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THE SPECTRA OF GENERAL DIFFERENTIAL OPERATORS IN THE DIRECT SUM SPACES

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Abstract. In this paper, the general ordinary quasi-differential expression M_p of n-th order with complex coefficients and its formal adjoint M_p^+ on any finite number of intervals $I_p = (a_p, b_p)$, $p = 1, \ldots, N$, are considered in the setting of the direct sums of $L^2_{w_p}(a_p, b_p)$-spaces of functions defined on each of the separate intervals, and a number of results concerning the location of the point spectra and the regularity fields of general differential operators generated by such expressions are obtained. Some of these are extensions or generalizations of those in a symmetric case in [1], [14], [15], [16], [17] and of a general case with one interval in [2], [11], [12], whilst others are new.

Keywords: quasi-differential expressions, essential spectra, joint field of regularity, regularly solvable operators, direct sum spaces

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1. Introduction

In [8] and [10], Everitt considered the problem of characterizing all self-adjoint operators which can be generated by a formally symmetric Sturm-Liouville differential (quasi-differential) expression M_p, defined on a finite number of intervals I_p, $p = 1, \ldots, N$, in the setting of direct sum spaces. In [12], Ibrahim considered the problem of the location of the point spectra and regularity fields of general ordinary quasi-differential operators in the one interval case with one regular end-point and the other end-point which may be regular or singular.

Our objective in this paper is to investigate the location of the point spectra and regularity fields of the operators generated by a general quasi-differential expression M_p on any finite number of intervals I_p, $p = 1, \ldots, N$, in the setting of direct sums of $L^2_{w_p}(a_p, b_p)$-space of functions defined on each of the separate intervals. These results extend those of formally symmetric expression studied in [1], [2],

[14], [15], [16] and [17], and also extend those proved in [5], [11] and [12] for general case with one interval.

The operators involved are no longer symmetric but direct sums as:

$$T_0(M) = \bigoplus_{p=1}^{N} T_0(M_p) \qquad \text{and} \qquad T_0(M^+) = \bigoplus_{p=1}^{N} T_0(M_p^+),$$

where $T_0(M_p)$ is the minimal operator generated by M_p on I_p and M_p^+ denotes the formal adjoint of M_p; these form an adjoint pair of closed operators in $\bigoplus_{p=1}^{N} L^2_{w_p}(I_p)$. This fact allows us to use the abstract theory developed in [1] for the operators which are regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$. Such an operator S satisfies $T_0(M_p) \subset S \subset [T_0(M_p^+)]^*$ and for some $\lambda \in \mathbb{C}$, $(S - \lambda I)$ is a Fredholm operator with zero index; this means that S has the desirable Fredholm property that the equation $(S - \lambda I)u = f$ has a solution if and only if f is orthogonal to the solutions of $(S^* - \overline{\lambda} I)v = 0$, and furthermore the solution spaces of $(S - \lambda I)u = 0$ and $(S^* - \overline{\lambda} I)v = 0$ have the same finite dimension. This notion was originally due to Visik [18].

We deal throughout with a quasi-differential expression M_p of arbitrary order n defined by a general Shin-Zettl matrix given in [3], [5] and [9], and the minimal operator $T_0(M_p)$ generated by $w_p^{-1} M_p[\cdot]$ in $L^2_{w_p}(I_p)$, $p = 1, \ldots, N$, where w_p is a positive weight function on the underlying interval I_p. The end-points of I_p may be regular or singular.

2. Preliminaries

In this section we give some definitions and results which will be needed later; see [2], [3], [4], [5] and [7].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$ and $R(T)$, respectively, and $N(T)$ will denote its null space. The nullity of T, written null(T), is the dimension of $N(T)$ and the deficiency of T, written def(T), is the co-dimension of $R(T)$ in H; thus if T is densely defined and $R(T)$ is closed, then $\operatorname{def}(T) = \operatorname{null}(T^*)$. The Fredholm domain of T is (in the notation of [3]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values $\lambda \in \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus, $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $\operatorname{ind}(T - \lambda I) = \operatorname{null}(T - \lambda I) - \operatorname{def}(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely defined operators A and B acting in H are said to form an adjoint pair if $A \subset B^*$ and, consequently, $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$ for all $x \in D(A)$ and $y \in D(B)$, where $(\cdot,\cdot)$ denotes the inner-product on H.

The field of regularity $\Pi(A)$ of A is the set of all $\lambda \in \mathbb{C}$ for which there exists a positive constant $K(\lambda)$ such that

$$\|(A - \lambda I)x\| \geqslant K(\lambda)\|x\| \quad \text{for all } x \in D(A),$$

or, equivalently, on using the Closed Graph Theorem, $\operatorname{null}(A-\lambda I) = 0$ and $R(A-\lambda I)$ is closed.

The joint field of regularity $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, $\overline{\lambda} \in \Pi(B)$ and both $\operatorname{def}(A - \lambda I)$ and $\operatorname{def}(B - \overline{\lambda} I)$ are finite. An adjoint pair A and B is said to be compatible if $\Pi(A, B) \neq \emptyset$.

Definition 2.1. A closed operator S in H is said to be regularly solvable with respect to the compatible adjoint pair A and B if $A \subset S \subset B^*$ and $\Pi(A, B) \cap \Delta_4(S) \neq \emptyset$, where $\Delta_4(S) = \{\lambda\colon \lambda \in \Delta_3(S),\ \operatorname{ind}(S - \lambda I) = 0\}$. The terminology "regularly solvable" comes from Visik's paper [18].

Definition 2.2. The resolvent set $\varrho(S)$ of a closed operator S in H consists of the complex numbers λ for which $(S - \lambda I)^{-1}$ exists, is defined on H and is bounded. The complement of $\varrho(S) \in \mathbb{C}$ is called the spectrum of S and written $\sigma(S)$. The point spectrum $\sigma_p(S)$, the continuous spectrum $\sigma_c(S)$ and the residual spectrum $\sigma_r(S)$ are the following subsets of $\sigma(S)$ (see [2] and [3]):

$\sigma_p(S) = \{\lambda \in \sigma(S)\colon\ (S - \lambda I) \text{ is not injective}\}$, i.e., the set of eigenvalues of S;

$\sigma_c(S) = \{\lambda \in \sigma(S)\colon\ (S - \lambda I) \text{ is injective},\ R(S - \lambda I) \underset{\neq}{\subset} \overline{R(S - \lambda I)} = H\}$;

$\sigma_r(S) = \{\lambda \in \sigma(S)\colon\ (S - \lambda I) \text{ is injective},\ \overline{R(S - \lambda I)} \neq H\}$.

For a closed operator S we have

$$\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S).$$

An important subset of the spectrum of a closed densely defined S in H is the so-called essential spectrum. The various essential spectra of S are defined as in [3, Chapter II] to be the sets:

$$\sigma_{ek}(S) = \mathbb{C} \setminus \Delta_k(S) \quad (k = 1, 2, 3, 4, 5), \tag{2.1}$$

where $\Delta_3(S)$ and $\Delta_4(S)$ have been defined earlier.

The sets $\sigma_{ek}(S)$ are closed and $\sigma_{ek}(S) \subset \sigma_{ej}(S)$ if $k < j$. The inclusion being strict in general. We refer the reader to [1], [2] and [3, Chapter IX] for further information about the sets $\sigma_{ek}(S)$.

3. Quasi-differential expressions

The quasi-differential expressions are defined in terms of a Shin-Zettl matrix F_p on an interval I_p. The set $Z_n(I_p)$ of Shin-Zettl matrices on I_p consists of $n \times n$-matrices $F_p = \{f^p_{rs}\}$, $1 \leqslant r, s \leqslant n$, $p = 1, \dots, N$, whose entries are complex-valued functions on I_p which satisfy the following conditions:

(3.1)
$$\begin{aligned} &f^p_{rs} \in L^2_{\text{loc}}(I_p) \quad (1 \leqslant r, s \leqslant n,\ n \geqslant 2),\\ &f^p_{r,r+1} \neq 0 \text{ a.e. on } I_p \quad (1 \leqslant r \leqslant n-1),\\ &f^p_{rs} = 0 \text{ a.e. on } I_p \quad (2 \leqslant r+1 < s \leqslant n),\ p = 1, \dots, N. \end{aligned}$$

For $F_p \in Z_n(I_p)$, the quasi-derivatives associated with F_p are defined by:

(3.2)
$$\begin{aligned} &y^{[0]} := y,\\ &y^{[r]} := (f^p_{r,r+1})^{-1}\Big\{\big(y^{[r-1]}\big)' - \sum_{s=1}^{r} f^p_{rs} y^{[s-1]}\Big\} \quad (1 \leqslant r \leqslant n-1),\\ &y^{[n]} := \big(y^{[n-1]}\big)' - \sum_{s=1}^{n} f^p_{ns} y^{[s-1]}, \end{aligned}$$

where the prime $'$ denotes differentiation.

The quasi-differential expression M_p associated with F_p is given by:

(3.3)
$$M_p[y] := i^n y^{[n]} \quad (n \geqslant 2),$$

this being defined on the set:

$$V(M_p) := \{y\colon\ y^{[r-1]} \in AC_{\text{loc}}(I_p),\ r = 1, \dots, n;\ p = 1, \dots, N\},$$

where $AC_{\text{loc}}(I_p)$ denotes the set of functions which are absolutely continuous on every compact subinterval of I_p.

The formal adjoint M^+_p of M_p is defined by the matrix $F^+_p \in Z_n(I_p)$ given by:

(3.4)
$$F^+_p := -L^{-1} F^*_p L,$$

where F^*_p is the conjugate transpose of F_p and $L_{n\times n}$ is the non-singular $n\times n$-matrix,

(3.5)
$$L_{n\times n} = \{(-1)^{r+s+1}\delta_{r,n+1-s}\} \quad (1 \leqslant r, s \leqslant n),$$

δ being the Kronecker delta. If $F^+_p = (f^p_{rs})^+$, then it follows that

(3.6)
$$(f^p_{rs})^+ = (-1)^{r+s+1}\overline{f^p}_{n-s+1,n-r+1}, \quad \text{for each } r \text{ and } s.$$

The quasi-derivatives associated with F_p^+ are therefore,

$$
\begin{aligned}
&y_+^{[0]} := y, \\
&y_+^{[r]} := (\overline{f^p}_{n-r,n-r+1})^{-1}\Big\{(y_+^{[r-1]})' - \sum_{s=1}^{r} \overline{f^p}_{n-s+1,n-r+1} y_+^{[s-1]}\Big\} \\
&\qquad (1 \leqslant r \leqslant n-1), \\
&y_+^{[n]} := (y_+^{[n-1]})' - \sum_{s=1}^{n} \overline{f^p}_{n-s+1,1} y_+^{[s-1]},
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
&M_p^+[y] := i^n y_+^{[n]},\ p = 1, \ldots, N,\ \text{for all } y \in V(M_p^+); \\
&V(M_p^+) := \Big\{y\colon\ y_+^{[r-1]} \in AC_{\mathrm{loc}}(I_p),\ r = 1, \ldots, n;\ p = 1, \ldots, N\Big\}.
\end{aligned} \tag{3.8}
$$

Note that: $(F_p^+)^+ = F_p$ and so $(M_p^+)^+ = M_p$. We refer to [5], [11], [12], [13] and [19] for a full account of the above and subsequent results on quasi-differential expressions.

Let the interval I_p have end-points a_p, b_p ($-\infty \leqslant a_p < b_p \leqslant \infty$), and let $w_p\colon I_p \to \mathbb{R}$ be a non-negative weight function with $w_p \in L^1_{\mathrm{loc}}(I_p)$ and $w_p(x) > 0$ (for almost all $x \in I_p$). Then $H_p = L^2_{w_p}(I_p)$ denotes the Hilbert function space of equivalence classes of Lebesgue measurable functions such that $\int_{I_p} w_p|f|^2 < \infty$; the inner-product is defined by:

$$
(f, g)_p := \int_{I_p} w_p(x) f(x) \overline{g(x)}\, \mathrm{d}x \quad (f, g \in L^2_{w_p}(I_p),\ p = 1, \ldots, N).
$$

The equation,

$$
M_p[u] - \lambda w_p u = 0 \quad (\lambda \in \mathbb{C}) \text{ on } I_p, \tag{3.9}
$$

is said to be regular at the left end-point $a_p \in \mathbb{R}$, if for all $X \in (a_p, b_p)$,

$$
a_p \in \mathbb{R}; \quad w_p, f^p_{rs} \in L^1(a_p, X) \quad (r, s = 1, \ldots, n; p = 1, \ldots, N).
$$

Otherwise (3.9) is said to be singular at a_p. If (3.9) is regular at both end-points, then it is said to be regular; in this case we have,

$$
a_p, b_p \in \mathbb{R};\ w_p, f^p_{rs} \in L^1(a_p, b_p), \quad (r, s = 1, \ldots, n; p = 1, \ldots, N).
$$

We shall be concerned with the case when a_p is a regular end-point of (3.9), the end-point b_p being allowed to be either regular or singular. Note that, in view of (3.6), an end-point of I_p is regular for (3.9), if and only if it is regular for the equation,

$$M_p^+[v] - \overline{\lambda} w_p v = 0 \quad (\lambda \in \mathbb{C}) \text{ on } I_p,\ p = 1, \dots, N. \tag{3.10}$$

Note that, at a regular end-point a_p, say, $u^{[r-1]}(a_p)\big(v_+^{[r-1]}(a_p)\big)$, $r = 1, \dots, n$, is defined for all $u \in V(M_p)$ ($v \in V(M_p^+)$. Set

$$\begin{aligned} D(M_p) &:= \{u\colon\ u \in V(M_p),\ u \text{ and } w_p^{-1} M_p[u] \in L^2_{w_p}(a_p, b_p)\},\\ D(M_p^+) &:= \{v\colon\ v \in V(M_p^+),\ v \text{ and } w_p^{-1} M_p^+[v] \in L^2_{w_p}(a_p, b_p)\},\\ &\quad p = 1, \dots, N. \end{aligned} \tag{3.11}$$

The subspaces $D(M_p)$ and $D(M_p^+)$ of $L^2_{w_p}(a_p, b_p)$ are domains of the so-called maximal operators $T(M_p)$ and $T(M_p^+)$ respectively, defined by:

$$T(M_p)u := w_p^{-1} M_p[u] \ \ (u \in D(M_p)) \quad \text{and} \quad T(M_p^+)v := w_p^{-1} M_p^+[v] \ \ (v \in D(M_p^+)).$$

For the regular problem the minimal operators $T_0(M_p)$ and $T_0(M_p^+)$, $p = 1, \dots, N$, are the restrictions of $w_p^{-1} M_p[u]$ and $w_p^{-1} M_p^+[v]$ to the subspaces:

$$\begin{aligned} D_0(M_p) &:= \big\{u\colon\ u \in D(M_p),\ u^{[r-1]}(a_p) = u^{[r-1]}(b_p) = 0,\ r = 1, \dots, n\big\},\\ D_0(M_p^+) &:= \big\{v\colon\ v \in D(M_p^+),\ v_+^{[r-1]}(a_p) = v_+^{[r-1]}(b_p) = 0,\ r = 1, \dots, n\big\};\\ &\quad p = 1, \dots, N, \end{aligned} \tag{3.12}$$

respectively. The subspaces $D_0(M_p)$ and $D_0(M_p^+)$ are dense in $L^2_{w_p}(a_p, b_p)$ and $T_0(M_p)$ and $T_0(M_p^+)$ are closed operators (see [3], [5], [11] and [19, Section 3]).

In the singular problem we first introduce the operators $T_0'(M_p)$ and $T_0'(M_p^+)$; $T_0'(M_p)$ being the restriction of $w_p^{-1} M_p[\cdot]$ to the subspace:

$$D_0'(M_p) := \{u\colon\ u \in D(M_p),\ \operatorname{supp} u \subset (a_p, b_p)\},\ p = 1, \dots, N, \tag{3.13}$$

and with $T_0'(M_p^+)$ defined similarly. These operators are densely-defined and closable in $L^2_{w_p}(a_p, b_p)$; and we define the minimal operators $T_0(M_p)$, $T_0(M_p^+)$ to be their respective closures (see [3], [5] and [19, Section 5]). We denote the domains of $T_0(M_p)$ and $T_0(M_p^+)$ by $D_0(M_p)$ and $D_0(M_p^+)$, respectively. It can be shown that:

$$\begin{aligned} u \in D_0(M_p) &\Longrightarrow u^{[r-1]}(a_p) = 0 \quad (r = 1, \dots, n;\ p = 1, \dots, N),\\ v \in D_0(M_p^+) &\Longrightarrow v_+^{[r-1]}(a_p) = 0 \quad (r = 1, \dots, n;\ p = 1, \dots, N), \end{aligned} \tag{3.14}$$

because we are assuming that a_p is a regular end-point. Moreover, in both regular and singular problems, we have

$$T_0^*(M_p) = T(M_p^+) \text{ and } T^*(M_p) = T_0(M_p^+), \quad p = 1, \ldots, N; \tag{3.15}$$

see [19, Section 5] in the case when $M_p = M_p^+$ and compare with treatment in [3, Section III.10.3] and [5] in general case.

In the case of two singular end-points, the problem on (a_p, b_p) is effectively reduced to the problems with one singular end-point on the intervals $(a_p, c_p]$ and $[c_p, b_p)$, where $c_p \in (a_p, b_p)$. We denote by $T(M_p; a_p)$ and $T(M_p; b_p)$ the maximal operators with domains $D(M_p; a_p)$ and $D(M_p; b_p)$, and denote $T_0(M_p; a_p)$ and $T_0(M_p; b_p)$ the closures of the operators $T_0'(M_p; a_p)$ and $T_0'(M_p; b_p)$ defined in (3.13) on the intervals $(a_p, c_p]$ and $[c_p, b_p)$, respectively, see [3], [7], [11], [13] and [14].

Let $\tilde{T}_0'(M_p)$, $p = 1, \ldots, N$, be the orthogonal sum as:

$$\tilde{T}_0'(M_p) = T_0'(M_p; a_p) \oplus T_0'(M_p; b_p)$$

in

$$L^2_{w_p}(a_p, b_p) = L^2_{w_p}(a_p, c_p) \oplus L^2_{w_p}(c_p, b_p),$$

$\tilde{T}_0'(M_p)$ is densely-defined and closable in $L^2_{w_p}(a_p, b_p)$ and its closure is given by

$$\tilde{T}_0(M_p) = T_0(M_p; a_p) \oplus T_0(M_p; b_p), \quad p = 1, \ldots, N.$$

Also,

$$\begin{aligned}
\text{null}[\tilde{T}_0(M_p) - \lambda I] &= \text{null}[T_0(M_p; a_p) - \lambda I] + \text{null}[T_0(M_p; b_p) - \lambda I], \\
\text{def}[\tilde{T}_0(M_p) - \lambda I] &= \text{def}[T_0(M_p; a_p) - \lambda I] + \text{def}[T_0(M_p; b_p) - \lambda I],
\end{aligned}$$

and $R[\tilde{T}_0(M_p) - \lambda I]$ is closed if and only if $R[T_0(M_p; a_p) - \lambda I]$ and $R[T_0(M_p; b_p) - \lambda I]$ are both closed. These results imply in particular that,

$$\Pi[\tilde{T}_0(M_p)] = \Pi[T(M_p; a_p)] \cap \Pi[T(M_p; b_p)], \quad p = 1, \ldots, N.$$

We refer to [3, Section 3.10.14], [11] and [13] for more details.

Remark 3.1. If $S_p^{a_p}$ is a regularly solvable extension of $T_0(M_p; a_p)$ and $S_p^{b_p}$ is a regularly solvable extension of $T_0(M_p; b_p)$, then $S = S_p^{a_p} \oplus S_p^{b_p}$ is a regularly solvable extension of $\tilde{T}_0(M_p)$, $p = 1, \ldots, N$. We refer to [3, Section 3.10.4], [11] and [13] for more details.

Next, we state the following results; the proof is similar to that in [3, Section 3.10.4], [11] and [13].

Theorem 3.2. *$\tilde{T}_0(M_p) \subset T_0(M_p)$, $T(M_p) \subset T_0(M_p; a_p) \oplus T_0(M_p; b_p)$ and*

$$\dim\{D[T_0(M_p)]/D[\tilde{T}_0(M_p)]\} = n, \quad p = 1, \ldots, N.$$

If $\lambda \in \Pi[\tilde{T}_0(M_p)] \cap \Delta_3[T_0(M_p) - \lambda I]$, then

$$\operatorname{ind}[T_0(M_p) - \lambda I] = n - \operatorname{def}[T_0(M_p; a_p) - \lambda I] - \operatorname{def}[T_0(M_p; b_p) - \lambda I],$$

and in particular, if $\lambda \in \Pi[T_0(M_p)]$,

$$\operatorname{def}[T_0(M_p) - \lambda I] = \operatorname{def}[T_0(M_p; a_p) - \lambda I] + \operatorname{def}[T_0(M_p; b_p) - \lambda I] - n.$$

Remark 3.3. It can be shown that

$$\begin{aligned} D[\tilde{T}_0(M_p)] &= \big\{u\colon\ u \in D[T_0(M_p)] \text{ and } u^{[r-1]}(c_p) = 0, \quad r = 1, \ldots, n\big\}, \\ D[\tilde{T}_0(M_p^+)] &= \big\{v\colon\ v \in D[T_0(M_p^+)] \text{ and } v_+^{[r-1]}(c_p) = 0, \quad r = 1, \ldots, n\big\}, \\ &\qquad p = 1, \ldots, N; \end{aligned} \tag{3.16}$$

see [3, Section 3.10.4].

Let H be the direct sum,

$$H = \bigoplus_{p=1}^{N} H_p = \bigoplus_{p=1}^{N} L^2_{w_p}(a_p, b_p). \tag{3.17}$$

The elements of H will be denoted by $\tilde{f} = \{f_1, \ldots, f_N\}$ with $f_1 \in H_1$, …, $f_N \in H_N$.

Remark 3.4. When $I_i \cap I_j = \emptyset$, $i \neq j$, $i, j = 1, \ldots, N$, the direct sum space $\bigoplus_{p=1}^{N} L^2_{w_p}(a_p, b_p)$ can be naturally identified with the space $L^2_w\Big(\bigcup_{p=1}^{N} I_p\Big)$, where $w_p = w$ on I_p, $p = 1, \ldots, N$. This remark is of significance when $\bigcup_{p=1}^{N} I_p$ may be taken as a single interval, see [8] and [10].

We now establish by [3], [8] and [13] some further notations,

$$
\begin{aligned}
(3.18) \qquad D_0(M) &= \bigoplus_{p=1}^{N} D_0(M_p), \quad D(M) = \bigoplus_{p=1}^{N} D(M_p),\\
D_0(M^+) &= \bigoplus_{p=1}^{N} D_0(M_p^+), \quad D(M^+) = \bigoplus_{p=1}^{N} D(M_p^+),\\
T_0(M)f &:= \{T_0(M_1)f_1, \dots, T_0(M_N)f_N\};\\
&\quad f_1 \in D_0(M_1), \dots, f_N \in D_0(M_N),\\
T_0(M^+)g &:= \{T_0(M_1^+)g_1, \dots, T_0(M_N^+)g_N\};\\
&\quad g_1 \in D_0(M_1^+), \dots, g_N \in D_0(M_N^+).
\end{aligned}
$$

Also,

$$
\begin{aligned}
T(M)f &:= \{T(M_1)f_1, \dots, T(M_N)f_N\}, \quad f_1 \in D(M_1), \dots, f_N \in D(M_N),\\
T(M^+)g &:= \{T(M_1^+)g_1, \dots, T(M_N^+)g_N\}, \quad g_1 \in D(M_1^+), \dots, g_N \in D(M_N^+).
\end{aligned}
$$

We summarize a few additional properties of $T_0(M)$ in the form of a Lemma.

Lemma 3.5. *We have,*

(a) $[T_0(M)]^* = \bigoplus_{p=1}^{N} [T_0(M_p)]^* = \bigoplus_{p=1}^{N} [T(M_p^+)],$

$[T_0(M^+)]^* = \bigoplus_{p=1}^{N} [T_0(M_p^+)]^* = \bigoplus_{p=1}^{N} [T(M_p)].$

In particular,

$D[T_0(M)]^* = D[T(M^+)] = \bigoplus_{p=1}^{N} D[T(M_p^+)],$

$D[T_0(M^+)]^* = D[T(M)] = \bigoplus_{p=1}^{N} D[T(M_p)],$

(b) $\text{null}[T_0(M) - \lambda I] = \sum_{p=1}^{N} \text{null}[T_0(M_p) - \lambda I)],$

$\text{null}[T_0(M^+) - \overline{\lambda} I] = \sum_{p=1}^{N} \text{null}[T_0(M_p^+) - \overline{\lambda} I)].$

(c) *The deficiency indices of* $T_0(M)$ *are given by:*

$$
\begin{aligned}
\text{def}[T_0(M) - \lambda I] &= \sum_{p=1}^{N} \text{def}[T_0(M_p) - \lambda I)] \quad \textit{for all } \lambda \in \Pi[T_0(M_p)],\\
\text{def}[T_0(M^+) - \overline{\lambda} I] &= \sum_{p=1}^{N} \text{def}[T_0(M_p^+) - \overline{\lambda} I)] \quad \textit{for all } \lambda \in \Pi[T_0(M_p^+)].
\end{aligned}
$$

P r o o f. Part (a) follows immediately from the definition of $T_0(M)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follow immediately from the definitions. □

Lemma 3.6. *For* $\lambda \in \Pi[T_0(M), T_0(M^+)]$,

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I]$$

is constant and

$$0 \leqslant \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \leqslant 2nN.$$

In the problem with one singular end-point,

$$nN \leqslant \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \leqslant 2nN, \quad \textit{for all } \lambda \in \Pi[T_0(M), T_0(M^+)].$$

In the regular problem,

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = 2nN, \quad \textit{for all } \lambda \in \Pi[T_0(M), T_0(M^+)].$$

P r o o f. For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, we obtain from Theorem 3.2 and Lemma 3.5 that

$$\begin{aligned}
&\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \\
&\quad = \left\{ \sum_{p=1}^{N} \operatorname{def}[T_0(M_p; a_p) - \lambda I)] + \sum_{p=1}^{N} \operatorname{def}[T_0(M_p; b_p) - \lambda I)] - nN \right\} \\
&\qquad + \left\{ \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+; a_p) - \overline{\lambda} I)] + \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+; b_p) - \overline{\lambda} I)] - nN \right\} \\
&\quad = \left\{ \sum_{p=1}^{N} \operatorname{null}[T(M_p^+; a_p) - \overline{\lambda} I)] + \sum_{p=1}^{N} \operatorname{null}[T(M_p^+; b_p) - \overline{\lambda} I)] - nN \right\} \\
&\qquad + \left\{ \sum_{p=1}^{N} \operatorname{null}[T(M_p; a_p) - \lambda I)] + \sum_{p=1}^{N} \operatorname{null}[T(M_p; b_p) - \lambda I)] - nN \right\} \\
&\quad \leqslant 2(2nN - nN) = 2nN,
\end{aligned}$$

with equality in the regular problem. In the problem with one singular end-point, the proof is similar to that in [3], and we have

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \geqslant nN.$$

For the problem with two singular end-points, we have

$$\begin{aligned}
&\operatorname{def}[T_0(M)-\lambda I]+\operatorname{def}[T_0(M^+)-\overline{\lambda} I]\\
&\quad=\left\{\sum_{p=1}^{N}\operatorname{def}[T_0(M_p;a_p)-\lambda I)]+\sum_{p=1}^{N}\operatorname{def}[T_0(M_p^+;a_p)-\overline{\lambda} I)]\right\}\\
&\qquad+\left\{\sum_{p=1}^{N}\operatorname{def}[T_0(M_p;b_p)-\lambda I)]+\sum_{p=1}^{N}\operatorname{def}[T_0(M_p^+;b_p)-\overline{\lambda} I)]\right\}-2nN\\
&\quad\geqslant 2nN-2nN=0.
\end{aligned}$$

The Lemma is therefore proved, we refer to [3], [5], [13, Lemma 2.4] for more details. □

Lemma 3.7. *Let* $T_0(M)=\bigoplus_{p=1}^{N} T_0(M_p)$ *be a closed densely-defined operator on* H. *Then,*

$$\Pi[T_0(M)]=\bigcap_{p=1}^{N}\Pi[T_0(M_p)].$$

P r o o f. The proof follows from Lemma 3.5 and since $R[T_0(M)-\lambda I)]$ is closed if and only if $R[T_0(M_p)-\lambda I)]$, $p=1,\dots,N$, are closed. □

Lemma 3.8. *If* S_p, $p=1,\dots,N$, *are regularly solvable with respect to* $T_0(M_p)$ *and* $T_0(M_p^+)$, *then* $S=\bigoplus_{p=1}^{N} S_p$ *is regularly solvable with respect to* $T_0(M)$ *and* $T_0(M^+)$.

P r o o f. The proof follows from Lemmas 3.5 and 3.7. □

Remark 3.9. Let $S=\bigoplus_{j=1}^{N} S_j$ be an arbitrary closed operator on H. Since $\lambda\in\varrho(S)$ if, and only if, $\operatorname{null}(S-\lambda I)=\operatorname{def}(S-\lambda I)=0$ (see [2, Theorem 1.3.2]), we have $\varrho(S)=\bigcap_{j=1}^{N}\varrho(S_j)$. We therefore have

$$\sigma(S)=\bigcup_{j=1}^{N}\sigma(S_j),\quad \sigma_p(S)=\bigcup_{j=1}^{N}\sigma_p(S_j)\ \text{ and }\ \sigma_r(S)=\bigcup_{j=1}^{N}\sigma_r(S_j). \tag{3.19}$$

Also,

$$\sigma_{ek}(S)=\bigcup_{j=1}^{N}\sigma_{ek}(S_j),\quad k=2,3. \tag{3.20}$$

We refer to [3, Chapter 9] for more details.

Theorem 3.10. *Suppose $f \in L^1_{\text{loc}}(I_p)$ and suppose that the conditions* (3.1) *are satisfied. Then given any complex numbers $c_j \in \mathbb{C}$, $j = 0, 1, \ldots, n-1$ and $x_0 \in (a_p, b_p)$ there exists a unique solution of $M_p[\varphi_p] = wf$ in (a_p, b_p) which satisfies*

$$\varphi_p^{[j]}(x_0) = c_j \quad (j = 0, 1, \ldots, n-1;\ p = 1, \ldots, N).$$

P r o o f. See [1], [3] and [14, Part II, Theorem 16.2.2]. □

Theorem 3.11 (cf. [3] and [14, Theorem II.2.5]). *Let M_p be a regular quasi-differential expression of order n on the closed interval $[a_p, b_p]$. For $f \in L^2_w(a_p, b_p)$, the equation $M_p[\varphi_p] = wf$ has a solution $\varphi_p \in V(M_p)$ satisfying*

$$\varphi_p^{[j]}(a_p) = \varphi_p^{[j]}(b_p) = 0 \quad (j = 0, 1, \ldots, n-1,\ p = 1, \ldots, N),$$

if, and only if, f is orthogonal in $L^2_w(a_p, b_p)$ to the solution space of $M_p^+[\varphi_p] = 0$, i.e.,

$$R[T_0(M_p) - \lambda I] = N[T(M_p^+) - \overline{\lambda} I]^{\perp}, \quad p = 1, \ldots, N. \tag{3.21}$$

Corollary 3.12 (cf. [14, Corollary II.2.6]). *As a result from Theorem* 3.11, *we have that*

$$R[T_0(M_p) - \lambda I]^{\perp} = N[T(M_p^+) - \overline{\lambda} I], \quad p = 1, \ldots, N. \tag{3.22}$$

Lemma 3.13 (cf. [3, Lemma IX.9.1]). *If $I_p = [a_p, b_p]$, with $-\infty < a_p < b_p < \infty$, $p = 1, \ldots, N$, then for any $\lambda \in \mathbb{C}$, the operator $[T_0(M_p) - \lambda I]$, $p = 1, \ldots, N$, has closed range, zero nullity and deficiency n. Hence,*

$$\sigma_{ek}[T_0(M_p)] = \begin{cases} \emptyset & (k = 1, 2, 3), \\ \mathbb{C} & (k = 4, 5),\ p = 1, \ldots, N. \end{cases}$$

4. The spectra of operators in direct sum spaces

In this section we shall consider our interval to be $I = [a, b)$. We denote by $T(M)$ and $T_0(M)$ the maximal and minimal operators defined on the interval I. Also, we deal with the various components of the spectra of $T_0(M)$ and $T_0(M^+)$ as the direct sum of differential operators $T_0(M_p)$ and $T_0(M_p^+)$, $p = 1, \dots, N$.

Lemma 4.1. *Let* $T_0(M) = \bigoplus_{j=1}^{N} T_0(M_j)$ *and* $T_0(M^+) = \bigoplus_{j=1}^{N} T_0(M_j^+)$ *be a regular differential operators, then the point spectra* $\sigma_p[T_0(M)]$ *and* $\sigma_p[T_0(M^+)]$ *of* $T_0(M)$ *and* $T_0(M^+)$ *are empty.*

Proof. Let $\lambda \in \sigma_p[T_0(M_j)]$. Then there exists a non-zero element $\varphi_j \in D_0(M_j)$, $j = 1, \dots, N$, such that

$$[T_0(M_j) - \lambda I]\varphi_j = 0, \quad j = 1, \dots, N.$$

In particular, this gives that

$$\begin{aligned} &M_j[\varphi_j] = \lambda w \varphi_j, \\ &\varphi_j^{[r]}(a_j) = \varphi_j^{[r]}(b_j) = 0 \quad (r = 0, 1, \dots, n-1;\ j = 1, \dots, N). \end{aligned}$$

From Theorem 3.10, it follows that $\varphi_j = 0$ and hence $\sigma_p[T_0(M_j)] = \emptyset$, $j = 1, \dots, N$. Similarly,

$$\sigma_p[T_0(M_j^+)] = \emptyset, \quad i = 1, \dots, N.$$

Therefore, by (3.19), we have,

$$\sigma_p[T_0(M)] = \bigcup_{j=1}^{N} \sigma_p[T_0(M_j)] = \emptyset \ \text{ and } \ \sigma_p[T_0(M^+)] = \bigcup_{j=1}^{N} \sigma_p[T_0(M_j^+)] = \emptyset,$$

see Naimark [14, part II, Section 19]. □

Theorem 4.2. *Let* $T_0(M) = \bigoplus_{j=1}^{N} T_0(M_j)$ *and* $T_0(M^+) = \bigoplus_{j=1}^{N} T_0(M_j^+)$*, then*

(i) $\varrho[T_0(M)] = \varrho[T_0(M^+)] = \emptyset$,

(ii) $\sigma_c[T_0(M)] = \sigma_c[T_0(M^+)] = \emptyset$,

(iii) $\sigma[T_0(M)] = \sigma[T_0(M^+)] = \mathbb{C}$, *and*
$\sigma_r[T_0(M)] = \sigma_r[T_0(M^+)] = \mathbb{C}$.

P r o o f. (i) Let $\lambda \in \mathbb{C}$, since $R[T_0(M_j) - \lambda I]$, $j = 1, \ldots, N$ are proper closed subspaces of $L^2_w(a_j, b_j)$, then the resolvent sets $\varrho[T_0(M_j)]$ are empty and hence

$$\varrho[T_0(M)] = \bigcap_{j=1}^{N} \varrho[T_0(M_j)] = \emptyset.$$

Similarly

$$\varrho[T_0(M^+)] = \bigcap_{j=1}^{N} \varrho[T_0(M_j^+)] = \emptyset.$$

(ii) Since $R[T_0(M_j) - \lambda I]$, $j = 1, \ldots, N$, are closed for any $\lambda \in \mathbb{C}$, then the continuous spectra of $T_0(M_j)$ are the empty sets, i.e., $\sigma_c[T_0(M_j)] = \emptyset$, $j = 1, \ldots, N$. Hence,

$$\sigma_c[T_0(M)] = \bigcup_{j=1}^{N} \sigma_c[T_0(M)] = \emptyset.$$

Similarly,

$$\sigma_c[T_0(M^+)] = \bigcup_{j=1}^{N} \sigma_c[T_0(M^+)] = \emptyset.$$

(iii) From (i), (ii) and Lemma 3.5, it follows that,

$$\sigma[T_0(M)] = \bigcup_{j=1}^{N} \sigma[T_0(M_j)] = \mathbb{C} \text{ and } \sigma_r[T_0(M)] = \bigcup_{j=1}^{N} \sigma_r[T_0(M_j)] = \mathbb{C}.$$

Similarly,

$$\sigma[T_0(M^+)] = \bigcup_{j=1}^{N} \sigma[T_0(M_j^+)] = \mathbb{C}$$

and

$$\sigma_r[T_0(M^+)] = \bigcup_{j=1}^{N} \sigma_r[T_0(M_j^+)] = \mathbb{C}.$$

□

Corollary 4.3. *Let* $T_0(M) = \bigoplus_{j=1}^{N} T_0(M_j)$ *and* $T_0(M^+) = \bigoplus_{j=1}^{N} T_0(M_j^+)$, *then*

(i) $\sigma_c[T(M)] = \sigma_c[T(M^+)] = \emptyset$ *and* $\sigma_r[T(M)] = \sigma_r[T(M^+)] = \emptyset$,

(ii) $\sigma[T(M)] = \sigma[T(M^+)] = \mathbb{C}$ *and* $\sigma_p[T(M)] = \sigma_p[T(M^+)] = \mathbb{C}$,

(iii) $\varrho[T(M)] = \varrho[T(M^+)] = \emptyset$.

P r o o f. From Theorem 3.11 and since $T(M_j) = [T_0(M_j^+)]^*$, $j = 1,\dots,N$, it follows that $R[T_0(M_j) - \lambda I]$, $j = 1,\dots,N$, are closed and, hence $R[T(M) - \lambda I] = \bigoplus_{p=1}^{N} R[T(M_j) - \lambda I]$ is closed for every $\lambda \in \mathbb{C}$; see [3, Theorem I.3.7]. Also by Lemma 3.5, we have

$$\operatorname{null}[T(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = \sum_{j=1}^{N} \operatorname{def}[T_0(M_j^+) - \overline{\lambda} I] = nN,$$

and

$$\operatorname{def}[T(M) - \lambda I] = \operatorname{null}[T_0(M^+) - \overline{\lambda} I] = \sum_{j=1}^{N} \operatorname{null}[T_0(M_j^+) - \overline{\lambda} I] = 0.$$

(i) Since $R[T(M_j) - \lambda I]$ are closed and $\operatorname{def}[T(M_j) - \lambda I] = 0$, $j = 1,\dots,N$, then by Lemma 3.5 $R[T(M) - \lambda I] = H$. This yields that $\sigma_c[T(M)] = \sigma_r[T(M)] = \emptyset$.

Similarly,

$$\sigma_r[T(M^+)] = \sigma_r[T(M^+)] = \emptyset.$$

(ii) Since $\operatorname{null}[T(M) - \lambda I] = \sum_{j=1}^{N} \operatorname{null}[T(M_j) - \lambda I] = nN$ and

$$\operatorname{null}[T(M^+) - \overline{\lambda} I] = \sum_{j=1}^{N} \operatorname{null}[T(M_j^+) - \overline{\lambda} I] = nN \quad \text{for every} \lambda \in \mathbb{C},$$

then we have that

$$\sigma_p[T(M)] = \bigcup_{j=1}^{N} \sigma_p[T(M_j)] = \mathbb{C} \text{ and } \sigma_p[T(M^+)] = \bigcup_{j=1}^{N} \sigma_p[T(M_j^+)] = \mathbb{C}.$$

It also follows that

$$\sigma[T(M)] = \bigcup_{j=1}^{N} \sigma[T(M_j)] = \mathbb{C}, \quad \sigma[T(M^+)] = \bigcup_{j=1}^{N} \sigma[T(M_j^+)] = \mathbb{C},$$

and, hence

$$\varrho[T(M)] = \varrho[T(M^+)] = \emptyset.$$

□

5. The field of regularity of operators in direct sum spaces

We now obtain some results which in fact are a natural consequence of those in Section 4.

Theorem 5.1. *Let* $T_0(M) = \bigoplus_{p=1}^{N} T_0(M_p)$ *and* $T_0(M^+) = \bigoplus_{p=1}^{N} T_0(M_p^+)$, *then*

(i) $\Pi[T_0(M)] = \Pi[T_0(M^+)]. = \mathbb{C}$, *and for every* $\lambda \in \mathbb{C}$,

$$\operatorname{def}[T_0(M) - \lambda I] = \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = nN,$$

(ii) $\Pi[T(M)] = \Pi[T(M^+)]. = \emptyset$, *and for every* $\lambda \in \mathbb{C}$,

$$\operatorname{null}[T(M) - \lambda I] = \operatorname{null}[T(M^+) - \overline{\lambda} I] = nN.$$

P r o o f. (i) We have from Theorem 3.11 and Lemma 4.1 that, for every $\lambda \in \mathbb{C}$, $[T_0(M_p) - \lambda I]^{-1}$ exists and its domain $R[T_0(M_p) - \lambda I]$ is a closed subspace of $L_w^2(a_p, b_p)$, $p = 1, \ldots, N$. Hence, since $T_0(M_p)$, $p = 1, \ldots, N$, are closed operators, then $[T_0(M_p) - \lambda I]^{-1}$ are also closed and so, it follows from the Closed Graph Theorem that $[T_0(M_p) - \lambda I]^{-1}$, $p = 1, \ldots, N$ are bounded, and hence

$$\Pi[T_0(M)] = \bigcap_{p=1}^{N} \Pi[T_0(M_p)]. = \mathbb{C}.$$

From Theorem 3.11, $R[T_0(M_p) - \lambda I]^{\perp}$, $p = 1, \ldots, N$, are n-dimensional subspaces of $L_w^2(a_p, b_p)$. Thus, by Lemma 3.5,

$$\begin{aligned} \operatorname{def}[T_0(M) - \lambda I] &= \sum_{p=1}^{N} \operatorname{def}[T_0(M_p) - \lambda I] \\ &= \sum_{p=1}^{N} \dim R[T_0(M_p) - \lambda I]^{\perp} = nN, \quad \text{for every } \lambda \in \mathbb{C} \end{aligned}$$

Similarly,

$$\begin{aligned} \operatorname{def}[T_0(M^+) - \overline{\lambda} I] &= \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+) - \lambda I] \\ &= \sum_{p=1}^{N} \dim R[T_0(M_p^+) - \overline{\lambda} I]^{\perp} = nN, \quad \text{for every} \lambda \in \mathbb{C}. \end{aligned}$$

(ii) As $\Pi[T_0(M^+)] = \mathbb{C}$, we have, for every $\lambda \in \mathbb{C}$, that $T_0(M^+) - \overline{\lambda} I$ has closed range, and so, since $T(M) = [T_0(M^+)]^*$, then $T(M) - \lambda I = \sum_{p=1}^{N} [T(M_p) - \lambda I]$ has closed range; see [3, Theorem I.3.7]. Furthermore, from (i),

$$\text{null}[T(M) - \lambda I] = \text{def}[T_0(M^+) - \overline{\lambda} I] = \sum_{p=1}^{N} \text{def}[T_0(M_p^+) - \overline{\lambda} I] = nN.$$

Hence, $\lambda \notin \Pi[T(M)]$, and so part (ii) of the theorem follows. □

Corollary 5.2. *The operators $T_0(M)$, $T_0(M^+)$ form a compatible adjoint pair with $\Pi[T_0(M), T_0(M^+)] = \mathbb{C}$.*

Proof. From part (i) of Theorem 5.1 and Lemma 3.7, it follows that

$$\Pi[T_0(M), T_0(M^+)] = \bigcap_{p=1}^{N} \Pi[T_0(M_p), T_0(M_p^+)] = \mathbb{C}.$$

Using (3.15), the corollary follows. □

Theorem 5.3. *If for some $\lambda_0 \in \mathbb{C}$, there are nN linearly independent solutions of the equations*

$$M[\varphi] = \lambda_0 w \varphi \quad \textit{and} \quad M^+[\varphi] = \overline{\lambda}_0 w \varphi,$$

in $L_w^2(a, b)$, then all solutions of the equations

$$M[\varphi] = \lambda w \varphi \quad \textit{and} \quad M^+[\varphi] = \overline{\lambda} w \varphi,$$

are in $L_w^2(a, b)$ for all $\lambda \in \mathbb{C}$.

Proof. The proof follows from Lemma 3.5 and Lemma 3.6. We refer to [6] and [13, Lemmas 3.3, 3.4] for more details. □

From Corollary 5.2 and Theorem 5.3 we have the following Lemma.

Lemma 5.4. *If, for some $\lambda_0 \in \mathbb{C}$, there are nN linearly independent solutions of the equations*

$$M[\varphi] = \lambda_0 w \varphi \quad \textit{and} \quad M^+[\varphi] = \overline{\lambda}_0 w \varphi,$$

in $L_w^2(a, b)$, then $\lambda_0 \in \Pi[T_0(M), T_0(M^+)]$; see also [15, Theorem 2.1] and [17, Lemma 5.1].

Theorem 5.5. *Let* $T_0(M) = \bigoplus_{p=1}^{N} T_0(M_p)$ *and* $T_0(M^+) = \bigoplus_{p=1}^{N} T_0(M_p^+)$ *be the minimal operators, defined on the interval* $[a,b)$. *If* $\Pi[T_0(M), T_0(M^+)]$ *is empty, then*

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \neq 2nN.$$

In particular, if $\Pi[T_0(M), T_0(M^+)]$ *is empty and* $n = 1$, *then*

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = N.$$

Proof. If for some $\lambda_0 \in \mathbb{C}$, $\operatorname{def}[T_0(M) - \lambda I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p) - \lambda I] = nN$ and

$$\operatorname{def}[T_0(M^+) - \overline{\lambda} I] = \sum_{p=1}^{N} \operatorname{def}[T_0(M_p^+) - \overline{\lambda} I] = nN,$$

then,

$$M[u] = \lambda_0 wu \quad \text{and} \quad M^+[v] = \overline{\lambda}_0 wv$$

each have nN solutions in $L_w^2(a,b)$ (see [6]). Hence by Theorem 5.3, we have that all solutions of

$$M[u] = \lambda wu \quad \text{and} \quad M^+[v] = \overline{\lambda} wv$$

are in $L_w^2(a,b)$ for all $\lambda \in \mathbb{C}$, and hence, by Corollary 5.2, we have that $\lambda \in \Pi[T_0(M), T_0(M^+)]$. Thus, if $\Pi[T_0(M), T_0(M^+)]$ is empty, we cannot have

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = 2nN.$$

In particular, if $n = 1$, then by Lemma 3.6 we have that

$$N \leqslant \operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] \leqslant 2N,$$

so if $\Pi[T_0(M), T_0(M^+)]$ is empty, we have

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = N.$$

□

For a regularly solvable operator, we have the following general theorem.

Theorem 5.6. *Suppose for a regularly solvable extension S of the minimal operator $T_0(M) = \bigoplus_{p=1}^{N} T_0(M_p)$ that*

$$\operatorname{def}[T_0(M) - \lambda I] + \operatorname{def}[T_0(M^+) - \overline{\lambda} I] = K, \quad nN \leqslant K \leqslant 2nN,$$
$$\text{for all } \lambda \in \Pi[T_0(M), T_0(M^+)].$$

Then,

$$\operatorname{null}[T(M) - \lambda I] + \operatorname{null}[T(M^+) - \overline{\lambda} I] \leqslant K, \quad \text{for all } \lambda \in \mathbb{C}.$$

If $\Pi[T_0(M), T_0(M^+)]$ is empty, then

$$\operatorname{null}[T(M) - \lambda I] + \operatorname{null}[T(M^+) - \overline{\lambda} I] < K.$$

Proof. Let $\operatorname{def}[T_0(M_p) - \lambda I] = r_p$, $\operatorname{def}[T_0(M_p^+) - \overline{\lambda} I] = s_p$, $p = 1, \ldots, N$, be such that

$$\operatorname{def}[T_0(M_p) - \lambda I] + \operatorname{def}[T_0(M_p^+) - \overline{\lambda} I] = r_p + s_p, \quad n \leqslant r_p + s_p \leqslant 2n,$$

for all $\lambda \in \Pi[T_0(M_p), T_0(M_p^+)]$, $p = 1, \ldots, N$. Then, for any closed extension S_p of $T_0(M_p)$ which is regularly solvable with respect to $T_0(M_p)$ and $T_0(M_p^+)$, we have from [3, Theorem III.3.5] that

$$\begin{aligned} \dim\{D(S_p)/D_0(M_p)\} &= \operatorname{def}[T_0(M_p) - \lambda I] = r_p, \quad p = 1, \ldots, N, \\ \dim\{D(S_p^*)/D_0(M_p^+)\} &= \operatorname{def}[T_0(M_p^+) - \overline{\lambda} I] = s_p, \quad p = 1, \ldots, N. \end{aligned}$$

Hence S_p and S_p^* are finite dimensional extensions of $T_0(M_p)$ and $T_0(M_p^+)$, respectively. Thus, from [3, Corollary IX.4.2], we get

$$\sigma_{ek}[T_0(M_p)] = \sigma_{ek}(S_p) \quad (k = 1, 2, 3;\ p = 1, \ldots, N). \tag{5.1}$$

Since $T_0(M_p) - \lambda I$ has closed range, zero nullity and deficiency r_p (see Lemma 3.13), then for any $\lambda \in \mathbb{C}$, we have that

$$\Pi[T_0(M_p)] \cap \sigma_{ek}[T_0(M_p)] = \emptyset \quad (k = 1, 2, 3;\ p = 1, \ldots, N).$$

Therefore,

$$\Delta_{ek}[T_0(M_p)] = \Delta_{ek}(S_p) = \mathbb{C} \quad (k = 1, 2, 3;\ p = 1, \ldots, N).$$

Similarly,

$$\Delta_{ek}[T_0(M_p^+)] = \Delta_{ek}(S_p^*) = \mathbb{C} \quad (k = 1, 2, 3;\ p = 1, \ldots, N).$$

Furthermore, the equations

$$M_p[\varphi_p] = \lambda_0 w\varphi_p \quad \text{and} \quad M_p^+[\varphi_p] = \overline{\lambda}_0 w\varphi_p, \quad p = 1, \ldots, N,$$

have at most r_p and s_p linearly independent solutions for $\lambda_0 \in \mathbb{C}$, respectively. Hence,

$$\begin{aligned} &\text{null}[T(M) - \lambda I] + \text{null}[T(M^+) - \overline{\lambda} I] \\ &= \sum_{p=1}^{N} \text{null}[T(M_p) - \lambda I] + \sum_{p=1}^{N} \text{null}[T(M_p^+) - \overline{\lambda} I] \\ &= \sum_{p=1}^{N} (r_p + s_p) \leqslant K, \quad nN \leqslant K \leqslant 2nN \ \text{ for all } \lambda \in \mathbb{C}. \end{aligned}$$

But for any $\lambda_0 \notin \Pi[T_0(M_p), T_0(M_p^+)]$, either $\lambda_0 \notin \Pi[T_0(M_p)]$ or $\overline{\lambda}_0 \notin \Pi[T_0(M_p^+)]$. If $\lambda_0 \notin \Pi[T_0(M_p)]$, then either λ_0 is an eigenvalue of $T_0(M_p)$ or $R[T_0(M_p) - \lambda I]$, $p = 1, \ldots, N$, are not closed. Similarly for $\overline{\lambda_0} \notin \Pi[T_0(M_p^+)]$. But $T_0(M_p)$ and $T_0(M_p^+)$ have no eigenvalues; thus if $\lambda_0 \notin \Pi[T_0(M_p), T_0(M_p^+)]$, then $R[T_0(M_p) - \lambda I]$ and $R[T_0(M_p^+) - \overline{\lambda} I]$, $p = 1, \ldots, N$, are both not closed, and so we can not have

$$\begin{aligned} &\text{null}[T(M) - \lambda I] + \text{null}[T(M^+) - \overline{\lambda} I] \\ &\qquad = \sum_{p=1}^{N} \text{null}[T(M_p) - \lambda I] + \sum_{p=1}^{N} \text{null}[T(M_p^+) - \overline{\lambda} I] = K. \end{aligned}$$

Hence,

$$\text{null}[T(M) - \lambda I] + \text{null}[T(M^+) - \overline{\lambda} I] < K, \quad nN \leqslant K \leqslant 2nN,$$

for all $\lambda \notin \Pi[T_0(M), T_0(M^+)] = \bigcap_{p=1}^{N} \Pi[T_0(M_p), T_0(M_p^+)]$. □

Remark 5.7. It remains an open question as to how many of the solutions of the equations:

$$M[u] = \lambda w u \quad \text{and} \quad M^+[v] = \overline{\lambda} w v,$$

may be in $L_w^2(a, b)$ for any $\lambda \in \mathbb{C}$, when $\Pi[T_0(M), T_0(M^+)]$ is empty, except that we know from the above that not all of them are in $L_w^2(a, b)$. We refer to [2], [6], [15], [17] for more details.

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