeta,\eta\}/R_{i,k}$). Neither ϑ_2 nor ι_2 can be in $(3-p,6)$ (in the 6-column there is no colour of $R_{i,j}$ and, considering β in $(i,4)$ and γ in $(i,5)$, both ϑ_1 and ι_1 provide at most one connection with $R_{i,j}$). That is why, because of connections $\{\vartheta,\iota\}/\{\beta,\gamma,\zeta\}$, ϑ_2 must be in $(p,5)$ and ι_2 in $(p,4)$. Now, η_2 must be in $(3-p,3+p)$ (connections $\eta/\{\beta,\gamma\}$). Moreover, the "rectangle" $\{j\} \times [4,5]$ must be filled in with colours of $R_{i,j}$ (connections $\{\vartheta,\iota\}/R_{i,j}$), and in $\{j,k\} \times [7,9]$ there are only 3-colours. However, $c_3 = 5$, at least one 3-colour, say κ, has two exemplars in $\{j,k\} \times [7,9]$, and at least one of connections β/κ, γ/κ is missing: in $(p,6-p)$ there is either ϑ_2 or ι_2, and in $(3-p,3+p)$ there is η_2.

Finally, suppose that ζ_1 is in $(k,3)$. Then, because of a connection $\varepsilon/R_{j,k}$, in (k,l) there is the unique colour of $R_{j,k}$, hence in $\{i,k\} \times \{3-l\}$ there are both colours of $R_{i,k}$ and in $\{j\} \times [1,2]$ there are both colours of $R_{i,j}$. The remaining $R_{i,j}$-exemplars are in $\{i\} \times [6,n]$, and so there is $\mu \in R_{i,j}$ such that a connection ζ/μ is missing.

Using the just proved inequality $q_1 + q_2 + q_3 + q(B) \leqslant 9 - r_{1,2} - r$ we obtain $\tilde{r} = c_2 - r_{1,2} - r = q([3,5] \times [1,n]) = (q_1 + q_2 + q_3 + q(B)) + q(\{i\} \times [3,5]) + q(\{j,k\} \times [4,5]) \leqslant (9 - r_{1,2} - r) + 3 + q(\{j,k\} \times [4,5])$, hence $q(\{j,k\} \times [4,5]) \geqslant c_2 - 12 \geqslant 3$ (Claim 3). Thus, at most one position in $\{j,k\} \times [4,5]$ is not occupied by a colour of $\tilde{R}$. We may suppose without loss of generality that there is $l \in [4,5]$ such that in (j,l), (k,l) and $(j,9-l)$ there are colours of $\tilde{R}$, say ε, ζ and η, respectively. Since ζ needs connections with $R_{i,j}$, ζ_2 cannot be in the $(9-l)$-column (in $\{i,j\} \times [4,5]$ there are $\beta,\gamma,\varepsilon_1,\eta_1 \notin R_{i,j}$). Therefore, ζ_2 must be in the $(6-l)$-row (connections $\zeta/\{\beta,\gamma\}$); we may suppose without loss of generality that ζ_2 is in $(6-l,6)$. Clearly, η_2 is not in $[1,2] \times [7,n]$ (connections $\eta/\{\beta,\gamma,\zeta\}$). Thus, η_2 is either in the l-column or in the 6-column.

If η_2 is in the l-column, all colours of $R_{i,k}$ are in the $[4,5]$-columns; however, there is only one "free" place for them, namely $(k,9-l)$. Thus, $r_{i,k} = 1$, $r_{i,j} = r_{j,k} = 2$ (Claim 15), $\{j,k\} \times \{3\}$ is filled in with colours of $R_{j,k}$ (connections $\beta/R_{j,k}$), $\{i,j\} \times \{6\}$ is filled in with colours of $R_{i,j}$ (connections $\zeta/R_{i,j}$), $r_{1,2} = 2$ (Claim 9), and $q_3 = 0$ (as above). Since $8 = 15 - 2 - 5 \leqslant c_2 - r_{1,2} - r = \tilde{r} = q_1 + (q([3,5] \times [3,5]) + q_3) + q(B) \leqslant q_1 + (6+0) + q(B) \leqslant 1 + 6 + 1 = 8$, we have $q_1 = q(B) = 1$, $c_2 = 15$, $n = 9$ and $c_3 = c_{3+} = 5$. Let ϑ and ι be colours contributing to q_1 and $q(B)$, respectively. Now, $\iota \notin R_j$: the assumption $\iota \in R_j$ means that ι_1 is in $\{j\} \times [7,9]$, ι_2 is in $(l-3, 9-l)$ (connections $\iota/(\{\beta,\gamma\} \cup R_{i,k})$), and a connection ζ/ι is missing. So, ι_1 is in $(k, 6)$ (connections $\iota/R_{i,j}$). Then in $\{j,k\} \times [7,9]$ there are only 3-colours, and at least one of them, say κ, appears there twice. Consider the distribution of colours in $[3,5] \times [1,2]$. Colours of $R_{i,j}$ occupy in that rectangle one i-row position and one j-row position (they are both in the 6-column). Analogously, colours of $R_{j,k}$ occupy there one j-row position and one k-row position. Finally, the unique colour of $R_{i,k}$ in $[3,5] \times [1,2]$ is in $\{i\} \times [1,2]$ (it is also in $(k, 9-l)$). Thus, ϑ_1 is in the k-row. Now, for two positions $(1,5)$ and $(2,4)$, providing both connections with β and γ, there are three "candidates", namely ϑ_2, ι_2 and κ_3.

If η_2 is in the 6-column, the only available position for it is $(l-3, 6)$. By Claim 16, ε_2 is in the "rectangle" $[1,2] \times \{9-l\}$. Therefore, $r_{i,k} = 2$ is impossible: in such a case colours of $R_{i,k}$ would fill in the "rectangles" $\{k\} \times [5,6]$ (connections $\eta/R_{i,k}$) and $\{i\} \times [1,2]$, and at least one of connections $\varepsilon/R_{i,k}$ would be missing.

Thus, $r_{i,k} = 1$, $r_{i,j} = r_{j,k} = 2$ (Claim 15), $r_{1,2} = 2$ (Claim 9), the square $[1,2] \times [1,2]$ is filled in with colours of $R_{1,2}$ (Claim 17), the set $\{j,k\} \times \{3\}$ is filled in with colours of $R_{j,k}$ (connections $\beta/R_{j,k}$), and the set $\{i,j\} \times \{6\}$ is filled in with colours of $R_{i,j}$ (connections $\zeta/R_{i,j}$).

Clearly, in $\{i\} \times [7,n]$ there are no colours of R_i (connections $R_i/R_{j,k}$) and in $\{k\} \times [7,n]$ there are no colours of R_k (connections $R_k/R_{i,j}$). Further, if in $\{j\} \times [7,n]$ there is a colour of R_j, say ϑ, then ϑ_2 must be in $[1,2] \times \{9-l\}$ (Claim 16) and, because of connections $\vartheta/\{\beta,\gamma\}$, it must be in $(l-3, 9-l)$. Then, however, a connection ϑ/ζ is missing.

So, any colour of $\tilde{R} = R_i \cup R_j \cup R_k$ has an exemplar in $[3,5] \times [1,6]$, hence $\tilde{r} \leqslant 3 \cdot 6 - 2r = 8$, $c_2 = w + \tilde{r} \leqslant 7 + 8$, $c_2 = 15$, $n = 9$, $c_3 = c_{3+} = 5$, $\tilde{r} = 8$, and in $[3,5] \times [1,6]$ there are exclusively colours of $R \cup \tilde{R}$. From $r_{i,j} = r_{j,k} = 2$ and $r_{i,k} = 1$ we see that $r_i = r_k = 3$ and $r_j = 2$. The rectangle $[3,5] \times [1,2]$ cannot contain both exemplars of a colour of $R_{i,k}$ (it would have no connections with R_j). Also, that rectangle does not contain a colour of $R_i = \{\beta,\gamma,\delta\}$. Therefore, it contains five colours of R and a colour of R_k, say ϑ. Consequently, $R_k = \{\zeta,\vartheta,\iota\}$, where ι occupies the position $(k,6)$ (connections $\iota/R_{i,j}$). Because of connections $\{\beta,\gamma\}/\{\vartheta,\iota\}$, ϑ_2 and ι_2 must occupy both places in $\{(1,5),(2,4)\}$. Now, the rectangle $[1,2] \times [7,9]$ contains no 2-colour: since $R_k = \{\zeta,\vartheta,\iota\}$, it could be only a colour of $R_i \cup R_j$, but such a colour would miss one of the connections with ϑ and ι. Because of $c_3 = c_{3+} = 5$ that rectangle contains two exemplars of a 3-colour, say κ. As κ_3 appears in the square $[3,5] \times [7,9]$, at least one of the connections κ/R is missing.

As all possibilities with $r_{1,2} \in [1,2]$ lead to a contradiction, to conclude the proof of the claim it is sufficient to use Claim 10. □

Claim 20. *If $i \in [1,5]$, then $\overline{w}(K(i)) \geqslant 3a+3$.*

Proof of Claim 20. From the definition it immediately follows that $\overline{w}(K(i)) = c_2 - w(K(i))$. Since $w(K(i)) \leqslant n$, with help of Claim 2 we obtain $\overline{w}(K(i)) \geqslant (n+3a+3) - n = 3a+3$. □

Claim 21. *Let $\{i,j,k\} = [3,5]$, $3 \leqslant \min\{r_{i,j}, r_{i,k}\} \leqslant \max\{r_{i,j}, r_{i,k}\} \leqslant 4$ and $l \in [1,2]$. If $r_{i,j} = r_{i,k} = 4$, then $r_{l,j} r_{3-l,k} = 0$. If $r_{i,j} + r_{i,k} \leqslant 7$ and $r_{l,j} r_{3-l,k} > 0$, then $r_{l,j} + r_{3-l,k} + r_{i,j} + r_{i,k} \leqslant 9$ and, for any $\alpha \in R_{l,j}$ and $\beta \in R_{3-l,k}$, a connection α/β is realized in a column containing at least one colour of $R_{i,j}$ and at least one colour of $R_{i,k}$.*

Proof of Claim 21. Suppose that the sets $R_{l,j}$ and $R_{3-l,k}$ are both non-empty and consider colours $\alpha \in R_{l,j}$, $\beta \in R_{3-l,k}$.

If $r_{i,j} = r_{i,k} = 4$, because of the connections $R_{l,j}/R_{i,k}$ (realized in columns of A) each S_α-column must contain two colours of $R_{i,k}$; analogously, any S_β-column contains two colours of $R_{i,j}$. As a consequence, the sets S_α and S_β are disjoint (note that any column of A has at most three colours of R) and there is no connection α/β in A, a contradiction.

Now, assume that $r_{i,j} + r_{i,k} \leqslant 7$. A connection α/β is realized in a p-column, $p \in [1,n]$. Since $\min\{r_{i,j}, r_{i,k}\} \geqslant 3$, the p-column contains at least one colour of $R_{i,j}$, at least one colour of $R_{i,k}$, and altogether at least $r_{i,j} + r_{i,k} - 4$ colours of $R_{i,j} \cup R_{i,k}$: α_2 can realize at most two connections $\alpha/R_{i,k}$ and β_2 at most two connections $\beta/R_{i,j}$.

Thus, if $r_{i,j} + r_{i,k} = 7$, the "rectangle" $[3,5] \times \{p\}$ is filled in with colours of $R_{i,j} \cup R_{i,k}$. If $\{q\} = S_\alpha - \{p\}$, then the q-column does not have an analogous property, as it has in (j,q) the colour α; therefore, it cannot provide any connection $R_{l,j}/R_{3-l,k}$. The same is true for the unique $(S_\beta - \{p\})$-column, so that $r_{l,j} = r_{3-l,k} = 1$ and $r_{l,j} + r_{3-l,k} + r_{i,j} + r_{i,k} = 9$.

Now, suppose that $r_{i,j} = r_{i,k} = 3$. If all connections $R_{l,j}/R_{3-l,k}$ are realized in the p-column, then $r_{l,j} + r_{3-l,k} \leqslant 3$ and $r_{l,j} + r_{3-l,k} + r_{i,j} + r_{i,k} \leqslant 9$. If $\{q\} = S_\alpha - \{p\}$ and the q-column provides a connection α/γ for a colour $\gamma \in R_{3-l,k} - \{\beta\}$, which is not realized in the p-column, then three positions in $[3,5] \times \{p,q\}$ are occupied by colours of $R_{i,k}$, two by colours of $R_{i,j}$ (one in the p-column and the other in the q-column), and one position is occupied by the colour α. Further, in $[1,2] \times \{p,q\}$ there are colours α, β, γ. That is why $S_\beta \cap S_\gamma = \emptyset$ (β_2 and γ_2 are in the k-row), four places in $[3,5] \times ((S_\beta \cup S_\gamma) - \{p,q\})$ are occupied by colours of $R_{i,j}$, and two by the colours β, γ. So, $S_{i,j} = S_\beta \cup S_\gamma$ and, besides colours of $R_{i,j}$, the set $\{i,j\} \times S_{i,j}$ contains

α and one colour of $R_{i,k}$. Therefore, $r_{l,j} = 1$ and $r_{3-l,k} = 2$: a colour of $R_{l,j} - \{\alpha\}$ would miss at least one of connections with β and γ , and a colour of $R_{3-l,k} - \{\beta, \delta\}$ would miss a connection with α. As a consequence, $r_{l,j} + r_{3-l,k} + r_{i,j} + r_{i,k} = 9$.

Similarly, if the unique $(S_\beta - \{p\})$-column provides a connection β/δ for a colour $\delta \in R_{l,j}$, we obtain $r_{l,j} = 2$, $r_{3-l,k} = 1$ and $r_{l,j} + r_{3-l,k} + r_{i,j} + r_{i,k} = 9$. □

Claim 22. *$w \leqslant n - a - 1$, and the equality can apply only if $c_2 = n + 3a + 3$ and $c_3 = c_{3+} = n - a - 2$.*

Proof of Claim 22. Using successively Claims 19 and 5, we obtain $w = r = c_2 - w(K(1)) - w(K(2)) \leqslant c_2 - 2(n - c_{3+}) = (c_2 + c_{3+}) + c_{3+} - 2n = (2n + a + 1) + c_{3+} - 2n$ and then, by Claim 2, $w - a - 1 \leqslant c_{3+} \leqslant n - 2a - 2$ so that $w \leqslant n - a - 1$. If the last inequality turns into equality, then $c_{3+} = n - 2a - 2$, $c_2 = (2n + a + 1) - (n - 2a - 2) = n + 3a + 3$ and, with help of Claim 2, $c_{4+} = 0$ and $c_3 = c_{3+}$. □

Claim 23. $w \geqslant \lceil \frac{1}{3}(c_2 + 2r_{3,4}) \rceil \geqslant \lceil \frac{1}{3}(n + 3a + 3 + 2r_{3,4}) \rceil$.

Proof of Claim 23. By the choice of $K(1,2)$ we have $3w \geqslant w(K(1,2)) + w(K(1,5)) + w(K(2,5)) = \sum_{i=1}^{4} \sum_{j=i+1}^{5} r_{i,j} + 2r_{3,4} \geqslant n + 3a + 3 + 2r_{3,4}$ where, for the last inequality, we have used Claim 2. □

Claim 24. $r_{3,5} \leqslant 4$.

Proof of Claim 24. Suppose that $r_{3,4} = r_{3,5} = 5$. Then, successively by Claims 11, 4 and 2, $r_{1,4} = r_{2,4} = r_{1,5} = r_{2,5} = 0$, $a = 0$ and $c_2 \geqslant n + 3 \geqslant 18$, hence $c_2 = w(K(3)) + r_{4,5}$ and, as $w(K(3)) \leqslant n$, $r_{4,5} \geqslant 3$. Now Claim 14 yields $\hat{r} := r_{4,5} + r_{1,3} + r_{2,3} \leqslant 8$ so that $18 \leqslant c_2 = (r_{3,4} + r_{3,5}) + \hat{r} \leqslant 2 \cdot 5 + 8$, $c_2 = 18$, $n = 15$, $\hat{r} = 8$ and, by Claim 14 again, $r_{4,5} = r_{1,3} + r_{2,3} = 4$. From Claim 11 it follows that the sets $S_{3,4}$, $S_{3,5}$, $S_{4,5}$ are pairwise disjoint. On the other hand, from $r_{4,5} = 4$ we see that $|S_{4,5}| \geqslant 4$. Thus, $n \geqslant |S_{3,4}| + |S_{3,5}| + |S_{4,5}| = 2 \cdot 6 + |S_{4,5}| \geqslant 16$, a contradiction. □

Claim 25. $r_{4,5} \geqslant 1$.

Proof of Claim 25. Suppose that $r_{4,5} = 0$. Since $w \geqslant 7$, we have $r_{3,4} \in [4, 5]$. If $r_{3,4} = 5$, then, by Claims 4 and 23, $w \geqslant \lceil \frac{1}{3}(15 + 3 \cdot 0 + 3 + 2 \cdot 5) \rceil = 10$, hence $r_{3,5} = 5$ in contradiction with Claim 24. If $r_{3,4} = 4$, Claims 23 and 3 imply $w \geqslant \lceil \frac{1}{3}(c_2 + 2 \cdot 4) \rceil \geqslant \lceil \frac{23}{3} \rceil = 8$ so that $r_{3,5} = 4$, $w = 8$, $c_2 \leqslant 16$, $n \in \{7, 9\}$ (see Claim 2) and $a \geqslant 1$. However, Claim 22 yields $w \leqslant n - a - 1 \leqslant 7$, a contradiction. □

Claim 26. $a = 1$.

Proof of Claim 26. If $a = 2$, by virtue of Claims 19, 23 and 22 we obtain $\frac{1}{3}(n+15) \leqslant \lceil \frac{1}{3}(n+15) \rceil \leqslant \lceil \frac{1}{3}(n+3\cdot 2+3+2r_{3,4}) \rceil \leqslant w \leqslant n-2-1$, hence $n \geqslant 12$, a contradiction.

So, suppose that $a = 0$. For $k \in [0,3]$, let $t^{(k)}$ be the number of colours appearing k times in the $[3,5]$-rows; then $t := t_{3,4} + t_{3,5} + t_{4,5} = t^{(2)} + 3t^{(3)}$. From Claims 25 and 4 we obtain $\max\{t_{3,4}, t_{3,5}, t_{4,5}\} \leqslant 5$ and $t \leqslant 15$. As $\sum\limits_{k=0}^{3} t^{(k)} = 2n+1$, we have also $3n = \sum\limits_{k=1}^{3} kt^{(k)} \leqslant \sum\limits_{k=1}^{3} t^{(k)} + t^{(2)} + 3t^{(3)} \leqslant (2n+1) + t \leqslant 2n + 16$, $n \in [15,16]$ and $t \geqslant n-1 \geqslant 14$. Thus, we know that $\min\{t_{3,4}, t_{3,5}, t_{4,5}\} \geqslant 4$ and at least two of the numbers $t_{3,4}$, $t_{3,5}$, $t_{4,5}$ are 5's.

First assume that there are i, j, k with $\{i,j,k\} = [3,5]$, $S_{i,j} \cap S_{i,k} \neq \emptyset$ and, without loss of generality, $t_{i,j} \geqslant t_{i,k}$ (so that $t_{i,j} = 5$). Consider colours $\alpha \in R_{i,j}$ and $\beta \in R_{i,k}$ present in an $(S_{i,j} \cap S_{i,k})$-column. We may suppose without loss of generality that $1 \in S_\alpha \cap S_\beta \subseteq S_{i,j} \cap S_{i,k}$. Let $c_{i,j}$ ($c_{i,k}$, respectively) be the number of colours in $\{1,2,k\} \times S_\alpha$ (in $\{1,2,j\} \times S_\beta$) that are missing in both $\{i,j\}$-rows ($\{i,k\}$-rows). Because of connections with α all colours must be present either in one of the $\{i,j\}$-rows or in $\{1,2,k\} \times S_\alpha$. That is why $2n+1 = (2n - t_{i,j}) + c_{i,j} = 2n - 5 + c_{i,j}$, $c_{i,j} = 6$, and both colours in $[1,2] \times \{1\}$, say γ and δ, are out of the $\{i,j\}$-rows. By Claim 13 we have $R_{1,2} = \emptyset$, hence both γ and δ are in the k-row. Then, however, $c_{i,k} \leqslant 4$ (note that both γ and δ are in one of the $\{i,k\}$-rows and in $\{1,2,j\} \times \{1\} \subseteq \{1,2,j\} \times S_\beta$ as well), and $2n+1 = (2n - t_{i,k}) + c_{i,k} \leqslant (2n-4) + 4$, a contradiction.

Henceforth we suppose that the sets $S_{3,4}$, $S_{3,5}$, $S_{4,5}$ are pairwise disjoint. Using Claim 24 we obtain $w \leqslant 5 + 2\cdot 4$, hence $r_3 + r_4 + r_5 = c_2 - w \geqslant 18 - 13 = 5$. If only one of the numbers r_3, r_4, r_5 is positive, say r_i, and $\{i,j,k\} = [3,5]$, then $r_i \geqslant 5$, Claim 12 yields $r_{j,k} \leqslant 2$, and consequently $c_2 = w(K(i)) + r_{j,k} \leqslant n+2$ in contradiction with Claim 2. Thus, we know that at least two of r_3, r_4, r_5 are positive. Claim 23 leads to the estimate $r_{3,5} \geqslant \lceil \frac{1}{2}(w - r_{3,4}) \rceil \geqslant \lceil \frac{1}{2}(\frac{1}{3}(18 + 2r_{3,4}) - r_{3,4}) \rceil = \lceil 3 - \frac{1}{6} r_{3,4} \rceil \geqslant \lceil 3 - \frac{5}{6} \rceil = 3$.

Suppose first that $r_4 r_5 > 0$ and consider colours $\alpha \in R_4$ and $\beta \in R_5$. Since α needs connections with $r_{3,5} \geqslant 3$ colours of $R_{3,5}$ and any $S_{3,5}$-column can provide at most two such connections, we have $S_\alpha \subseteq S_{3,5}$; analogously, $r_{3,4} \geqslant 3$ implies $S_\beta \subseteq S_{3,4}$. However, $S_{3,4} \cap S_{3,5} = \emptyset$ and so the connection α/β is realized in an l-row, $l \in [1,2]$; then, clearly, all colours of $R_4 \cup R_5$ are in the l-row, and $r_{3-l,4} = r_{3-l,5} = 0$. By Claim 5, $w(K(3-l)) = r_{3-l,3} \geqslant 2$. A colour $\gamma \in R_{3-l,3}$ needs connections with α, β and $R_{4,5}$, therefore all the sets $S_\gamma \cap S_\alpha$, $S_\gamma \cap S_\beta$, $S_\gamma \cap S_{4,5}$ are non-empty, and $|S_\gamma| \geqslant |S_\gamma \cap (S_{3,4} \cup S_{3,5} \cup S_{4,5})| = |S_\gamma \cap S_{3,4}| + |S_\gamma \cap S_{3,5}| + |S_\gamma \cap S_{4,5}| \geqslant$

$|S_\gamma \cap S_\beta| + |S_\gamma \cap S_\alpha| + |S_\gamma \cap S_{4,5}| \geqslant 1+1+1$ in contradiction with the fact that γ is a 2-colour.

Thus, we may suppose that $r_3 > 0$ and there is $i \in [4,5]$ such that $r_i > 0$ and $r_{9-i} = 0$. Provided that $r_{4,5} \geqslant 3$, we repeat the above considerations leading to a contradiction. Therefore, we assume that $r_{4,5} \in [1,2]$ (Claim 25). By Claim 2 we have $18 \leqslant c_2 = r_3 + r_i + w \leqslant r_3 + r_i + 5 + r_{3,5} + 2$, hence $r_3 + r_i + r_{3,5} \geqslant 11$. Consider a colour $\alpha \in R_{4,5}$.

If $r_{1,i} r_{2,i} > 0$, then any colour of $R_{l,3}$, $l \in [1,2]$, must have one exemplar in an S_α-column (and hence in an $S_{4,5}$-column) and the other in an $S_{3,9-i}$-column: it needs connections with $R_{3-l,i}$, and $r_{3,9-i} \geqslant 3$ implies $S_{3-l,i} \subseteq S_{3,9-i}$; note that the obtained inclusion together with Claim 11 yield $r_{3,9-i} \leqslant 4$. The number of colours of R_3 with an exemplar in $[1,2] \times S_{3,9-i}$ is at most 2, since the second exemplar of each such colour must be in $\{3\} \times S_\alpha$. On the other hand, the number of colours of R_3 with an exemplar in $\{3\} \times S_{3,9-i}$ is at most $4 - r_{3,9-i}$: if $r_{3,9-i} = 4$ and $\mu \in R_i$, all four places in $\{3, 9-i\} \times S_\mu$ are occupied by colours of $R_{3,9-i}$; if $r_{3,9-i} = 3$, then a colour $\mu \in R_i$ must appear in an $S^{(2)}_{3,9-i}$-column, and so μ_2 can provide a column connection with a 3-row exemplar of a colour of R_3 only if its column contains in the $(9-i)$-row the last colour of $R_{3,9-i}$. Thus, $r_3 = r_{1,3} + r_{2,3} \leqslant 2 + (4 - r_{3,9-i})$ and, using Claim 12, $r_3 + r_i + r_{3,5} \leqslant r_3 + r_i + r_{3,9-i} = (r_3 + r_{3,9-i}) + (r_{1,i} + r_{2,i}) \leqslant 6 + 4 = 10$ in contradiction with $r_3 + r_i + r_{3,5} \geqslant 11$.

If $r_{1,i} r_{2,i} = 0$, there is $l \in [1,2]$ with $r_{l,i} > 0$ and $r_{3-l,i} = 0$. In such a case consider a colour $\beta \in R_{l,i}$. Any colour of $R_{3-l,3}$ has one exemplar in an S_α-column, $S_\alpha \subseteq S_{4,5}$, and the other in an S_β-column, $S_\beta \subseteq S_{l,i} \subseteq S_{3,9-i}$. As above, the number of colours of $R_{3-l,3}$ with an exemplar in $\{3\} \times S_{3,9-i}$ is at most $4 - r_{3,9-i}$. The number of colours of $R_{3-l,3}$ with an exemplar in $\{3-l\} \times S_{3,9-i}$ is at most $4 - r_{l,3}$, because any such colour as well as any colour of $R_{l,3}$ must have an exemplar in $\{l, 3\} \times S_\alpha$. Thus, $r_{3-l,3} \leqslant (4 - r_{3,9-i}) + (4 - r_{l,3})$. Since $r_{3-l,i} = r_{3-l,9-i} = r_{3-l,l} = 0$, Claim 5 yields $r_{3-l,3} \geqslant 2$. A colour $\gamma \in R_{3-l,3}$ can realize its connections with $R_{l,i}$ only in the unique $(S_\gamma \cap S_{3,9-i})$-column, hence $r_{l,i} \leqslant 2$. Using the last two inequalities containing the symbol $\leqslant$ we obtain $r_3 + r_i + r_{3,5} \leqslant r_3 + r_i + r_{3,9-i} = (r_3 + r_{3,9-i}) + r_{l,i} \leqslant 8 + 2 = 10$, a contradiction. □

Claim 27. $r_i \geqslant 1$, $i = 3, 4, 5$.

P r o o f of Claim 27. Suppose that $r_i = 0$ and $\{i, j, k\} = [3,5]$. If there are $l \in [1,2]$ and $p \in \{j, k\}$ with $r_{l,p} = 0$, then, provided that $\{p, q\} = \{j, k\}$, Claim 5 with respect to $r_{l,p} = r_{l,3-l} = 0$ yields $r_{l,q} \geqslant 4$. As a consequence, $r_{i,p} + r_{3-l,p} \leqslant 4$ (Claim 12) and $c_2 = w(K(q)) + r_{i,p} + r_{3-l,p} \leqslant n + 4$ in contradiction with Claim 2. Thus, we may assume that $r_{1,j} r_{2,j} r_{1,k} r_{2,k} > 0$.

Suppose first that the following condition $(*)$ is fulfilled: There are $p \in \{j,k\}$ and colours $\alpha \in R_{1,p}$, $\beta \in R_{2,p}$ such that α_1, β_1 share the p-row and α_2, β_2 share a column. Let $\{p,q\} = \{j,k\}$ and, without loss of generality, $S_\alpha = [1,2]$, $S_\beta = \{1,3\}$. By Claim 20, $\overline{w}(K(p)) = r_q + r_{i,q} \geqslant 6$. Let $\hat{C}$ be the set of colours of $R_q \cup R_{i,q}$ having an exemplar in $\{q\} \times [4,n]$. If $\mu \in \hat{C}$, then μ_2 must provide both connections with α and β. However, in the $\{1,2,i\}$-rows there are only three appropriate positions for colours of $\hat{C}$, namely (1,3), (2,2) and $(i,1)$. Therefore, $|\hat{C}| = 3$, $r_q + r_{i,q} = 6$, and we may assume without loss of generality that all positions in $\{q\} \times [1,6]$ are filled in with colours of $R_q \cup R_{i,q}$. We have also $r_p + r_{i,p} \geqslant 6$. Clearly, each colour of $R_p \cup R_{i,p}$ has an exemplar in $\{p\} \times [1,6]$, since any position in the $\{1,2,i\}$-rows provides at most two connections with $\hat{C}$; consequently, $r_p + r_{i,p} = 6$. As $r_{1,2} = r_{1,i} = r_{2,i} = 0$, 2-colours occupy altogether 6+6=12 positions in the $\{1,2,i\}$-rows. By Claim 2, the number of places in A occupied by 2-colours is at least $2(n+6)$, hence the $\{p,q\}$-rows are filled in with 2-colours. Therefore, colours appearing in $\{p,q\} \times [7,n]$ are there twice, i.e., $r_{p,q} = n - 6 \leqslant 4$ (Claim 4) so that $n = 9$ (Claim 26) and $r_{p,q} = 3$. Thus, the set of colours missing in both $\{p,q\}$-rows is of cardinality $2n + a + 1 - (2n - t_{p,q}) = t_{p,q} + 2 = r_{p,q} + 2 = 5$. However, any colour of that set must have two exemplars in $\{1,2,i\} \times S_{p,q} = \{1,2,i\} \times [7,9]$, a contradiction.

Now, suppose that $(*)$ is not fulfilled. Then any S_α-column with $\alpha \in R_{i,j}$ contains at most two colours of R_k (and if two, one of them is in the k-row), and so $r_k \leqslant 2 + 2 = 4$. Analogously, analyzing the situation of a colour $\beta \in R_{i,k}$, we obtain $r_j \leqslant 4$. On the other hand, by Claim 5, $4 \leqslant r_{l,j} + r_{l,k}$, $l = 1,2$ and, consequently, $8 \leqslant (r_{1,j} + r_{1,k}) + (r_{2,j} + r_{2,k}) = r_j + r_k \leqslant 8$, hence $r_j = r_k = r_{l,j} + r_{l,k} = 4$, $l = 1,2$. Furthermore, if $S_\alpha = \{p,q\}$, all of the following four sets contain exactly two colours of R_k: $[1,2] \times S_\alpha$, $\{k\} \times S_\alpha$, $\{1,2,k\} \times \{p\}$, and $\{1,2,k\} \times \{q\}$. Similarly, if $S_\beta = \{x,y\}$, exactly two colours of R_j are present in the sets $[1,2] \times S_\beta$, $\{j\} \times S_\beta$, $\{1,2,j\} \times \{x\}$ and $\{1,2,j\} \times \{y\}$. Thus, $S_\alpha \cap S_\beta \subseteq S_{i,j} \cap S_{i,k} = \emptyset$: an $(S_{i,j} \cap S_{i,k})$-column should contain at least one colour of each of the sets $R_{i,j}$, $R_{i,k}$ and exactly two colours of each of the sets R_j, R_k, which is impossible. By Claim 20, $\overline{w}(K(k)) = r_j + r_{i,j} = 4 + r_{i,j} \geqslant 6$, hence $r_{i,j} \geqslant 2$ and, analogously, $r_{i,k} \geqslant 2$.

Let us show that $r_{i,j} = r_{i,k} = 2$. Indeed, if e.g. $r_{i,j} \geqslant 3$, then, according to the above considerations, $s_{i,j} \leqslant 4$: with $s_{i,j} \geqslant 5$ we would have $r_k \geqslant 5$. Connections $R_{1,j}/R_{2,k}$ and $R_{1,k}/R_{2,j}$ (note that $r_{1,j}r_{2,k} > 0$ and $r_{1,k}r_{2,j} > 0$) can be realized (since $S_{i,j} \cap S_{i,k} = \emptyset$ and $r_{i,j} \geqslant 3$) only in $S_{i,j}$-columns and connections β/R_j in S_β-columns. Therefore, for any colour $\mu \in R_j$ with μ_1 in $[1,2] \times S_\beta$, μ_2 is in $\{j\} \times S_{i,j}$, and the number of such colours is at most $s_{i,j} - r_{i,j} \leqslant 4 - r_{i,j}$. The number of colours of R_j with an exemplar in $\{j\} \times S_\beta$ is at most 2, hence $r_j \leqslant (4 - r_{i,j}) + 2 = 6 - r_{i,j} \leqslant 3$, a contradiction.

Thus, by Claim 2, $r_{j,k} = c_2 - r_j - r_k - (r_{i,j} + r_{i,k}) = c_2 - 4 - 4 - 4 \geqslant (n+6) - 12$. From Claim 4 it follows that $4 \geqslant r_{j,k} \geqslant n - 6$, hence $n = 9$ and $r_{j,k} \geqslant 3$, so that $r_{j,k} = r_{3,4}$ and $w = r_{i,j} + r_{i,k} + r_{j,k} = r_{3,4} + 4$. By Claim 22 we have $w \leqslant 7$, hence $w = 7$ (Claim 9), $r_{3,4} = 3$, $c_2 = 15$ and $c_3 = c_{3+} = 5$. As $n = 9 = w(K(j)) = w(K(k))$, the $\{j,k\}$-rows are filled in with 2-colours; three colours of $R_{j,k}$ appear there twice and the remaining twelve colours just once. Therefore, $c_3 = r_{1,2,i}$ and then $s_{j,k} \geqslant 4$ since the colours of $R_{1,2,i}$ need at least ten places in $\{1,2,i\} \times S_{j,k}$. We have $S_{i,j} \cap S_{j,k} = \emptyset$: if $\mu \in R_{i,j}$, $\nu \in R_{j,k}$ and both μ, ν are in a common $(S_{i,j} \cap S_{j,k})$-column, that column should contain μ, ν, two colours of R_k and at least two colours of $R_{1,2,i}$ (as $r_{1,2,i} = 5$). Similarly, $S_{i,k} \cap S_{j,k} = \emptyset$, and so using $S_{i,j} \cap S_{i,k} = \emptyset$ we obtain $s_{j,k} \leqslant 9 - s_{i,j} - s_{i,k} \leqslant 5$.

If $s_{j,k} = 5$, consider colours $\gamma, \delta \in R_k$ present in $[1,2] \times S_\alpha$ and colours $\varepsilon, \zeta \in R_j$ present in $[1,2] \times S_\beta$. From $s_{i,j} = r_{i,j} = 2 = s_{i,k} = r_{i,k}$ it follows that $S_{i,j} = S_\alpha$, $S_{i,k} = S_\beta$, hence the sets $\{j\} \times S_\alpha$ and $\{k\} \times S_\beta$ are filled in with colours of $R_{i,j}$ and $R_{i,k}$, respectively. That is why γ_2 and δ_2 are in $\{k\} \times ([1,9] - S_\alpha - S_\beta)$, while ε_2 and ζ_2 are in $\{j\} \times ([1,9] - S_\alpha - S_\beta)$. Moreover, as $s_{j,k} = 5$, γ_2, δ_2, ε_2 and ζ_2 cover four $([1,9] - S_\alpha - S_\beta)$-columns. Because of connections $\{\gamma,\delta\}/\{\varepsilon,\zeta\}$, there is $l \in [1,2]$ such that γ_1, δ_1, ε_1 and ζ_1 share the l-row. If η, ϑ are colours of R_k in $\{k\} \times S_\alpha$ and ι, κ are colours of R_j in $\{j\} \times S_\beta$, then, because of connections $\{\varepsilon,\zeta\}/\{\eta,\vartheta\}$ and $\{\gamma,\delta\}/\{\iota,\kappa\}$, η_2, ϑ_2, ι_2 and κ_2 must occur in $[1,2] \times ([1,9] - S_\alpha - S_\beta)$. On the other hand, the number of colours of $R_{1,2,i}$ that appear in only two $S_{j,k}$-columns is at most 3 (only the colours of $R_{1,2,i}$ in the unique column with two colours of $R_{j,k}$ can have this property), and the total number of places occupied by $R_{1,2,i}$ in $S_{j,k}$-columns is at least $3 \cdot 2 + 2 \cdot 3 = 12$; this is a contradiction since $|\{1,2,i\} \times ([1,9] - S_\alpha - S_\beta)| = 15 < 12 + |\{\eta_2, \vartheta_2, \iota_2, \kappa_2\}|$.

Thus, $s_{j,k} = 4$. There are two colours $\gamma, \delta \notin R_{j,k}$ having an exemplar in $\{j,k\} \times S_{j,k}$. Evidently, γ_1 and δ_1 are in independent positions; we may suppose without loss of generality that γ_1 is in the j-row and δ_1 in the k-row. Because of connections β/γ and α/δ, γ_2 must be in an S_β-column and δ_2 must be in an S_α-column. That is why (note that the sets $S_{i,j}$, $S_{i,k}$, $S_{j,k}$ are pairwise disjoint) γ_2 and δ_2 must share an l-row, $l \in [1,2]$. Since $(*)$ is not fulfilled, we can replace α by $\alpha' \in R_{i,j} - \{\alpha\}$ and/or β by $\beta' \in R_{i,k} - \{\beta\}$ and repeat the above analysis. Therefore, if ε and ζ are colours in (j,m) and (k,m), respectively, where m is the unique element of the set $[1,9] - S_\alpha - S_\beta - S_{j,k}$, there are only the following three possibilities: $\varepsilon \in R_{i,j}$ and $\zeta \in R_k$, $\varepsilon \in R_j$ and $\zeta \in R_{i,k}$, $\varepsilon \in R_j$ and $\zeta \in R_k$.

If $\varepsilon \in R_j$, then, because of connections $\varepsilon/\{\beta,\delta\}$, ε_2 must be in $\{l\} \times S_\beta$. As $w(K(l)) = 4$, at least one of the two colours of R_k appearing in $\{k\} \times S_\alpha$ has its second exemplar in the $(3-l)$-row, and so misses at least one of connections with γ and ε.

If $\zeta \in R_k$, then, analogously, there is a colour of R_j in $\{j\} \times S_\beta$ missing at least one of connections with δ and ζ. □

Claim 28. $r_{3,4} = 3$.

P r o o f of Claim 28. By Claims 26 and 4, we have $r_{3,4} \leqslant 4$. If $r_{3,4} = 4$, Claims 22 and 23 yield $n-2 \geqslant w \geqslant \lceil \frac{1}{3}(n+14) \rceil \geqslant \frac{1}{3}(n+14)$, hence $n \geqslant 10$, even $n \geqslant 11$ (Claim 26), and $w \geqslant 9$, so that $r_{3,5} \in [3,4]$.

Suppose first that $r_{3,5} = 4$. We know that $r_4 \geqslant 1$ and $r_5 \geqslant 1$ (Claim 27). On the other hand, by Claim 21, $r_{1,4}r_{2,5} = r_{1,5}r_{2,4} = 0$, hence there is $l \in [1,2]$ such that $r_{l,4}r_{l,5} > 0$ and $r_{3-l,4} = r_{3-l,5} = 0$. As $r_{3-l,l} = 0$, with help of Claims 26, 5 and 4 we obtain $r_{3-l,3} = 4$ so that, by the choice of $K(1,2)$, $w = 8 + r_{4,5} > w(K(3-l,3)) = r_{l,4} + r_{l,5} + r_{4,5} + 4$, $r_{l,4} + r_{l,5} \leqslant 3$ and, by Claim 5, $r_{l,3} \geqslant 1$. By Claim 20, $\overline{w}(K(3)) = r_{l,4} + r_{l,5} + r_{4,5} \geqslant 6$, hence $r_{4,5} \geqslant 6 - (r_{l,4} + r_{l,5}) \geqslant 3$. However, the inequalities $r_{4,5} \geqslant 3$ and $r_{l,3} + r_{3-l,3} \geqslant 1 + 4 = 5$ are in contradiction with Claim 12.

Now, assume that $r_{3,5} = 3$. If there is $l \in [1,2]$ with $r_{l,5} \geqslant 1$ and $r_{3-l,4} = 0$, then $r_{3-l,3} + r_{3-l,5} \geqslant 4$ (Claim 5), $r_{3-l,3} \leqslant 2$ (Claim 13), $r_{3-l,5} \geqslant 2$, $r_{l,4} \geqslant 1$ (Claim 27) and so $r_{l,4} + r_{3-l,5} + r_{3,4} + r_{3,5} \geqslant 1 + 2 + 4 + 3 = 10$ in contradiction with Claim 21. Thus, we know that $r_{l,5} \geqslant 1$ implies $r_{3-l,4} \geqslant 1$ for $l = 1,2$; moreover, allowing for symmetry, we may suppose that, in the case $r_{4,5} = r_{3,5} = 3$, $r_{l,5} \geqslant 1$ implies also $r_{3-l,3} \geqslant 1$ for $l = 1,2$.

By Claim 27, there is $l \in [1,2]$ such that $r_{l,5} \geqslant 1$, hence $r_{3-l,4} \geqslant 1$ and, by Claim 21, this is possible only if $r_{l,5} = r_{3-l,4} = 1$. By the choice of $K(1,2)$, $w(K(l,5)) = 1 + (r_{3-l,3} + 1 + 4) < w = 4 + 3 + r_{4,5}$, $r_{3-l,3} \leqslant r_{4,5}$ and $w(K(3-l)) = r_{3-l,3} + 1 + r_{3-l,5} \leqslant r_{4,5} + 1 + r_{3-l,5}$. With respect to Claim 5, $r_{3-l,5} = 0$ implies $r_{3-l,3} \geqslant 3$ and, consequently, $r_{4,5} = r_{3-l,3} = 3$; in such a case, however, $r_{3,3-l} + r_{3,4} = 7$ in contradiction with Claim 13 (as $r_{l,5} \geqslant 1$). So, we may suppose that $r_{3-l,5} \geqslant 1$.

If $r_{4,5} = 3$, then by the above symmetry $r_{3-l,5} = r_{l,3} = 1$ and $w(K(l)) = r_{l,4} + 2$, $w(K(3-l)) = r_{3-l,3} + 2$. Then Claim 5 yields $r_{l,4}r_{3-l,3} > 0$ and $r_{l,4} + r_{3-l,3} \geqslant 4$, hence $r_{l,4} + r_{3-l,3} + r_{3,5} + r_{4,5} \geqslant 10$ in contradiction with Claim 21.

Finally, for $r_{4,5} = 2$ we obtain $r_{3-l,3} \leqslant 2$, $w(K(3-l,5)) = r_{3-l,5} + (r_{l,3} + r_{l,4} + 4) < w = 9$, $r_{3-l,5} + r_{l,3} + r_{l,4} \leqslant 4$, $r_{l,3} + r_{l,4} \geqslant 3$ (Claim 5) and $(r_{l,3} + r_{l,4}) + r_{3,4} \geqslant 3 + 4 = 7$ in contradiction with Claim 13 (since $r_{3-l,5} \geqslant 1$).

Now, the claim follows from Claim 19. □

Put $d := \sum_{l=1}^{2} \sum_{i=3}^{5} d(l,i)$, where $d(l,i) := w - w(K(l,i))$.

Claim 29. $d = 7w - 3c_2$.

P r o o f of Claim 29. If $\{i,j,k\} = [3,5]$, then $w(K(1,i)) + w(K(2,i)) = 2r_{j,k} + \sum_{l=1}^{2} \sum_{m=3}^{5} r_{l,m} = 2r_{j,k} + c_2 - w$, hence $-d(1,i) - d(2,i) = 2r_{j,k} + c_2 - 3w$. Analogously,

$-d(1,j)-d(2,j)=2r_{i,k}+c_2-3w$ and $-d(1,k)-d(2,k)=2r_{i,j}+c_2-3w$. Summing the last three equalities we obtain $-d=2(r_{j,k}+r_{i,k}+r_{i,j})+3c_2-9w=3c_2-7w$. □

Claim 30. $r_{3,5}=2$.

P r o o f of Claim 30. By Claim 28, we have $3=r_{3,4}\geqslant r_{3,5}$. Suppose that $r_{3,5}=3$. If $w=7$, then $c_2=15$ (Claim 23), $n=9$ (Claim 2) and $\min\{w(K(1)),w(K(2))\}\geqslant 4$ (Claim 5). Therefore, $14=2w\geqslant w(K(1,5))+w(K(2,5))=2r_{3,4}+r_3+r_4+r_5=6+w(K(1))+w(K(2))\geqslant 14$ and $w(K(1,5))=w(K(2,5))=7$. By the choice of $K(1,2)$, we see that then necessarily $r_{1,5}=r_{2,5}=0$. Since $r_4\leqslant 3$ (Claim 7), we have $r_3=c_2-w-r_4-r_5\geqslant 15-7-3-0=5$ and $9\geqslant w(K(3))=r_3+r_{3,4}+r_{3,5}\geqslant 5+3+3=11$, a contradiction.

If $w\geqslant 8$, then, by Claim 22, $n\geqslant 10$, hence $n\geqslant 11$ and $c_2\geqslant 17$ (Claim 2). Consider first the case $w=8$, i.e., $r_{4,5}=2$. From Claim 29 we know that $d=56-3c_2\leqslant 5$. By the choice of $K(1,2)$, $d(l,i)=0$ implies $r_{l,i}=0$. By Claim 27, at most three summands of d are 0's, so $d\geqslant 3, c_2=17$, $n=11$ and $d=5$. There must be $l\in[1,2]$ and $i\in[3,5]$ with $d(l,i)=0=r_{l,i}$; let $\{i,j,k\}=[3,5]$. Claim 27 yields $r_{3-l,i}\geqslant 1$ so that $7\geqslant w(K(3-l,i))=r_{3-l,i}+(r_{l,j}+r_{l,k}+r_{j,k})\geqslant 1+(4+r_{j,k})$ (Claim 5) and $r_{j,k}=2$. Thus, $8=w(K(l,i))=r_{3-l,j}+r_{3-l,k}+r_{j,k}=r_{3-l,j}+r_{3-l,k}+2$. With help of Claim 5, $c_2=8+w(K(l))+w(K(3-l))\geqslant 8+4+7=19$, a contradiction.

If $w=9$ (and $r_{4,5}=3$), then $r_{l,i}\in[0,2]$ for any $l\in[1,2]$ and $i\in[3,5]$. Indeed, the assumptions $r_{l,i}\geqslant 3$ and $\{i,j,k\}=[3,5]$ would lead, by Claim 21, to $r_{3-l,j}=r_{3-l,k}=0$. Then $r_{3-l,i}\geqslant 4$ (Claim 5) and $r_{l,i}+r_{3-l,i}\geqslant 7$; since $r_{j,k}=3$, we have obtained a contradiction with Claim 12. By Claim 5, we know that at least one summand of the sum $r_{l,3}+r_{l,4}+r_{l,5}$ is 2 for both $l=1,2$. If there are $i,j\in[3,5]$, $i\neq j$, such that $r_{1,i}=r_{2,j}=2$, we obtain an immediate contradiction with Claim 21.

Therefore, we may suppose that there is $j\in[3,5]$ with $r_{1,j}=r_{2,j}=2$, and the remaining summands in $\sum\limits_{l=1}^{2}\sum\limits_{m=3}^{5}r_{l,m}$ are 1's. Let $\{i,j,k\}=[3,5]$ and consider colours $\alpha,\gamma\in R_{1,j}$, $\beta\in R_{2,k}$, $\delta\in R_{2,i}$. By Claim 21, the connections α/β and α/δ cannot be realized in the same column: in such a column there would be α,β,δ and at least one colour of each of the sets $R_{i,j}$, $R_{i,k}$, $R_{j,k}$, a contradiction. Therefore, with help of the same claim, positions in $[3,5]\times S_\alpha$ are occupied by α, all three colours of $R_{i,k}$, one colour of $R_{i,j}$ and one colour of $R_{j,k}$. Similarly, places in $[3,5]\times S_\gamma$ are occupied by γ, all three colours of $R_{i,k}$, one colour of $R_{i,j}$ and one colour of $R_{j,k}$. As a consequence, $S_\alpha\cap S_\gamma=\emptyset$ (if $S_\alpha\cap S_\gamma\neq\emptyset$, then for at least one colour $\varepsilon\in\{\alpha,\gamma\}$ the set $\{j\}\times S_\varepsilon$ is filled in with α and γ), and at least one of connections $\beta/\{\alpha,\gamma\}$ is missing. □

To conclude the proof of Theorem 3, we are left with the case $r_{3,5}=r_{4,5}=2$. By Claim 23, we have $7=w\geqslant\lceil\frac{1}{3}(n+12)\rceil\geqslant\frac{1}{3}(n+12)$, hence $n=9$. Claim 27 implies

$r_5 \geqslant 1$, therefore, by the choice of $K(1,2)$, $14 = 2w > w(K(1,5)) + w(K(2,5)) = 2r_{3,4} + w(K(1)) + w(K(2)) \geqslant 6 + 4 + 4 = 14$, where, for the last inequality, we have used Claim 5. □

To resume the results of the analysis of the achromatic number of $K_5 \times K_n$, recall that $I_3 = \{1,6\}$, $I_2 = \{2,4,5,7,8,10\}$, $I_1 = \{3,9\} \cup [11,14]$, $I_0 = [15,24]$, and put $I_{-1} := \{25\}$, $I_{-2} := [26,28]$.

Theorem 4. *Let n be a positive integer and $a \in [-2,3]$.*

1. *If $n \in I_a$, then* $\operatorname{achr}(K_5 \times K_n) = 2n + a$.
2. *If $n \in [29,36]$, then* $\operatorname{achr}(K_5 \times K_n) = \lfloor \frac{3}{2}n \rfloor + 12$.
3. *If $n \in [37,42]$, then* $\operatorname{achr}(K_5 \times K_n) = \lfloor \frac{5}{3}n \rfloor + 6$.
4. *If $n \geqslant 43$, then* $\operatorname{achr}(K_5 \times K_n) = \lfloor \frac{9}{5}n \rfloor$.

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