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A GENERALIZATION OF LERCH'S FORMULA

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Abstract. We give higher-power generalizations of the classical Lerch formula for the gamma function.

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1. Results

About a century ago, Matyas Lerch [6, p. 13] proved a famous formula saying

$$(1) \qquad \prod_{n=0}^{\infty}(n+x) = \frac{\sqrt{2\pi}}{\Gamma(x)},$$

where $\Gamma(x)$ is the gamma function. Here, we use the notation due to Deninger [2] of the regularized product

$$\prod_{n=0}^{\infty} a_n = \exp\left(-\frac{\mathrm{d}}{\mathrm{d}s} \sum_{n=0}^{\infty} a_n^{-s}\bigg|_{s=0}\right)$$

where $\sum_{n=0}^{\infty} a_n^{-s}$ is analytically continued to a holomorphic function at $s = 0$ (see Manin [7] for a survey). In the case above,

$$\sum_{n=0}^{\infty}(n+x)^{-s}$$

is nothing but the Hurwitz zeta function $\zeta(s, x)$, and the Lerch formula was originally written as

$$\zeta'(s, x)|_{s=0} = \log \frac{\Gamma(x)}{\sqrt{2\pi}},$$

where the differentiation is with respect to the first variable s.

On the other hand, it is not so well-known that Lerch [7, p. 60] also proved that

$$(2) \qquad \prod_{n=0}^{\infty} \left((n+x)^2 + y^2\right) = \frac{2\pi}{\Gamma(x+iy)\Gamma(x-iy)}.$$

This is remarkable from the following two points of view. Firstly, let $x > 0$ and $y = 0$ in (2), then we get (1) again, since

$$\prod_{n=0}^{\infty} (n+x)^2 = \left(\prod_{n=0}^{\infty} (n+x)\right)^2.$$

Secondly, it gives an example of the non-trivial equality

$$(3) \qquad \prod_{n=0}^{\infty} (a_n b_n) = \left(\prod_{n=0}^{\infty} a_n\right)\left(\prod_{n=0}^{\infty} b_n\right)$$

by setting $a_n = n + x + iy$ and $b_n = n + x - iy$. It should be noted that there exist also counter examples to (3). Moreover, it is worth noting that in the study of the higher Selberg zeta function $z_\Gamma(s)$ which are studied in [4], the formula (2) becomes essential to obtain the analytic properties, especially the determinant expression of $z_\Gamma(s)$.

In this paper we show that the formulas (1) and (2) can be generalized to the higher power case in the following form:

Theorem. *The following formulas hold:*

(i) $\qquad \displaystyle\prod_{n=0}^{\infty} \left((n+x)^m - y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=1} \Gamma(x - \zeta y)} \quad (m = 1, 2, \ldots).$

(ii) $\qquad \displaystyle\prod_{n=0}^{\infty} \left((n+x)^m + y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=1} \Gamma(x + \zeta y)} \quad$ (*for odd $m > 0$*).

(iii) $\qquad \displaystyle\prod_{n=0}^{\infty} \left((n+x)^m + y^m\right) = \frac{(2\pi)^{m/2}}{\prod_{\zeta^m=-1} \Gamma(x + \zeta y)} \quad$ (*for even $m > 0$*).

Corollary. *We have*

(i)
$$\prod_{n=0}^{\infty} (n^2 + a^2) = a(e^{\pi a} - e^{-\pi a}) = 2a \sinh(\pi a).$$

$$\prod_{n=0}^{\infty} (n^4 + a^4) = a^2 \left(e^{\sqrt{2}\pi a} + e^{-\sqrt{2}\pi a} - 2\cos\left(\sqrt{2}\pi a\right) \right)$$

(ii)
$$= 2a^2 \left(\cosh\left(\sqrt{2}\pi a\right) - \cos\left(\sqrt{2}\pi a\right) \right).$$

Using the theorem and the corollary above one can get a number of interesting explicit identities as follows.

Examples.

(1)
$$\prod_{n=0}^{\infty} (n^2 + 1) = e^{\pi} - e^{-\pi}.$$

(2)
$$\prod_{n=0}^{\infty} (n^2 - n + 1) = e^{\frac{\sqrt{3}}{2}\pi} + e^{-\frac{\sqrt{3}}{2}\pi}.$$

(3)
$$\prod_{n=0}^{\infty} (n^3 + 1) = \sqrt{2\pi} \left(e^{\frac{\sqrt{3}}{2}\pi} + e^{-\frac{\sqrt{3}}{2}\pi} \right).$$

(4)
$$\prod_{n=0}^{\infty} (n^4 + 1) = 2 \left(\cosh\left(\sqrt{2}\pi\right) - \cos\left(\sqrt{2}\pi\right) \right).$$

2. PROOF OF THE THEOREM

In this section, we give a proof of the theorem.

P r o o f of Theorem. We prove (i). The formulas (ii) and (iii) follow from (i) easily, so we omit them. We assume $m \geqslant 2$ from now on, since the case $m = 1$ is nothing but the original result of Lerch. Let

$$\varphi_m(s, x, y) = \sum_{n=0}^{\infty} ((n + x)^m - y^m)^{-s},$$

then by Bochner [1], $\varphi_m(s, x, y)$ is analytically continued to all $s \in \mathbb{C}$ as a meromorphic function in s, and it is holomorphic around $s = 0$ and $s = 1$. Hence we have expressions

$$\varphi_m(s, x, y) = a_0(x, y) + a_1(x, y)s + \ldots \quad \text{(around } s = 0)$$

and

$$\varphi_m(s, x, y) = b_0(x, y) + b_1(x, y)(s - 1) + \ldots \quad \text{(around } s = 1\text{)}.$$

Looking now at the relation

$$\frac{\partial}{\partial y}\varphi_m(s, x, y) = msy^{m-1}\varphi_m(s + 1, x, y),$$

we see that

$$\frac{\partial}{\partial y}a_1(x, y) = my^{m-1}b_0(x, y)$$

$$= my^{m-1}\varphi_m(1, x, y)$$

$$= \sum_{n=0}^{\infty} \frac{my^{m-1}}{(n + x)^m - y^m}.$$

It is easy to verify the following decomposition:

$$\frac{my^{m-1}}{(n + x)^m - y^m} = \sum_{\zeta^m=1} \frac{\zeta}{(n + x) - \zeta y}.$$

Hence we have

$$\frac{\partial}{\partial y}a_1(x, y) = \sum_{\zeta^m=1} \zeta\left[\frac{1}{x - \zeta y} + \sum_{n=1}^{\infty}\left(\frac{1}{n + (x - \zeta y)} - \frac{1}{n}\right)\right],$$

since

$$\sum_{\zeta^m=1} \zeta = 0.$$

Moreover, let

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right) = 0.5772156649\ldots$$

be the Euler constant, then we of course have

$$\frac{\partial}{\partial y}a_1(x, y) = \sum_{\zeta^m=1} \zeta\left[\frac{1}{x - \zeta y} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n + (x - \zeta y)} - \frac{1}{n}\right)\right],$$

again by $\sum_{\zeta^m=1} \zeta = 0$.

On the other hand, by the infinite product expression for the gamma function

$$\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(1 + \frac{x}{n}\right)e^{-\frac{x}{n}},$$

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we see that

$$-\frac{\Gamma'}{\Gamma}(x) = \frac{1}{x} + \gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n+x} - \frac{1}{n} \right).$$

Thus we obtain

$$\frac{\partial}{\partial y} a_1(x,y) = \sum_{\zeta^m=1} (-\zeta) \cdot \frac{\Gamma'}{\Gamma}(x - \zeta y)$$

$$= \frac{\partial}{\partial y} \log\left(\prod_{\zeta^m=1} \Gamma(x - \zeta y) \right).$$

Hence

$$a_1(x,y) = \log\left(C \prod_{\zeta^m=1} \Gamma(x - \zeta y) \right)$$

for a constant C. Since

$$\prod_{n=0}^{\infty} \left((n+x)^m - y^m \right) = \exp\left(-\frac{\partial}{\partial s} \varphi_m(s,x,y) \Big|_{s=0} \right)$$

$$= \exp\left(-a_1(x,y) \right),$$

we get

$$\prod_{n=0}^{\infty} \left((n+x)^m - y^m \right) = \frac{1}{C \prod_{\zeta^m=1} \Gamma(x - \zeta y)}.$$

Letting $y \to 0$, we have

$$C^{-1} = \left(\prod_{n=0}^{\infty} (n+x)^m \right) \prod_{\zeta^m=1} \Gamma(x)$$

$$= \left(\prod_{n=0}^{\infty} (n+x) \right)^m \Gamma(x)^m$$

$$= (2\pi)^{m/2}.$$

Here in the last equality we used the Lerch formula (1) in the Introduction. This completes the proof of the statement (i) of the theorem. □

3. Calculation for the corollary and examples

P r o o f. First we prove the corollary:

(i) Put $m = 2, x = 0$ and $y = a$ in (iii) of the theorem. Then we have immediately

$$\prod_{n=0}^{\infty} (n^2 + a^2) = \frac{2\pi}{\Gamma(ia)\Gamma(-ia)} = a(e^{\pi a} - e^{-\pi a}),$$

where we used the relation

$$\frac{1}{\Gamma(x)\Gamma(-x)} = -\frac{x\sin(\pi x)}{\pi} = -\frac{x(e^{\pi ix} - e^{-\pi ix})}{2\pi i}.$$

(ii) Put $m = 4$, $x = 0$ and $y = \frac{1+i}{\sqrt{2}}a$ in (iii) of the theorem. Then we obtain

$$\prod_{n=0}^{\infty} (n^4 + a^4) = \frac{(2\pi)^2}{\prod_{\zeta^4=1} \Gamma(-\zeta \frac{1+i}{\sqrt{2}} a)}.$$

The remaining calculation is quite similar to (i) above. This completes the proof of the corollary. $\square$

We now make calculation for examples: It would be sufficient to remark on (3). Put $m = 3$, $x = 0$ and $y = -1$ in (i) of the theorem. Then

$$\prod_{n=0}^{\infty} (n^3 + 1) = \frac{(2\pi)^{3/2}}{\Gamma(1)\Gamma(\omega)\Gamma(\overline{\omega})}$$

where $\omega = \frac{-1+\sqrt{3}i}{2}$. We use

$$\frac{1}{\Gamma(\omega)\Gamma(\overline{\omega})} = \frac{\omega \cdot \overline{\omega}}{\Gamma(\omega+1)\Gamma(\overline{\omega}+1)} = \frac{1}{\Gamma(\omega+1)\Gamma(\overline{\omega}+1)}$$

$$= \frac{1}{\Gamma(\omega+1)\Gamma(-\omega)} = \frac{\sin(-\omega\pi)}{\pi}$$

to get the example (3).

We now make small remarks for further study.

Remarks.

(a) From examples (2), (3) and Lerch's result

$$\prod_{n=0}^{\infty} (n+1) = \sqrt{2\pi},$$

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we see that when a_n, b_n are given by $a_n = n + 1$, $b_n = n^2 - n + 1$, respectively, the following equality is true.

$$\prod_{n=0}^{\infty}(a_n b_n) = \left(\prod_{n=0}^{\infty} a_n\right) \cdot \left(\prod_{n=0}^{\infty} b_n\right).$$

(b) We can obtain a "multiple version" of our theorem using multiple gamma functions (see [3]) instead of the usual gamma function.

(c) To obtain a q-analogue of the Lerch formula appropriately we need to extend the notion of the (zeta) regularized products. See [5].

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