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ON SOME DISCRETE INEQUALITIES IN TWO INDEPENDENT VARIABLES

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Abstract. In this paper we establish some new nonlinear difference inequalities. We also present an application of one inequality to certain nonlinear sum-difference equation.

Keywords: discrete inequality, two independent variables, difference equations, subadditive and submultiplicative functions

MSC 2000: 26D15, 39A10

1. Introduction

The theory of finite difference equation is in a process of continuous development and it has become significant for its various applications. Finite difference inequalities which give explicit bounds on unknown functions provide a very useful and important technique in the study of many qualitative as well as quantitative properties of solutions of nonlinear difference equations. During the past few years, various investigators have discovered many useful and new finite difference inequalities, mainly inspired by their applications in various branches of finite difference equations; see [1]–[9] and the references cited therein. In this paper, our main objective is to establish some new discrete inequalities involving functions of two independent variables which can be used in the analysis of certain classes of difference equations.

2. Main results

For discrete inequalities it is characteristic that the functions occurring in them are defined on countable sets, which can, without loss of generality, be assumed to be subsets of the set $\mathbb{Z}$ of integers. We shall introduce some notation; $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}_+ = [0,\infty)$. Let $\alpha, \beta \in \mathbb{Z}$, $\alpha \leqslant \beta$, $\mathbb{N}_\alpha = \{n \in \mathbb{Z} : n \geqslant \alpha\}$, $\mathbb{N}_{\alpha,\beta} = \{n \in \mathbb{Z} : \alpha \leqslant n \leqslant \beta\}$. Let $\sum_{j=\alpha}^{\beta} x(j)$, and $\prod_{j=\alpha}^{\beta} x(j)$, be the sum, respectively the product of $x(j)$, $j \in \mathbb{N}_{\alpha,\beta}$, and assume that $\sum_{j=\alpha}^{\alpha-1} x(j) = 0$, $\prod_{j=\alpha}^{\alpha-1} x(j) = 1$. We also assume that all the sums and products involved throughout the discussion exist on the respective domains of their definitions.

We need the inequality in the following Lemma 2.1, which appears in [1, p. 161].

Lemma 2.1. *Let $u(n)$, $a(n)$, $b(n)$, $q(n)$ be nonnegative functions defined for $n \in \mathbb{N}_\alpha$ satisfying the inequality*

$$u(n) \leqslant a(n) + q(n) \sum_{s=\alpha}^{n-1} b(s)u(s), \quad n \in \mathbb{N}_\alpha.$$

Then

$$u(n) \leqslant a(n) + q(n) \sum_{s=\alpha}^{n-1} b(s)a(s) \prod_{i=s+1}^{n-1} \bigl(1 + b(i)q(i)\bigr), \quad n \in \mathbb{N}_\alpha.$$

The inequality in the following lemma is an analogue of the inequality in Lemma 2.1 involving infinite series and products.

Lemma 2.2. *Let $u(n)$, $a(n)$, $b(n)$, $q(n)$ be nonnegative functions defined for $n \in \mathbb{N}_\alpha$, satisfying the inequality*

$$u(n) \leqslant a(n) + q(n) \sum_{s=n+1}^{\infty} b(s)u(s), \quad n \in \mathbb{N}_\alpha, \tag{2.1}$$

and let $0 < \sum_{s=n+1}^{\infty} b(s)a(s) < \infty$. Then

$$u(n) \leqslant a(n) + q(n) \sum_{s=n+1}^{\infty} b(s)a(s) \prod_{i=s}^{\infty} \bigl(1 + b(i)q(i)\bigr), \quad n \in \mathbb{N}_\alpha. \tag{2.2}$$

Proof. From (2.1) we have

$$u(n) \leqslant a(n) + q(n)w(n), \tag{2.3}$$

where the function $w(n)$ is defined by $w(n) = \sum_{s=n+1}^{\infty} b(s)u(s)$. From (2.3) we get

$$w(n) \leqslant \sum_{s=n+1}^{\infty} b(s)\big(a(s) + q(s)w(s)\big) = f(n) + \sum_{s=n+1}^{\infty} b(s)q(s)w(s), \tag{2.4}$$

where $f(n) = \sum_{s=n+1}^{\infty} b(s)a(s)$. Clearly, $f(n)$ is nonincreasing in the variable n on each $\mathbb{N}_\alpha$. First, we assume that $f(n) > 0$ for $n \in \mathbb{N}_\alpha$. From (2.4) we observe that

$$\frac{w(n)}{f(n)} \leqslant 1 + \sum_{s=n+1}^{\infty} b(s)\frac{w(s)}{f(s)}.$$

Define a function $v(n)$ by

$$v(n) = 1 + \sum_{s=n+1}^{\infty} b(s)q(s)\frac{w(s)}{f(s)},$$

then $w(n)/f(n) \leqslant v(n)$ and

$$\Delta v(n) = v(n) - v(n-1) = -b(n)q(n)\frac{w(n)}{f(n)} \geqslant -b(n)q(n)v(n). \tag{2.5}$$

From (2.5) and using the fact that $v(n) > 0$ for $n \in \mathbb{N}_\alpha$, we observe that $v(n-1) \leqslant \big(1 + b(n)q(n)\big)v(n)$, or

$$v(n) \leqslant \big(1 + b(n+1)q(n+1)\big)v(n+1). \tag{2.6}$$

Now, setting $n = i$ in (2.6) and substituting $i = n, n+1, n+2, \dots$ successively and using the fact that $v(\infty) = 1$, we get

$$v(n) \leqslant \prod_{i=n+1}^{\infty} \big(1 + b(i)q(i)\big). \tag{2.7}$$

Using (2.7) in $w(n)/f(n) \leqslant v(n)$ we have

$$w(n) \leqslant f(n) \prod_{i=n+1}^{\infty} \big(1 + b(i)q(i)\big). \tag{2.8}$$

The required inequality in (2.2) follows from (2.3) and (2.8). □

In 2001, Pachpatte proved the following Lemma 2.3 (see [9, Theorem 1]).

Lemma 2.3. *Let $u(m,n)$, $a(m,n)$, $f(m,n)$ be nonnegative functions defined for $m,n \in \mathbb{N}_0$.*

(i) *Assume that $f(m,n)$ is nondecreasing in m and nonincreasing in n for $m,n \in \mathbb{N}_0$. If $u(m,n) \leqslant f(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} a(s,t)u(s,t)$ for all $m,n \in \mathbb{N}_0$, then*

$$u(m,n) \leqslant f(m,n) \prod_{s=0}^{m-1} \Big(1 + \sum_{t=n+1}^{\infty} a(s,t)\Big).$$

(ii) *Assume that $\overline{f}(m,n)$ is nonincreasing in each of the variables $m,n \in \mathbb{N}_0$. If $u(m,n) \leqslant \overline{f}(m,n) + \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} a(s,t)u(s,t)$ for all $m,n \in \mathbb{N}_0$, then*

$$u(m,n) \leqslant \overline{f}(m,n) \prod_{s=m+1}^{\infty} \Big(1 + \sum_{t=n+1}^{\infty} a(s,t)\Big).$$

Now, we use the notation; a function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is said to be subadditive if $f(x+y) \leqslant f(x) + f(y)$, $x,y \in \mathbb{R}_+$; submultiplicative if $f(xy) \leqslant f(x)f(y)$, $x,y \in \mathbb{R}_+$. Our main results on discrete inequalities are established in the following theorems.

Theorem 2.4. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$, $d(m,n)$ be real-valued nonnegative functions defined for $m,n \in \mathbb{N}_0$ and $u(m,n) \geqslant u_0 > 0$, where u_0 is a real constant. Let $W(r)$ be a real-valued continuous, positive, nondecreasing, subadditive, and submultiplicative function on $I = [u_0,\infty)$ and let $H(r)$ be real-valued, continuous, positive, and nondecreasing function on I. Assume that $a(m,n)$ is nondecreasing in m for $m \in \mathbb{N}_0$. If*

$$u(m,n) \leqslant a(m,n) + b(m,n) \sum_{s=0}^{m-1} c(s,n)u(s,n) + H\Big(\sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} d(s,t)W\big(u(s,t)\big)\Big), \tag{2.9}$$

for $m,n \in \mathbb{N}_0$, then for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, $m,m_1,n,n_1 \in \mathbb{N}_0$,

$$u(m,n) \leqslant q(m,n)[a(m,n) + H(G^{-1}[G\big(V_1(\infty,0)\big) + V_2(m,n)])], \tag{2.10}$$

where

$$q(m,n) = 1 + b(m,n)\sum_{s=0}^{m-1} c(s,n)\prod_{i=s+1}^{m-1}\big(1 + c(i,n)b(i,n)\big), \tag{2.11}$$

$$V_1(\infty,0) = \sum_{s=0}^{\infty}\sum_{t=1}^{\infty} d(s,t)W\big(q(s,t)a(s,t)\big), \tag{2.12}$$

$$V_2(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} d(s,t)W\big(q(s,t)\big),$$

$$G(r) = \int_{r_0}^{r}\frac{\mathrm{d}s}{W\big(H(s)\big)}, \quad r \geqslant u_0 \ \ \textit{with}\ \ r_0 \geqslant u_0, \tag{2.13}$$

G^{-1} is the inverse function of G and

$$G\Big(\sum_{s=0}^{\infty}\sum_{t=1}^{\infty} d(s,t)W\big(q(s,t)a(s,t)\big)\Big) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} d(s,t)W\big(q(s,t)\big) \in \mathrm{Dom}(G^{-1})$$

for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, $m, m_1, n, n_1 \in \mathbb{N}_0$.

P r o o f. Define a function $z(m,n)$ by

$$z(m,n) = a(m,n) + H\Big(\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} d(s,t)W\big(u(s,t)\big)\Big). \tag{2.14}$$

Then(2.9) can be restated as

$$u(m,n) \leqslant z(m,n) + b(m,n)\sum_{s=0}^{m-1} c(s,n)u(s,n). \tag{2.15}$$

Clearly $z(m,n)$ is nonnegative and nondecreasing in m, $m \in \mathbb{N}_0$. Treating n, $n \in \mathbb{N}_0$ as a fixed in (2.15) and using Lemma 2.1 to (2.15), we get

$$\begin{aligned} u(m,n) &\leqslant z(m,n) + b(m,n)\sum_{s=0}^{m-1} c(s,n)z(s,n)\prod_{i=s+1}^{m-1}\big(1 + c(i,n)b(i,n)\big) \\ &\leqslant z(m,n)q(m,n), \end{aligned}$$

where $q(m,n)$ is defined by (2.11). From (2.14) and the last estimate we have

$$u(m,n) \leqslant q(m,n)\big(a(m,n) + H\big(v(m,n)\big)\big), \tag{2.16}$$

where $v(m,n)$ is defined by

$$v(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} d(s,t) W\big(u(s,t)\big).$$

From (2.16) we get

$$\begin{aligned} \text{(2.17)} \qquad v(m,n) &\leqslant \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} d(s,t) W\big(q(s,t)\big[a(s,t) + H\big(v(s,t)\big)\big]\big) \\ &\leqslant V_1(\infty,0) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} d(s,t) W\big(q(s,t)\big) W\big(H(v(s,t))\big) \end{aligned}$$

where $V_1(\infty,0)$ is defined by (2.12). Define a function $k(m,n)$ by the right-hand side of (2.17). Then, $v(m,n) \leqslant k(m,n)$ and

$$\begin{aligned} \text{(2.18)} \qquad & [k(m,n) - k(m-1,n)] - [k(m,n-1) - k(m-1,n-1)] \\ &= -\, d(m-1,n) W\big(q(m-1,n)\big) W\big(H\big(v(m-1,n)\big)\big) \\ &\geqslant -\, d(m-1,n) W\big(q(m-1,n)\big) W\big(H\big(k(m-1,n)\big)\big). \end{aligned}$$

From (2.18) and the fact that $k(m-1,n) \leqslant k(m-1,n-1)$, we observe that

$$\begin{aligned} \text{(2.19)} \qquad & \frac{k(m,n) - k(m-1,n)}{W\big(H\big(k(m-1,n)\big)\big)} - \frac{k(m,n-1) - k(m-1,n-1)}{W\big(H\big(k(m-1,n-1)\big)\big)} \\ &\qquad \geqslant -\, d(m-1,n) W\big(q(m-1,n)\big). \end{aligned}$$

Keeping m fixed in (2.19), substituting $n = t$, and taking the sum over $t = n+1$, $n+2, \dots, r$ ($r \geqslant n+1$ is arbitrary in $\mathbb{N}_0$), we obtain

$$\begin{aligned} \text{(2.20)} \qquad & \frac{k(m,n) - k(m-1,n)}{W\big(H\big(k(m-1,n)\big)\big)} - \frac{k(m,r) - k(m-1,r)}{W\big(H\big(k(m-1,r)\big)\big)} \\ &\qquad \leqslant \sum_{t=n+1}^{r} d(m-1,t) W\big(q(m-1,t)\big). \end{aligned}$$

Noting that $\lim_{r\to\infty} k(m+1,r) = \lim_{r\to\infty} k(m,r) = V_1(\infty,0)$ and letting $r \to \infty$ in (2.20), we have

$$\text{(2.21)} \qquad \frac{k(m,n) - k(m-1,n)}{W\big(H\big(k(m-1,n)\big)\big)} \leqslant \sum_{t=n+1}^{\infty} d(m-1,t) W\big(q(m-1,t)\big).$$

From (2.13) and (2.21), we have

$$G(k(m,n)) - G(k(m-1,n)) = \int_{k(m-1,n)}^{k(m,n)} \frac{\mathrm{d}s}{W(H(s))} \leqslant \frac{k(m,n)-k(m-1,n)}{W\big(H(k(m-1,n))\big)} \leqslant \sum_{t=n+1}^{\infty} d(m-1,t)W(q(m-1,t)). \tag{2.22}$$

Now, keeping n fixed in (2.22), substituting $m = s+1$, and taking the sum over $s = 0, 1, \ldots, m-1$ ($m \geqslant 1$ is arbitrary in $\mathbb{N}_0$), we obtain

$$G(k(m,n)) - G(k(0,n)) \leqslant \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} d(s,t)W(q(s,t)). \tag{2.23}$$

Noting that $k(0,n) = V_1(\infty, 0)$ in (2.23), we get

$$k(m,n) \leqslant G^{-1}\Bigg(G(V_1(\infty,0)) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} d(s,t)W(q(s,t))\Bigg). \tag{2.24}$$

The required inequality in (2.10) follows from the fact that $v(m,n) \leqslant k(m,n)$, (2.16) and (2.24). The subdomain $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$ is obvious. □

Theorem 2.5. *Under the conditions of Theorem 2.4, assume that $a(m,n)$ is nonincreasing in m for $m \in \mathbb{N}_0$, and $0 < \sum\limits_{s=m+1}^{\infty} c(s,n) < \infty$. If*

$$u(m,n) \leqslant a(m,n) + b(m,n) \sum_{s=m+1}^{\infty} c(s,n)u(s,n) + H\Bigg(\sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} d(s,t)W(u(s,t))\Bigg)$$

for $m, n \in \mathbb{N}_0$, then for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, $m, m_1, n, n_1 \in \mathbb{N}_0$,

$$u(m,n) \leqslant \bar{q}(m,n)[a(m,n) + H(G^{-1}[G\big(\overline{V_1}(\infty,0)\big) + \overline{V_2}(m,n)])],$$

where

$$\bar{q}(m,n) = 1 + b(m,n)\sum_{s=m+1}^{\infty} c(s,n)\prod_{i=m+1}^{\infty}\big(1 + c(i,n)b(i,n)\big),$$

$$\overline{V_1}(\infty,0) = \sum_{s=1}^{\infty}\sum_{t=1}^{\infty} d(s,t)W\big(\bar{q}(s,t)a(s,t)\big),$$

$$\overline{V_2}(m,n) = \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} d(s,t)W\big(\bar{q}(s,t)\big),$$

G is defined by (2.13), *G^{-1} is the inverse function of G and*

$$G\bigg(\sum_{s=1}^{\infty}\sum_{t=1}^{\infty} d(s,t)W\big(\bar{q}(s,t)a(s,t)\big)\bigg) + \sum_{s=m+1}^{\infty}\sum_{t=n+1}^{\infty} d(s,t)W\big(\bar{q}(s,t)\big) \in \mathrm{Dom}(G^{-1})$$

for $0 \leqslant m \leqslant m_1$, $0 \leqslant n \leqslant n_1$, $m, m_1, n, n_1 \in \mathbb{N}_0$.

The proof of Theorem 2.5 can be given along the lines of the proof of Theorem 2.4, and we omit it. In the proof of Theorem 2.5 we have to use Lemma 2.2 instead of Lemma 2.1 appearing in the proof of Theorem 2.4.

Theorem 2.6. *Let $u(m,n)$, $a(m,n)$, $b(m,n)$, $c(m,n)$ be real-valued nonnegative functions defined for $m,n \in \mathbb{N}_0$ and $L\colon \mathbb{N}_0^2 \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function which satisfies the condition*

$$0 \leqslant L(m,n,u) - L(m,n,v) \leqslant M(m,n,v)\varphi^{-1}(u-v),$$

for $u \geqslant v \geqslant 0$, where $M(m,n,v)$ is a real-valued nonnegative function defined for $m,n \in \mathbb{N}_0$, $v \in \mathbb{R}_+$. Let $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and strictly increasing function with $\varphi(0) = 0$, φ^{-1} the inverse function of φ and $\varphi^{-1}(uv) \leqslant \varphi^{-1}(u)\varphi^{-1}(v)$ for $u, v \in \mathbb{R}_+$. Assume that $a(m,n)$ is nondecreasing in m for $m \in \mathbb{N}_0$. If

$$u(m,n) \leqslant a(m,n) + b(m,n)\sum_{s=0}^{m-1} c(s,n)u(s,n) + \varphi\bigg(\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} L\big(s,t,u(s,t)\big)\bigg) \tag{2.25}$$

for $m,n \in \mathbb{N}_0$, then

$$u(m,n) \leqslant q(m,n)\bigg\{a(m,n) + \varphi\bigg(e(m,n)\prod_{s=0}^{m-1}\bigg[1 + \sum_{t=n+1}^{\infty} M\big(s,t,q(s,t)a(s,t)\big)\varphi^{-1}(q(s,t))\bigg]\bigg)\bigg\} \tag{2.26}$$

for $m,n\in\mathbb{N}_0$, *where*

$$e(m,n)=\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}L\big(s,t,q(s,t)a(s,t)\big) \tag{2.27}$$

for $m,n\in\mathbb{N}_0$, *and* $q(m,n)$ *is defined by* (2.11).

Proof. Define a function $z(m,n)$ by

$$z(m,n)=a(m,n)+\varphi\bigg(\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}L\big(s,t,u(s,t)\big)\bigg). \tag{2.28}$$

Then (2.25) can be restated as

$$u(m,n)\leqslant z(m,n)+b(m,n)\sum_{s=0}^{m-1}c(s,n)u(s,n). \tag{2.29}$$

Clearly $z(m,n)$ is a nonnegative and nondecreasing in m, $m\in\mathbb{N}_0$. Treating n, $n\in\mathbb{N}_0$ as fixed in (2.29) and applying Lemma 2.1 to (2.29), we get

$$\begin{aligned}u(m,n)&\leqslant z(m,n)+b(m,n)\sum_{s=0}^{m-1}c(s,n)z(s,n)\prod_{i=s+1}^{m-1}\big(1+c(i,n)b(i,n)\big)\\&\leqslant z(m,n)q(m,n),\end{aligned}$$

where $q(m,n)$ is defined by (2.11). From (2.28) and the last estimate we have

$$u(m,n)\leqslant q(m,n)\big(a(m,n)+\varphi(w(m,n))\big), \tag{2.30}$$

where $w(m,n)$ is defined by $w(m,n)=\sum\limits_{s=0}^{m-1}\sum\limits_{t=n+1}^{\infty}L\big(s,t,u(s,t)\big)$. From (2.30) we get

$$\begin{aligned}w(m,n)&\leqslant\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}L(s,t,q(s,t)\big[a(s,t)+\varphi(w(s,t))\big])\\&\leqslant e(m,n)+\sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty}M\big(s,t,q(s,t)a(s,t)\big)\varphi^{-1}(q(s,t))w(s,t),\end{aligned}$$

where $e(m,n)$ is defined by (2.27). Clearly, $e(m,n)$ is nonnegative, nondecreasing in m, $m\in\mathbb{N}_0$, and nonincreasing in n, $n\in\mathbb{N}_0$. Now, by (i) of Lemma 2.3, we obtain

$$w(m,n)\leqslant e(m,n)\prod_{s=0}^{m-1}\bigg(1+\sum_{t=n+1}^{\infty}M\big(s,t,q(s,t)a(s,t)\big)\varphi^{-1}(q(s,t))\bigg). \tag{2.31}$$

Using (2.31) in (2.30) we get the required inequality in (2.26). □

Theorem 2.7. *Under the conditions of Theorem 2.6, assume that $a(m,n)$ is nonincreasing in m for $m \in \mathbb{N}_0$, and $0 < \sum_{s=m+1}^{\infty} c(s,n)u(s,n) < \infty$. If*

$$u(m,n) \leqslant a(m,n) + b(m,n) \sum_{s=m+1}^{\infty} c(s,n)u(s,n) + \varphi\Big(\sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L\big(s,t,u(s,t)\big)\Big)$$

for $m,n \in \mathbb{N}_0$, then

$$u(m,n) \leqslant \bar{q}(m,n)\Big\{a(m,n) + \varphi\Big(\bar{e}(m,n) \times \prod_{s=m+1}^{\infty} \Big[1 + \sum_{t=n+1}^{\infty} M\big(s,t,\bar{q}(s,t)a(s,t)\big)\varphi^{-1}\big(\bar{q}(s,t)\big)\Big]\Big)\Big\}$$

for $m,n \in \mathbb{N}_0$, where

$$\bar{e}(m,n) = \sum_{s=m+1}^{\infty} \sum_{t=n+1}^{\infty} L\big(s,t,\bar{q}(s,t)a(s,t)\big)$$

for $m,n \in \mathbb{N}_0$, and $\bar{q}(m,n)$ is defined in Theorem 2.5.

The proof of Theorem 2.7 can be given along the lines of the proof of Theorem 2.6, and we omit it. In the proof of Theorem 2.7 we have to use Lemma 2.2 (respectively, Lemma 2.3 (ii)) instead of Lemma 2.1 (respectively, Lemma 2.3 (i)) appearing in the proof of Theorem 2.6.

3. Some applications

In this section we present some immediate applications of Theorem 2.4 to study certain properties of solutions of the following a nonlinear sum-difference equation,

$$u(m,n) = F(m,n) + \sum_{s=0}^{m-1} A\big(m,n,s,t,u(s,t)\big) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{\infty} B\big(m,n,s,t,u(s,t)\big) \tag{3.1}$$

where $u\colon \mathbb{N}_0^2 \to \mathbb{R} - \{0\}$, $F\colon \mathbb{N}_0^2 \to \mathbb{R}$, $A, B\colon \mathbb{N}_0^2 \times \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R} - \{0\}$, and

$$|F(m,n)| \leqslant a(m,n), \tag{3.2}$$

$$\bigl|A\bigl(m,n,s,t,u(s,t)\bigr)\bigr| \leqslant b(m,n)c(s,n)|u(s,n)|, \tag{3.3}$$

$$|B(m,n,s,t,u(s,t)| \leqslant d(s,t)W(|u(s,t)|), \tag{3.4}$$

where $a(m,n)$, $b(m,n)$, $c(s,n)$ and $d(s,t)$ are as defined in Theorem 2.4, $W(r)$ is a real-valued continuous, positive, nondecreasing, subadditive, and submultiplicative function on $I = [u_0, \infty)$ and $|u| \geqslant u_0 > 0$ where u_0 is real constant. Let $u(m,n)$ be a solution of (3.1). From (3.1)–(3.4), we have

$$|u(m,n)| \leqslant a(m,n) + b(m,n)\sum_{s=0}^{m-1} c(s,n)|u(s,n)| + \sum_{s=0}^{m-1}\sum_{t=n+1}^{\infty} d(s,t)W(|u(s,t)|). \tag{3.5}$$

Now, a suitable application of Theorem 2.4 to (3.5) yields the required estimate as follows

$$|u(m,n)| \leqslant q(m,n)\bigl[a(m,n) + G^{-1}\bigl[G(V_1(\infty,0)) + V_2(m,n)\bigr]\bigr].$$

The right-hand side of the above inequality gives an upper bound on the solution $u(m,n)$ of (3.1) in terms of the known functions.

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References

[1] *D. Bainov and P. Simeonov*: Integral Inequalities and Applications. Kluwer Academic Publishers, Dordrecht, 1992.

[2] *S. S. Dragomir*: On Gronwall type lemmas and applications. Monografii Matematics. Univ. Timişoara No. 29, 1987.

[3] *A. Mate and P. Neval*: Sublinear perturbations of the differential equation $y^{(n)} = 0$ and of the analogous difference equation. J. Differential Equations *52* (1984), 234–257.

[4] *D. S. Mitrinović, J. E. Pečarić and A. M. Fink*: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic Publishers, Dordrecht-Boston-London, 1991.

[5] *B. G. Pachpatte*: On certain new finite difference inequalities. Indian J. Pure Appl. Math. *24* (1993), 373–384.

[6] *B. G. Pachpatte*: Some new finite difference inequalities. Computer Math. Appl. *28* (1994), 227–241.
[7] *B. G. Pachpatte*: On some new discrete inequalities useful in the theory of partial finite difference equations. Ann. Differential Equations *12* (1996), 1–12.
[8] *B. G. Pachpatte*: On some fundamental integral inequalities and their discrete analogues. J. Inequal. Pure Appl. Math. *2* (2001), Paper No. 15.
[9] *B. G. Pachpatte*: On some finite difference inequalities in two independent variables. J. Math. Anal. Appl. *254* (2001), 587–598.

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