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STRICTLY CYCLIC ALGEBRA OF OPERATORS ACTING ON BANACH SPACES $H^p(\beta)$

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Dedicated to the memory of Karim Seddighi

Abstract. Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers and $1 \leqslant p < \infty$. We consider the space $H^p(\beta)$ of all power series $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n) z^n$ such that $\sum\limits_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty$. We investigate strict cyclicity of $H_p^{\infty}(\beta)$, the weakly closed algebra generated by the operator of multiplication by z acting on $H^p(\beta)$, and determine the maximal ideal space, the dual space and the reflexivity of the algebra $H_p^{\infty}(\beta)$. We also give a necessary condition for a composition operator to be bounded on $H^p(\beta)$ when $H_p^{\infty}(\beta)$ is strictly cyclic.

Keywords: the Banach space of formal power series associated with a sequence β, bounded point evaluation, strictly cyclic maximal ideal space, Schatten p-class, reflexive algebra, semisimple algebra, composition operator

MSC 2000: 47B37, 47A25

Introduction

First, in the following we generalize the definitions from [4].

Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 \leqslant p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_{\beta}^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^p(\beta)$ denote the space of all such formal power seires. These are reflexive Banach spaces with the norm $\|\cdot\|_\beta$ ([3]) and the dual of $H^p(\beta)$ is $H^q\big(\beta^{p/q}\big)$ where $1/p+1/q=1$ and $\beta^{p/q}=\big\{\beta(n)^{p/q}\big\}_n$ ([5]). Also if $g(z)=\sum\limits_{n=0}^{\infty}\hat{g}(n)z^n\in H^q\big(\beta^{p/q}\big)$, then $\|g\|^q=\sum\limits_{n=0}^{\infty}|\hat{g}(n)|^q\beta(n)^p$. The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p=2$ and respectively $\beta(n)=1$, $\beta(n)=(n+1)^{-1/2}$ and $\beta(n)=(n+1)^{1/2}$. If $\lim\limits_n\beta(n+1)/\beta(n)=1$ or $\liminf\limits_n\beta(n)^{1/n}=1$, then $H^p(\beta)$ consists of functions analytic on the open unit disc U. It is convenient and helpful to introduce the notation $\langle f,g\rangle$ to stand for $g(f)$ where $f\in H^p(\beta)$ and $g\in H^p(\beta)^*$. Note that $\langle f,g\rangle=\sum\limits_{n=0}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$.

Let $\hat{f}_k(n)=\delta_k(n)$. So $f_k(z)=z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\|=\beta(k)$. Clearly M_z, the operator of multiplication by z on $H^p(\beta)$, shifts the basis $\{f_k\}_k$.

A function φ in $H^p(\beta)$ that maps the unit disc U into itself induces a composition operator C_φ on $H^p(\beta)$ defined by $C_\varphi f=f\circ\varphi$.

We denote the set of multipliers $\{\varphi\in H^p(\beta)\colon \varphi H^p(\beta)\subseteq H^p(\beta)\}$ by $H_p^\infty(\beta)$ and the linear transformation of multiplication by φ on $H^p(\beta)$ by M_φ. The space $H_p^\infty(\beta)$ is a commutative Banach algebra under the norm $\|\varphi\|_\infty=\|M_\varphi\|$ and also $H_p^\infty(\beta)$ is equal to the weakly closed algebra generated by M_z.

Let $\mathscr{X}$ be a Banach space. We denote by $\mathscr{B}(\mathscr{X})$ the set of all bounded operators on $\mathscr{X}$. A subalgebra $\mathscr{A}$ of $\mathscr{B}(\mathscr{X})$ is cyclic if $\mathscr{A}x_0$ is dense in $\mathscr{X}$ for some x_0 in $\mathscr{X}$. $\mathscr{A}$ is strictly cyclic if $\mathscr{A}x_0=\mathscr{X}$. The vector x_0 is called cyclic for $\mathscr{A}$ in the former case and strictly cyclic in the latter case. We say that M_z is strictly cyclic on $H^p(\beta)$ if $H_p^\infty(\beta)$ is strictly cyclic. In this case f_0 ($f_0=1$) is a strictly cyclic vector and $H_p^\infty(\beta)=H^p(\beta)$. This implies that M_z is strictly cyclic if and only if $fg\in H^p(\beta)$ for all f and g in $H^p(\beta)$.

Remember that a complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. The functional of evaluation of the j-th derivative at λ is denoted by $e_\lambda^{(j)}$.

If Ω is a bounded domain in the complex domain $\mathbb{C}$, then by $H(\Omega)$ and $H^\infty(\Omega)$ we mean respectively the set of all analytic functions and the set of all bounded analytic functions on Ω. By $\|\cdot\|_\Omega$ we denote the supremum norm on Ω.

Main results

In this section we investigate strict cyclicity of the operator M_z and characterize the maximal ideal space and the dual space of $H_p^\infty(\beta)$ and study the reflexivity of $H_p^\infty(\beta)$. Also a sufficient condition for a composition operator to be bounded on $H^p(\beta)$ will be given.

Note that the spectral radius of M_z is denoted by $r(M_z)$.

Lemma 1. *If M_z is strictly cyclic on $H^p(\beta)$, then* $\liminf_n \beta(n)^{1/n} = r(M_z)$.

Proof. It follows from the fact that $\Omega_1 = \left\{z\colon |z| < \liminf_n \beta(n)^{1/n}\right\}$ is the largest open disc such that $H^p(\beta) \subset H(\Omega_1)$ and $\Omega_2 = \{z\colon |z| < r(M_z)\}$ is the largest open disc such that $H_p^\infty(\beta) \subset H^\infty(\Omega_2)$ (see Theorems 1 and 3 in [6]). □

Proposition 2. *If M_z is strictly cyclic on $H^p(\beta)$, then for all $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ in $H^p(\beta)$, $\|f\|_p \leqslant \|f\|_\infty \leqslant c\|f\|_p$ and $\sum_{n=0}^{\infty} |\hat{f}(n)|(r(M_z))^n \leqslant c\|f\|_p$ for some $c > 0$.*

Proof. Since M_z is strictly cyclic, $H^p(\beta) = H_p^\infty(\beta) f_0 = H_p^\infty(\beta)$. Let $\varrho\colon H_p^\infty(\beta) \longrightarrow H^p(\beta)$ be the map $\varrho(M_f) = M_f f_0$. Then clearly $\|\varrho\| \leqslant \|f_0\| = 1$ and so ϱ is continuous and $\|f\|_p \leqslant \|f\|_\infty$. Since ϱ is bijective, by the Inverse Mapping Theorem ϱ^{-1} is bounded and so for some constant $c > 0$, $\|f\|_\infty \leqslant c\|f\|_p$. Thus indeed $\|f\|_p \leqslant \|f\|_\infty \leqslant c\|f\|_p$. Now let $|\lambda_0| = r(M_z)$. We show that the functional of evaluation at λ_0 is bounded. Let s be a polynomial. From theorem (3) in [6], $|s(\lambda_0)| \leqslant \|M_s\|$. But $\|M_s\| = \|s\|_\infty$ and as we saw $\|s\|_\infty \leqslant c\|s\|_p$. Thus for all polynomials s, $|s(\lambda_0)| \leqslant c\|s\|_p$. Since polynomials are dense in $H^p(\beta)$, the point evaluation at λ_0, e_{λ_0}, is bounded and

$$\|e_{\lambda_0}\|^q = \sum_{n=0}^{\infty} \frac{|\lambda_0|^{nq}}{\beta(n)^q} = \sum_{n=0}^{\infty} \frac{(r(M_z))^{nq}}{\beta(n)^q} < \infty,$$

where $1/p + 1/q = 1$ ([5]). Now by the Hölder inequality we have

$$\begin{aligned}\left|\sum_{n=0}^{\infty} \hat{f}(n) r(M_z)^n\right| &\leqslant \left(\sum_{n=0}^{\infty} |\hat{f}(n)|\beta(n)^p\right)^{1/p} \left(\sum_{n=0}^{\infty} \frac{(r(M_z))^{nq}}{\beta(n)^q}\right)^{1/q} \\ &= \|f\|_p \|e_{\lambda_0}\|.\end{aligned}$$

This completes the proof. □

Theorem 3. *Suppose that M_z is strictly cyclic. Then a linear functional L on $H_p^\infty(\beta)$ is multiplicative if and only if L is the functional of point evaluation at some point of $\{z \in \mathbb{C}\colon |z| \leqslant r(M_z)\}$.*

Proof. Let L be multiplicative and put $L(f_1) = \lambda_1$ $(f_1(z) = z^1)$, hence $L(f_n) = \lambda_1^n$ for all n and so $L(p) = p(\lambda_1)$ for all polynomials p. Since L is bounded and the polynomials are dense in $H^p(\beta)$, if follows that λ_1 is a bounded point evaluation on $H^p(\beta)$ and indeed $L = e_{\lambda_1}$.

Conversely, let $\lambda \in \{z\colon\ |z| \leqslant r(M_z)\}$. Then

$$\sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q} \leqslant \sum_{n=0}^{\infty} \frac{(r(M_z))^{nq}}{\beta(n)^q} < \infty.$$

So the functionals of point evaluation at λ, e_λ, are bounded for all λ in $\{z\colon\ |z| \leqslant r(M_z)\}$ and $e_\lambda(fg) = (fg)(\lambda) = e_\lambda(f)e_\lambda(g)$. Thus e_λ is multiplicative. □

In the following we denote the spectrum of φ by $\sigma(\varphi)$. Recall that the maximal ideal space of $H_p^\infty(\beta)$ is the set of all nonzero homomorphisms of $H_p^\infty(\beta) \longrightarrow \mathbb{C}$ with w^* topology.

Corollary 4. *Suppose that M_z is strictly cyclic. Then the maximal ideal space of $H_p^\infty(\beta)$ is the set $\{e_\lambda\colon\ \lambda \in \overline{\Omega}\}$ where $\Omega = \{z\colon\ |z| < r(M_z)\}$. Also for $\varphi \in H_p^\infty(\beta)$, $\sigma(\varphi) = \varphi(\overline{\Omega})$ and so φ is a cyclic vector for M_z if and only if φ never vanishes on $\overline{\Omega}$.*

Proof. The first part follows immediately from the above theorem and for the second part, by Theorem 8.6 of Chapter VII in [1], we have

$$\begin{aligned}\sigma(\varphi) &= \{h(\varphi)\colon\ h \text{ is a nonzero homomorphism}\}\\ &= \{e_\lambda(\varphi)\colon\ \lambda \in \overline{\Omega}\} = \{\varphi(\lambda)\colon\ \lambda \in \overline{\Omega}\} = \varphi(\overline{\Omega}).\end{aligned}$$

Finally we note that φ is cyclic if and only if it is invertible in $H_p^\infty(\beta)$. This completes the proof. □

Theorem 5. *Let $\liminf_n \beta(n)^{1/n} = 1$, M_z be strictly cyclic on $H^p(\beta)$ and the function φ in $H^p(\beta)$ be such that $\|\varphi\|_U < 1$. Then the composition operator on $H^p(\beta)$ induced by φ, C_φ, is bounded.*

Proof. By the above corollary the spectrum of each element φ of the Banach algebra $H_p^\infty(\beta)$ is equal to $\varphi(\overline{\Omega})$ where $\overline{\Omega} = \bigl\{z\colon\ |z| \leqslant \liminf_n \beta(n)^{1/n} = 1\bigr\} = \overline{U}$. But the spectrum of φ as an element of $H_p^\infty(\beta)$ is the same as the spectrum of the multiplication operator M_φ on $H^p(\beta)$, so we have

$$\lim_n \|M_\varphi^n\|^{1/n} = r(M_\varphi) = \sup\{|\lambda|\colon\ \lambda \in \varphi(\overline{\Omega})\} = \|\varphi\|_U < 1.$$

Now let $f = \sum_{n=0}^{\infty} \hat{f}(n) f_n \in H^p(\beta)$, then $C_\varphi f = f \circ \varphi = \sum_{n=0}^{\infty} \hat{f}(n)\varphi^n$ and since $\|\varphi^n\|_\beta = \|M_{\varphi^n} f_0\| \leqslant \|M_{\varphi^n}\|$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} |\hat{f}(n)|\, \|\varphi^n\|_\beta &\leqslant \sum_{n=0}^{\infty} |\hat{f}(n)|\, \|M_\varphi^n\| \leqslant \Bigl(\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p\Bigr)^{1/p} \Bigl(\sum_{n=0}^{\infty} \frac{\|M_\varphi^n\|^q}{\beta(n)^q}\Bigr)^{1/q}\\ &= \|f\|_p \Bigl(\sum_{n=0}^{\infty} \frac{\|M_\varphi^n\|^q}{\beta(n)^q}\Bigr)^{1/q}.\end{aligned}$$

But by the Root Test $\sum\limits_{n=0}^{\infty} \|M_\varphi^n\|^q/\beta(n)^q$ converges and so $C_\varphi f \in H^p(\beta)$ for all f in $H^p(\beta)$ and C_φ is bounded. $\square$

For the definition of the Schatten p-class for $p > 0$ see [2].

Corollary 6. *If φ satisfies the conditions of the Theorem, then the operator C_φ is in every Schatten p-class of $H^p(\beta)$.*

P r o o f. Let $\|\varphi\|_U < \alpha < 1$ and put $h = \alpha f_1$. Then h belongs to every $H^p(\beta)$ space. Also $C_h f_n = h^n = \alpha^n f_n$ and $\{\alpha^n\} \in \ell^p$ for all p. Thus C_h is in every Schatten p-class of $H^p(\beta)$ which we denote by $s_p(\beta)$. Let $g = \alpha^{-1}\varphi$. Then g belongs to the given space $H^p(\beta)$ and $\|g\|_U = \|\varphi\|_U/\alpha < 1$. So by the above theorem C_g is bounded on $H^p(\beta)$ and since $\varphi = h \circ g$, we have $C_\varphi = C_g C_h$. But C_g is bounded and $C_h \in s_p(\beta)$ for all p. Thus indeed C_φ is in $s_p(\beta)$ for all p. $\square$

Lemma 7. *M_z is strictly cyclic on $H^p(\beta)$ if and only if the dual space of $H_p^\infty(\beta)$ is exactly $\bigl\{L_g\colon\ g \in H^q\bigl(\beta^{\frac{p}{q}}\bigr),\ L_g(f) = \langle f, g\rangle\bigr\}$.*

P r o o f. This follows from the fact that $(H^p(\beta))^* = H^q\bigl(\beta^{p/q}\bigr)$. $\square$

If M_z is strictly cyclic, then $H_p^\infty(\beta) f_0 = H^p(\beta)$ and so for all x in $H^p(\beta)$ there exists $f_x \in H_p^\infty(\beta)$ such that $M_{f_x} f_0 = x$ (in fact, $f_x = x$). So in this case we can consider $H_p^\infty(\beta)$ as the set $\{M_f\colon\ f \in H^p(\beta)\}$ and the linear functional L_g that is defined in the lemma as $L_g(M_f) = \langle f, g\rangle$.

Lemma 8. *Let M_z be strictly cyclic on $H^p(\beta)$. Then there is a g in $H^q\bigl(\beta^{p/q}\bigr)$ such that $M_f^* g = \langle f, g\rangle\, g$ for every f in $H^p(\beta)$.*

P r o o f. Since $H^p(\beta) = H_p^\infty(\beta)$ and $H_p^\infty(\beta)$ is a commutative Banach algebra with identity, there is a nonzero multiplicative linear functional F on $H_p^\infty(\beta)$. So $F \in H_p^\infty(\beta)^* = \bigl\{L_g\colon\ g \in H^q\bigl(\beta^{p/q}\bigr)\bigr\}$. Thus there is a g in $H^q\bigl(\beta^{p/q}\bigr)$ such that $F = L_g$. Now for f and h in $H^p(\beta)$ we have

$$\begin{aligned}
\bigl\langle h, M_f^* g\bigr\rangle &= \langle M_f h, g\rangle = \langle hf, g\rangle = L_g(M_{hf}) \\
&= L_g(M_h M_f) = L_g(M_h) L_g(M_f) \\
&= \langle h, g\rangle \langle f, g\rangle = \langle h, \langle f, g\rangle\, g\rangle
\end{aligned}$$

for all h in $H^p(\beta)$. So $M_f^* g = \langle f, g\rangle\, g$ for all f in $H^p(\beta)$. $\square$

For $g \in H^q\bigl(\beta^{p/q}\bigr)$ we denote by $[g]$ the closed linear subspace of $H^q\bigl(\beta^{p/q}\bigr)$ generated by g.

Corollary 9. *If M_z is strictly cyclic and L_g is multiplicative where $g \in H^q\big(\beta^{p/q}\big)$, then $M_f^*[g] \subseteq [g]$ for every f in $H^p(\beta)$.*

Proof. This is an immediate consequence of the Lemma. $\square$

Remember that a subaglebra $\mathscr{A}$ of bounded operators on a Banach space is called reflexive if $\operatorname{Lat} \mathscr{A} \subseteq \operatorname{Lat} B$ implies that $B \in \mathscr{A}$. Also a commutative Banach algebra $\mathscr{A}$ is semisimple if for every f in $\mathscr{A}$, there is a multiplicative linear functional L on $\mathscr{A}$ such that $L(f) \neq 0$.

Theorem 10. *If M_z is strictly cyclic and $H_p^\infty(\beta)$ is semisimple, then $H_p^\infty(\beta)$ is reflexive.*

Proof. It is easy to see that the algebra $H_p^\infty(\beta)$ is reflexive if and only if the algebra $\mathscr{B} = \{M_f^* \colon f \in H^p(\beta)\}$ is reflexive. We show that $\mathscr{B}$ is reflexive. Put

$$\mathscr{N} = \big\{g \in H^q\big(\beta^{p/q}\big) \colon L_g \text{ is multiplicative}\big\}.$$

Since $H_p^\infty(\beta)$ is semisimple, $\mathscr{N}$ spans $H^q\big(\beta^{p/q}\big)$. Now let $g \in \mathscr{N}$, $A \in B\big(H^q\big(\beta^{p/q}\big)\big)$ and $\operatorname{Lat} \mathscr{B} \subseteq \operatorname{Lat} A$. Since L_g is multiplicative, by Corollary 9, $M_f^*[g] \subseteq [g]$ for every f in $H^p(\beta)$. Thus $A[g] \subseteq [g]$ for all g in $\mathscr{N}$. Therefore $Ag = \lambda g$ and $M_f^* g = \lambda_f g$ and so

$$AM_f^* g = A(\lambda_f g) = \lambda_f \lambda_g = \lambda M_f^* g = M_f^*(\lambda g) = M_f^* A g.$$

Since g is arbitrary, thus $AM_f^* = M_f^* A$. But since M_z is strictly cyclic, $H_p^\infty(\beta)$ is an abelian Banach algebra with identity which is maximal and so $\mathscr{B}$ is also a maximal abelian algebra. Thus $A \in \mathscr{B}$. This says that $\mathscr{B}$ and so $H_p^\infty(\beta)$ is reflexive. $\square$

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