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THE OMEGA LIMIT SETS OF SUBSETS IN A METRIC SPACE

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Abstract. In this paper, we discuss the properties of limit sets of subsets and attractors in a compact metric space. It is shown that the ω-limit set $\omega(Y)$ of Y is the limit point of the sequence $\{(\mathrm{Cl}\, Y)\cdot[i,\infty)\}_{i=1}^{\infty}$ in 2^X and also a quasi-attractor is the limit point of attractors with respect to the Hausdorff metric. It is shown that if a component of an attractor is not an attractor, then it must be a real quasi-attractor.

Keywords: limit set of a set, attractor, quasi-attractor, hyperspace

MSC 2000: 34C35, 54H20

1. Introduction

The pair attractor-repeller plays an important role in Conley's Theory [3], [4], which leads to applications of Conley decomposition and chain recurrence. However, Conley's definition of an attractor is discrepant with that of [1], [2]. The limit set of a neighborhood is used in the definition of an attractor [3], also Hale and Waterman [6] emphasize the importance of the limit set of a set in the analysis of the limiting behavior of a system. In this article we present some basic properties of the limit set of a subset in Section 2; it includes some considerations in the space of closed subsets which is called the hyperspace. In Section 3 the boundary conditions of an attractor neighborhood are discussed by the limit set of a subset. In the last section, we prove three theorems about quasi-attractors.

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2. The omega limit sets of subsets

Let (X, d) be a metric space with metric d, on which there is a flow $f\colon X \times \mathbb{R} \to X$. For $Y \subset X$ and $J \subset \mathbb{R}$ we denote $Y \cdot J = \{x \cdot t = f(x,t)\colon x \in Y,\ t \in J\}$. A set Y is invariant under f if Y is a subset in X with $Y \cdot \mathbb{R} = Y$. Throughout the paper for $Y \subset X$, $\operatorname{Cl} Y$, ∂Y, $\operatorname{Int} Y$ and $\operatorname{Ext} Y$ denote respectively the closure, boundary, interior and the interior of the complement of Y. If Y is a subset of X, the ω-limit set of Y is defined to be the set $\omega(Y) = \bigcap\limits_{t \geqslant 0} \operatorname{Cl}\{Y \cdot [t, \infty)\}$ (see [3]). When we wish to emphasize the dependence on f, we will write $\omega(Y, f)$. It is easy to prove that $\omega(Y)$ is the maximal invariant set in $\operatorname{Cl}\{Y \cdot [0, \infty)\}$ ([4], [9]).

Lemma 2.1. *The following facts about $\omega(Y)$ hold.*

(1) $\omega(Y) = \bigcap\limits_{n \geqslant 0} \operatorname{Cl}\{Y \cdot [n, \infty)\}$;

(2) *Let $Y_i \subset X$ $(i = 1, 2)$, then $\omega(Y_1 \cup Y_2) = \omega(Y_1) \cup \omega(Y_2)$; in particular, if $Y \subset Z$, $\omega(Y) \subset \omega(Z)$;*

(3) *$z \in \omega(Y)$ if and only if there are sequences $y_n \in Y$ and $t_n \in \mathbb{R}$ $(n = 1, 2, \ldots)$, $t_n \to +\infty$ as $n \to \infty$, such that $\lim\limits_{n\to\infty} y_n \cdot t_n = z$ ([6]).*

P r o o f. We only prove the necessity of (3), the other parts are straightforward. Since $z \in \omega(Y) = \bigcap\limits_{n \geqslant 0} \operatorname{Cl}\{Y \cdot [n, \infty)\}$, z is in $\operatorname{Cl}\{Y \cdot [n, \infty)\}$ for each n, thus there are $y_n \in Y$ and $t_n \geqslant n$ such that $d(z, y_n \cdot t_n) < 1/n$, i.e., $\lim\limits_{n\to\infty} d(z, y_n \cdot t_n) = 0$ and $\lim\limits_{n\to\infty} t_n \geqslant \lim\limits_{n\to\infty} n = +\infty$. $\square$

Lemma 2.2. *For any $Y \subset X$, $\omega(Y) = \omega(\operatorname{Cl} Y)$ holds.*

P r o o f. Since the flow $f\colon X \times \mathbb{R} \to X$ is continuous, $\operatorname{Cl}\{f(Y \times [t, +\infty))\} \supset f(\operatorname{Cl}\{Y \times [t, +\infty)\})$ for each $t \geqslant 0$, which implies $\operatorname{Cl}\{Y \cdot [t, +\infty)\} \supset (\operatorname{Cl} Y) \cdot [t, +\infty)$. So it follows that $\operatorname{Cl}\{Y \cdot [t, +\infty)\} \supset \operatorname{Cl}\{(\operatorname{Cl} Y) \cdot [t, +\infty)\}$. The converse inclusion is obvious, thus for each $t \geqslant 0$, $\operatorname{Cl}\{Y \cdot [t, +\infty)\} = \operatorname{Cl}\{(\operatorname{Cl} Y) \cdot [t, +\infty)\}$. Now $\bigcap\limits_{t \geqslant 0} \operatorname{Cl}\{Y \cdot [t, +\infty)\} = \bigcap\limits_{t \geqslant 0} \operatorname{Cl}\{(\operatorname{Cl}(Y) \cdot [t, +\infty)\}$ or $\omega(Y) = \omega(\operatorname{Cl} Y)$ follows. $\square$

It is worth noting that if $Y \subset X$ is connected and $\operatorname{Cl}\{Y \cdot [0, +\infty)\}$ is compact, then $\omega(Y)$ is connected and compact. The compactness of $\operatorname{Cl}\{Y \cdot [0, +\infty)\}$ is crucial, it is always true if X is compact. On the other hand, it is easy to find a counterexample such that $\omega(Y)$ is disconnected if $\operatorname{Cl}\{Y \cdot [0, +\infty)\}$ is not compact. According to Lemma 2.1 (3) it follows that $\omega(\{x\}) = \omega(x)$, that is, $\omega(\{x\})$ is just the usual ω-limit set of x. However, the set $\Lambda(Y) = \bigcup\limits_{x \in Y} \omega(x)$ is generally much smaller than $\omega(Y)$.

From now on we assume that (X, d) is a compact metric space. Let $2^X = \{A\colon\ A$ is a nonempty closed subset of $X\}$ be the hyperspace of X. Given two sets A and B in 2^X, let $H_d(A,B) = \inf\{\varepsilon > 0\colon\ A \subset N_d(B,\varepsilon)$ and $B \subset N_d(A,\varepsilon)\}$, where $N_d(A,\varepsilon) = \{x \in X\colon\ d(x,a) < \varepsilon$ for some $a \in A\}$. H_d is a metric on 2^X, called the Hausdorff metric. The topology it induces makes 2^X a compact space and is consistent with the Vietoris topology on 2^X. For these basic facts we refer to [8, Chapter 4]. For the flow f defined on X we put $S = \{A\colon\ A$ is nonempty and invariant under $f\}$, which is the collection of invariant sets of f and is a closed subset in 2^X [5].

Definition 2.4. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of (X, T), where T is the topology induced by d. We define $\limsup A_i$ and $\liminf A_i$ as follows [8, P56]:

$$\begin{aligned}\limsup A_i = \{x \in X\colon\ &\text{for each } U \in T \text{ such that } x \in U,\\ &U \cap A_i \neq \emptyset \text{ for infinitely many } i\};\\ \liminf A_i = \{x \in X\colon\ &\text{for each } U \in T \text{ such that } x \in U,\ U \cap A_i \neq \emptyset\\ &\text{for all but finitely many } i\}.\end{aligned}$$

If $\limsup A_i = \liminf A_i = A$, we write $\lim A_i = A$. In addition, we define $M^1 = \limsup A_i$, $M^2 = \bigcap_{k=1}^{\infty} \Big\{\bigcup_{i\geqslant k} A_i\Big\}$ and $M^3 = \bigcap_{k=1}^{\infty} \mathrm{Cl}\Big\{\bigcup_{i\geqslant k} A_i\Big\}$.

Lemma 2.5. *For any sequence $\{A_i\}_{i=1}^{\infty}$ of subsets of X, the equality $M^1 = M^3$ holds.*

Proof. For any $x \in M^3$, by the definition we have $x \in \mathrm{Cl}\Big\{\bigcap_{i\geqslant k} A_i\Big\}$ for each $k \geqslant 1$, which implies that for any open neighborhood U of x, $U \cap \Big(\bigcup_{i\geqslant k} A_i\Big) \neq \emptyset$. Thus for each k there is an $i_k \geqslant k$ with $U \cap A_{i_k} \neq \emptyset$, and it follows obviously that $x \in M^1$ and $M^3 \subset M^1$.

Conversely, for any $x \in M^1$ and any open neighborhood U of x there are infinitely many positive integers i_k such that $U \cap A_{i_k} \neq \emptyset$. Then $U \cap \Big(\bigcup_{i\geqslant k} A_i\Big) \neq \emptyset$ for each k, that is, $x \in \mathrm{Cl}\Big\{\bigcup_{i\geqslant k} A_i\Big\}$. It follows that $x \in \bigcap_{k=1}^{\infty} \mathrm{Cl}\Big\{\bigcup_{i\geqslant k} A_i\Big\} = M^3$, so $M^1 \subset M^3$. $\square$

By the definition $M^2 \subset M^3$ holds; furthermore, $\mathrm{Cl}\{M^2\} \subset M^3$ since

$$\mathrm{Cl}\Big\{\bigcap_{k=1}^{\infty}\bigcup_{i\geqslant k} A_i\Big\} \subset \bigcap_{k=1}^{\infty}\mathrm{Cl}\Big\{\bigcup_{i\geqslant k} A_i\Big\},$$

but the converse is not true. The following example shows that $\mathrm{Cl}\{M^2\}$ may be a proper subset of $M^1 = M^3$.

Example. Let $X = [0, 2]$, $A_i = \big[1/(i+1), 1/i\big] \cap \{2\}$, then $\bigcup\limits_{i \geqslant k} A_i = \big(0, 1/k\big] \cap \{2\}$, which implies $M^2 = \{2\}$. On the other hand, for any open neighborhood $U = [0, a)$ $(a > 0)$ there are infinitely many i satisfying $U \cap A_i \neq \emptyset$, which implies $0 \in M^1$. In fact $M^1 = M^3 = \{0, 2\}$, thus $\mathrm{Cl}\{M^2\}$ is a proper set of M^1.

Lemma 2.6 [8, P57]. *If X is a compact metric space and $\{A_i\}_{i=1}^{\infty}$ is a sequence of closed subsets in 2^X, then $\lim A_i = A$ if and only if $\{A_i\}_{i=1}^{\infty}$ converges to A in 2^X with respect to the Hausdorff metric.*

Theorem 2.7. *For $Y \subset X$, $\omega(Y)$ is the limit point in 2^X of the sequence $\{(\mathrm{Cl}\, Y) \cdot [i, \infty)\}_{i=1}^{\infty}$.*

P r o o f. Since $A_i = (\mathrm{Cl}\, Y) \cdot [i, \infty)$ is decreasing, it is easy to verify that $\liminf A_i = \limsup A_i$. Hence by Lemma 2.6 $M^1 = M^3$ is the limit point of $\{(\mathrm{Cl}\, Y) \cdot [i, \infty)\}$ in 2^X. Now since $\omega(Y) = \omega(\mathrm{Cl}\, Y)$ (Lemma 2.2), the conclusion follows. □

Remark. We can also consider the α-limit set of a set defined by $\alpha(Y) = \bigcap\limits_{t \geqslant 0} \mathrm{Cl}\{Y \cdot (-\infty, -t]\}$, $\alpha(\{x\}) = \alpha(x)$, and similar results hold.

3. Attractors and attractor neighborhoods

To the flow f on the compact metric space X there corresponds the backward flow f^* defined by $f^*(x, t) = f(x, -t)$, then $\omega(x, f^*) = \alpha(x, f)$. If A is a closed subset, we define the set $\tilde{\alpha}(A, f) = \{x \colon \omega(x, f) \cap A \neq \emptyset\}$.

Definition 3.1 [3]. A set A is called an attractor for f if A admits a neighborhood N such that $A = \omega(N, f)$.

Lemma 3.2. *Assume that N is a closed subset of X such that $x \in \partial N$ implies $x \cdot \mathbb{R}^- \cap \mathrm{Ext}\, N \neq \emptyset$ and let $A = \{x \colon x \cdot \mathbb{R} \subset N\}$. Then if $A \neq \emptyset$, A is an attractor.*

P r o o f. Let $N_1 = \{x \in N \colon x \cdot \mathbb{R}^+ \subset N\}$, then $A \subset N_1 \subset N$. Also it is easy to verify that $N_1 \cdot \mathbb{R}^+ \subset N$ and $\mathrm{Cl}\{N_1 \cdot \mathbb{R}^+\} \subset N$, which implies $\omega(N_1) \subset N$ and $\omega(N_1) = A$. Now we only need to prove that N_1 is a neighborhood of A. If on the contrary, $A \cap \partial N_1 \neq \emptyset$, let $x_0 \in A \cap \partial N_1$. Obviously N is a neighborhood of A. From $x_0 \in A$ it follows that $x_0 \cdot \mathbb{R} \subset \mathrm{Int}\, N$; furthermore, for $x_0 \in \partial N_1$ there exists a sequence $\{x_n\}$ satisfying $x_n \in N \setminus N_1$ and $\lim\limits_{n \to \infty} x_n = x_0$. Define $t_n = \max\{t \geqslant 0 \colon x_n \cdot [0, t] \subset N\}$, then $\lim\limits_{n \to \infty} t_n = +\infty$ since $x_0 \cdot \mathbb{R} \subset \mathrm{Int}\, N$. Because of $x_n \notin N_1$, t_n is finite and $x_n \cdot t_n \in \partial N$. The compactness of ∂N implies that there is a convergent subsequence of $\{x_n \cdot t_n\}$, by restriction to a subsequence we may suppose that $\lim\limits_{n\to\infty} x_n \cdot t_n = y \in \partial N$. Thus for $y \in \partial N$ there exists a $T > 0$ such that $y \cdot (-T) \in \operatorname{Ext} N$; choose n_0 large enough such that $(x_{n_0} \cdot t_{n_0}) \cdot (-T) \in \operatorname{Ext} N$ and $t_{n_0} > T$, that is $x_{n_0} \cdot (t_{n_0} - T) \in \operatorname{Ext} N$ and $0 < t_{n_0} - T < t_{n_0}$, which is contradictory to the definition of t_n. Hence $A \cap \partial N_1 = \emptyset$ and N_1 is a neighborhood of A. □

Remark 3.3. (1) If N satisfies $\omega(x, f^*) \cap \operatorname{Ext} N \neq \emptyset$ for each $x \in \partial N$, the conclusion of Lemma 3.2 is also true.

(2) In the proof of Lemma 3.2 we only need the compactness of ∂N, so the result is also true if ∂N is compact and X is not.

Lemma 3.4. *A nonempty compact invariant set A is an attractor if and only if A has an open neighborhood N satisfying $\bigcap\limits_{t\geqslant 0} N \cdot t = A$.*

P r o o f. If A is an attractor, A admits an open neighborhood N such that $\omega(N, f) = A$. Since A is an invariant subset of N, $A \cdot t = A$ for each $t \geqslant 0$, thus $N \cdot t \supset A \cdot t = A$ and $\bigcap\limits_{t\geqslant 0} N \cdot t \supset A$. On the other hand $N \cdot [t, +\infty) \supset N \cdot t$, thus $A = \bigcap\limits_{t\geqslant 0} \operatorname{Cl}\{N \cdot [t, +\infty)\} \supset \bigcap\limits_{t\geqslant 0} N \cdot t$. Thus it follows that $A = \bigcap\limits_{t\geqslant 0} N \cdot t$.

Conversely if A has an open neighborhood N satisfying $\bigcap\limits_{t\geqslant 0} N \cdot t = A$, choose a closed neighborhood M of A satisfying $A \subset M \subset N$. Since $A \cap \partial M = \emptyset$, for each $x \in \partial M$ there is a $t_x > 0$ such that $x \notin N \cdot t_x$, which implies $x \cdot (-t_x) \notin N$. Then $x \cdot (-t_x) \notin M$, by Lemma 3.2 we see that A is an attractor. □

Definition 3.5 [3]. An attractor neighborhood means a closed subset N of X such that for $x \in \partial N$, $\omega(x, f^*) \subset \operatorname{Ext} N$.

If A is an attractor, the set $\tilde{\alpha}(A, f) = \{x\colon\ \omega(x, f) \cap A \neq \emptyset\}$ is called the basin of attraction of A. As a matter of fact, $\tilde{\alpha}(A, f)$ is just the set $\{x\colon\ \omega(x, f) \subset A\}$ for an attractor A and it is an open invariant set.

Remark. Moeckel [7] pointed out that a better definition of attractor neighborhood is: a closed subset N of X such that $\omega(N, f)$ lies in the interior of N.

Note that the maximal invariant set $A = \{x\colon\ x \cdot \mathbb{R} \subset N\}$ in an attractor neighborhood N may be empty, then $N \subset \tilde{\alpha}(A, f)$ does not hold. The result of [3, Lemma 2.2A] is wrong. In general $N \subset \tilde{\alpha}(A, f)$ is not true even when A is nonempty. For example, see the following figure where the closed region *abcd* is an attractor neighborhood, $A = \{O\}$ is a stable focus, but $N \not\subset \tilde{\alpha}(A, f)$ since $\tilde{\alpha}(A, f)$ is the open disc which does not contain the arc $\widehat{ab}$.

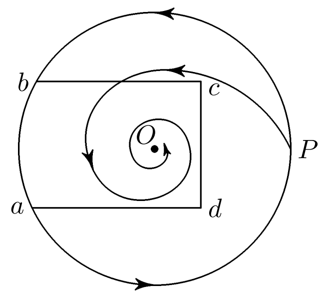

Fig. 1

Theorem 3.6. *Assume that N is a closed subset of X, then for each $x \in \partial N$, $\alpha(x) \cap \operatorname{Ext} N \neq \emptyset$ holds if and only if $x \cdot \mathbb{R}^- \cap \operatorname{Ext} N \neq \emptyset$ $(x \in \partial N)$.*

Proof. The necessity is clear, we only prove the sufficiency. If $x \cdot \mathbb{R}^- \cap \operatorname{Ext} N \neq \emptyset$ for each $x \in \partial N$, then there is a $t_1 < 0$ such that $x \cdot t_1 \in \operatorname{Ext} N$. If $x \cdot (-\infty, t_1] \cap N = \emptyset$, it follows that $\alpha(x) \subset \operatorname{Cl}\{\operatorname{Ext} N\}$; otherwise, by the connectedness of $x \cdot (-\infty, t_1]$ there is a $t_2 (< t_1)$ such that $x \cdot t_2 \in \partial N$. Thus there is a $t_3 (< t_2)$ such that $x \cdot t_3 \in \operatorname{Ext} N$. Now we proceed as above and consider two cases:

(I) If $x \cdot (-\infty, T] \cap N = \emptyset$ for some $T (< 0)$, then $\alpha(x) \subset \operatorname{Cl}\{\operatorname{Ext} N\}$. Since $\alpha(x)$ is invariant, it follows from $x \cdot \mathbb{R}^- \cap \operatorname{Ext} N \neq \emptyset$ that $\alpha(x) \cap \operatorname{Ext} N \neq \emptyset$, for every $x \in \partial N$.

(II) There is a sequence $t_{2n+1} \to -\infty$ such that $x \cdot t_{2n+1} \in \operatorname{Ext} N$, then by the compactness of $\operatorname{Cl}\{\operatorname{Ext} N\}$ it follows that $\alpha(x) \cap \operatorname{Cl}\{\operatorname{Ext} N\} \neq \emptyset$. Hence $\alpha(x) \cap \operatorname{Ext} N \neq \emptyset$. □

The following Fig. 2 shows that the condition $\alpha(x) \subset \operatorname{Ext} N \neq \emptyset$ for each $x \in \partial N$ is stronger than $x \cdot \mathbb{R}^- \cap \operatorname{Ext} N \neq \emptyset$ $(x \in \partial N)$, where N is the closed region *abcd*.

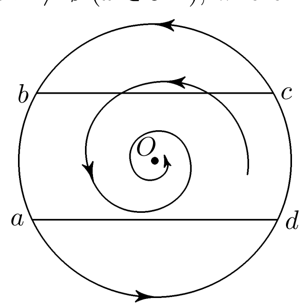

Fig. 2

Theorem 3.7. *If N is a closed subset of X satisfying $x \cdot \mathbb{R}^- \cap \operatorname{Ext} N \neq \emptyset$ for any $x \in \partial N$, let $A = \{x \in N \colon x \cdot \mathbb{R} \subset N\} \neq \emptyset$. Then there is a subset M of N such that $A \subset M$ and $\alpha(x) \subset \operatorname{Ext} M$ for each $x \in \partial M$, i.e., N has a subset M that is an attractor neighborhood.*

Proof. Let $N_1 = \{x \in N\colon\ x \cdot \mathbb{R}^+ \subset N\}$, then by the proof of Lemma 3.2, N_1 is a neighborhood of A and $\omega(N_1) = A$. Since for any neighborhood U of A there exists a $t > 0$ such that $N_1 \cdot [t, \infty) \subset U$, we see that $N_1 \cdot [T, \infty) \subset \operatorname{Int} N_1$ for some T. Now it is easy to conclude that $x \cdot (-\infty, -\tau] \cap N_1 = \emptyset$ for any $x \in \partial N_1$ and some $\tau > 0$; otherwise, for any $t_n > 0$ there is a $x_n \in \partial N_1$ such that $x_n \cdot (-\infty, -t_n] \cap N_1 \neq \emptyset$. Choose a $t_n > T$ and $z \in x_n \cdot (-\infty, -t_n] \cap N_1$, then $z = x_n \cdot (-t'_n) \in N_1$ with $t'_n \geqslant t_n > T$. Hence $z \cdot [T, \infty) = (x_n \cdot (-t'_n)) \cdot [T, \infty) = x_n \cdot [T - t'_n, \infty) \subset \operatorname{Int} N_1$. On the other hand, because of $T - t'_n < 0$, it follows that $x_n \in x_n \cdot [T - t'_n, \infty) \subset \operatorname{Int} N_1$, which is contradictory to $x_n \in \partial N_1$. Define $M = N_1 \cdot T \subset \operatorname{Int} N_1$. Since A is an invariant set in $\operatorname{Int} N_1$, $M = N_1 \cdot T \supset A \cdot T = A$, and the map $f(\cdot, -T)\colon\ \partial M \to \partial N_1$ is a homeomorphism. Thus for any $x \in \partial M$, $x \cdot (-\infty, -\tau - T] \cap N_1 = \emptyset$ holds, it follows that $\alpha(x) \subset \operatorname{Cl}\{\operatorname{Ext} N_1\}$ and $\alpha(x) \cap M = \emptyset$. □

According to the above argument, it is not assured that $\omega(N) \subset N$ for an attractor neighborhood N. We will prove the next theorem to get a condition for $\omega(N) \subset N$.

Theorem 3.8. *If N is a closed subset of X satisfying $x \cdot \mathbb{R}^- \cap \operatorname{Ext} N \neq \emptyset$ for any $x \in \partial N$, let $A = \{x \in X\colon\ x \cdot \mathbb{R} \subset N\} \neq \emptyset$. Then $N \subset \tilde{\alpha}(A, f)$ if and only if $\omega(N) \subset N$.*

Proof. Suppose $\omega(N) \subset N$. For any $x \in N$, $\omega(x) = \omega(\{x\}) \subset \omega(N) \subset N$. Since A is the maximal invariant set in N, it follows $\omega(x) \subset \omega(N) \subset A$. Hence by the definition we have $N \subset \tilde{\alpha}(A, f)$.

Next assume that $N \subset \tilde{\alpha}(A, f)$. By the proof of Lemma 3.2, A is an attractor; at the same time, the positively invariant set $N_1 = \{x \in N\colon\ x \cdot \mathbb{R}^+ \subset N\}$ is a neighborhood of A satisfying $A = \omega(N_1)$. Since $N \subset \tilde{\alpha}(A, f)$, i.e., $\omega(x) \subset A$ for every $x \in N$, there is a $t_x > 0$ such that $x \cdot t_x \in \operatorname{Int} N_1$. It follows that there is a neighborhood $B_x(\delta_x) = \{y\colon\ d(x, y) < \delta_x\}$ such that $B_x(\delta_x) \cdot t_x \subset \operatorname{Int} N_1$. Now by the compactness of N there exists a $T > 0$ such that $N \cdot [T, +\infty) \subset N_1$, which implies $\omega(N) \subset \omega(N_1) = A \subset N$. □

4. Quasi-attractors

Definition 4.1. An invariant set A is called a quasi-attractor if A is the intersection of attractors. A quasi-repeller of f means a quasi-attractor of f^*.

If the compact metric space X is connected, the only sets which are both attractors and repellers are X and $\emptyset$; however, a non-trivial invariant set may be both a quasi-attractor and a quasi-repeller. Also it is not generally true that a component of an attractor is an atrractor. Conley [3] proved the following result:

Lemma 4.2 [3, Theorem 2.4A]. *Any closed-open subset of an attractor is an attractor.*

Theorem 4.3. *If a component A_0 of an attractor A is not an attractor, then A_0 is a real quasi-attractor.*

Proof. Firstly we assert that any open neighborhood of A_0 meets the other components of A. Otherwise, let U be an open neighborhood of A_0 disjoint with the other components of A. Thus $U \cap A = A_0$. Since A_0 is a component of the closed set A, it is also a closed subset in U, so we may choose U to be a closed neighborhood with $U \cap A = A_0$, hence A_0 is a closed-open subset of A. It follows from Lemma 4.2 that A_0 is an attractor, which is a contradiction.

Define $U_n = N\big(A_0, 1/n\big) = \big\{x\colon\ d(x, A_0) < 1/n\big\}$ $(n = 1, 2, \ldots)$ and $A_n = \bigcup\big\{A_\alpha\colon A_\alpha$ is a component of A such that $A_\alpha \cap U_n \neq \emptyset\big\}$. Then we assert that $\bigcap\limits_{n\geqslant 1} A_n = A_0$. Actually, $A_0 \subset \bigcap\limits_{n\geqslant 1} A_n$ is clear. Conversely, let $x \in \bigcap\limits_{n\geqslant 1} A_n \subset A$, if $x \notin A_0$, denote $\overline{A}$ the component of A such that $x \in \overline{A}$. Since the attractor A is closed, all the components of A are closed subsets. Thus A_0 and $\overline{A}$ are disjoint closed subsets, or $\inf\{d(x, y)\colon\ x \in A_0,\ y \in \overline{A}\} = \delta > 0$. Choose n large enough such that $1/n < \delta$, then $A_n \cap \overline{A} = \emptyset$, which implies $x \notin A_n$. This is a contradiction. Hence $\bigcap\limits_{n\geqslant 1} A_n = A_0$ holds.

Since A is an attractor, it admits an open neighborhood N with $\omega(N) = A$. Choose n_0 large enough such that $U_n \subset N$ for $n \geqslant n_0$, and in the sequel we always assume $n \geqslant n_0$. Define $N_n^1 = \{x \in N\colon\ \omega(x) \subset A_n\}$ and $N_n^2 = \{x \in N\colon\ \omega(x) \subset (A \setminus A_n)\}$. Then $N = N_n^1 \cup N_n^2$ and $N_n^1 \cap N_n^2 = \emptyset$ since $\omega(x)$ is connected and lies in a component of A. Now for any $x \in N_n^1$ there exists a ball $B_\delta(x) = \{y\colon\ d(y, x) < \delta\}$ with $B_\delta(x) \subset N_n^1$, otherwise, there is a sequence $\{x_n\}$ such that $x_n \to x$ $(n \to \infty)$ and $\omega(x_n) \subset A \setminus A_n$. Let L be a component of A with $\omega(x) \subset L$, $L \subset A_n$ is clear. Thus by the dependence of initial value there is a sequence $\{t_n\}$ such that $d(x_n \cdot t_n, L) \to 0$ and $t_n \to \infty$. From Lemma 2.1 (3) the limit points of $\{x_n \cdot t_n\}$ $(t_n \to \infty)$ belong to $\omega(N_n^2)$, so it follows that $\omega(N_n^2) \cap L \neq \emptyset$. This is a contradiction, since N_n^2 is positively invariant and $N_n^2 \supset \omega(N_n^2) = A \setminus A_n$. Hence N_n^1 is an open set with $\omega(N_n^1) = A_n$, i.e., A_n is an attractor, which implies that A_0 is a real quasi-attractor. □

Lemma 4.4 [2, P39, 6.4A]. *In a compact metric space X there are at most countably many attractors.*

Theorem 4.5. *A quasi-attractor is a limit point of attractors in 2^X with respect to the Hausdorff metric.*

Proof. Let A be a quasi-attractor, then by Lemma 4.4 we have $A = \bigcap_{i=1}^{\infty} A_i$, where A_i are attractors. Define $B_i = \bigcap_{k=1}^{i} A_k$. Since the intersection of finite attractors is still an attractor, hence $\{B_i\}$ is a decreasing sequence of attractors and $A = \bigcap_{i=1}^{\infty} B_i$. A similar argument as in Theorem 2.7 shows that A is the limit point of $\{B_i\}$ in 2^X with respect to the Hausdorff metric. $\square$

Definition 4.6 [3]**.** The intersection of an attractor and a repeller is called a Morse set.

Theorem 4.7. *If a closed invariant set A is both a quasi-attractor and a quasi-repeller, then A is the limit point of a sequence of Morse sets.*

Proof. By Definition 4.1 and Lemma 4.4, $A = \bigcap_{i=1}^{\infty} A_i$ and $A = \bigcap_{i=1}^{\infty} B_i$, where A_i are attractors and B_i are repellers; furthermore, we can assume that A_i and B_i are decreasing sequences. Now denote $C_i = A_i \cap B_i$ $(i = 1, 2, \ldots)$, then $\{C_i\}$ are Morse sets. It is easy to verify that $A = \bigcap_{i=1}^{\infty} C_i$, and C_i is decreasing. Thus it follows from Lemma 2.6 that A is the limit point of C_i in 2^X with respect to the Hausdorff metric. $\square$

Remark. All the similar results on quasi-repellers are also true.

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References

[1] *N. P. Bhatia and G. P. Szegö*: Stability Theory of Dynamical Systems. Springer-Verlag, Berlin, 1970.
[2] *G. Butler and P. Waltmann*: Persistence in dynamical systems. J. Differential Equations *63* (1986), 255–263.
[3] *C. C. Conley*: The gradient structure of a flow: I. Ergod. Th. & Dynam. Sys. 8* (1988), 11–26.
[4] *C. C. Conley*: Isolated invariant sets and Morse index. Conf. Board Math. Sci., No 38. Amer. Math. Sci., Providence, 1978.
[5] *C. C. Conley*: Some abstract properties of the set of invariant sets of a flow. Illinois J. Math. *16* (1972), 663–668.
[6] *J. K. Hale and P. Waltmann*: Persistence in infinite-dimensional systems. SIAM J. Math. Anal. *20* (1989), 388–395.

[7] *R. Moeckel*: Some comments on "The gradient structure of a flow: I". vol. 8*, Ergod. Th. & Dynam. Sys., 1988.
[8] *S. B. Nadler, Jr.*: Continuum Theory: An Introduction. Marcel Dekker, New York-Basel-Hong Kong, 1992.
[9] *T. Huang*: Some global properties in dynamical systems. PhD. thesis. Inst. of Math., Academia Sinica, 1998.

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