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EXTENSIONS, DILATIONS AND FUNCTIONAL MODELS OF INFINITE JACOBI MATRIX

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Abstract. A space of boundary values is constructed for the minimal symmetric operator generated by an infinite Jacobi matrix in the limit-circle case. A description of all maximal dissipative, accretive and selfadjoint extensions of such a symmetric operator is given in terms of boundary conditions at infinity. We construct a selfadjoint dilation of maximal dissipative operator and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of dilation. We construct a functional model of the dissipative operator and define its characteristic function. We prove a theorem on the completeness of the system of eigenvectors and associated vectors of dissipative operators.

Keywords: infinite Jacobi matrix, symmetric operator, selfadjoint and nonselfadjoint extensions, maximal dissipative operator, selfadjoint dilation, scattering matrix, functional model, characteristic function, completeness of the system of eigenvectors and associated vectors

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1. Introduction

The theory of extensions of symmetric operators is one of the main branches of operator theory. The first fundamental results in this area were obtained by von Neumann [13], although the apparent origins can be found in the famous works of Weyl (see [15], [17]), and also in numerous papers on the classical problem of moments (see [1], [4]).

To describe the various classes of extensions of symmetric operators, the theorems on the representation of linear relations have proved to be useful. The first result of this type belongs to Rofe-Beketov [14]. Independently, in [5] and [9], the notion of a 'space of boundary values' was introduced and all maximal dissipative, accretive, selfadjoint, and other extensions of symmetric operators were described (see [8] and also in the survey article [7]).

However, regardless of the general scheme, the problem of description of the maximal dissipative (accretive), selfadjoint and other extensions of a given symmetric operator via boundary conditions is of considerable interest. This problem is particularly interesting in the case of singular operators, because at the singular ends of the interval under consideration the usual boundary conditions are in general are meaningless.

It is known [10], [12] that the theory of dilations with application of operator models gives an adequate approach to the spectral theory of dissipative (contractive) operators. A central part in this theory is played by the characteristic function, which carries the complete information on the spectral properties of the dissipative operator. Thus, in the incoming spectral representation of the dilation, the dissipative operator becomes the model. The problem of the completeness of the system of eigenvectors and associated vectors is solved in terms of the factorization of the characteristic function. The computation of the characteristic functions of dissipative operators is preceded by the construction and investigation of a selfadjoint dilation and of the corresponding scattering problem, in which the characteristic function is realized as the scattering matrix.

In this paper we consider the minimal symmetric operator in the space $\ell_w^2(\mathbb{N})$ ($\mathbb{N} := \{0, 1, 2, \ldots\}$) with defect index (1,1) (in Weyl's limit-circle case), generated by an infinite Jacobi matrix. We construct a space of boundary values of the minimal operator and describe all maximal dissipative, maximal accretive and selfadjoint extensions of minimal operator in terms of the boundary conditions at ∞. We construct a selfadjoint dilation of the maximal dissipative operators and their incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of dilation according to the scheme of Lax and Phillips [11]. With the help of the incoming spectral representation we construct a functional model of the dissipative operator and define its characteristic function. Finally, on the basis of the results obtained regarding the theory of the characteristic function we prove a theorem on completeness of the system of eigenvectors and associated vectors of dissipative operators.

2. Extensions of the symmetric operator generated by an infinite Jacobi matrix

An *infinite Jacobi matrix* is defined to be a matrix of the form

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \dots \\ a_0 & b_1 & a_1 & 0 & 0 & \dots \\ 0 & a_1 & b_2 & a_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $a_n \neq 0$ and $\operatorname{Im} a_n = \operatorname{Im} b_n = 0$ $(n \in \mathbb{N})$.

For every sequence $y = \{y_n\}$ $(n \in \mathbb{N})$ of complex numbers $y_0, y_1, \dots$ denote by ℓy the sequence with components $(\ell y)_n$ $(n \in \mathbb{N})$ defined by

$$\begin{aligned} (\ell y)_0 &:= \frac{1}{w_0}(Jy)_0 = \frac{1}{w_0}(b_0 y_0 + a_0 y_1), \\ (\ell y)_n &:= \frac{1}{w_n}(Jy)_n = \frac{1}{w_n}(a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}), \quad n \geqslant 1, \end{aligned}$$

where $w_n > 0$ $(n \in \mathbb{N})$.

For two arbitrary sequences $y = \{y_n\}$ and $z = \{z_n\}$ $(n \in \mathbb{N})$, denote by $[y, z]$ the sequence with components $[y, z]_n$ $(n \in \mathbb{N})$ defined by

$$[y, z]_n = a_n(y_n \overline{z}_{n+1} - y_{n+1}\overline{z}_n) \quad (n \in \mathbb{N}). \tag{2.1}$$

Then we have Green's formula

$$\sum_{j=0}^{j} \{w_j(\ell y)_j \bar{z}_j - w_j y_j (\ell \overline{z})_j\} = -[y, z]_n \quad (n \in \mathbb{N}). \tag{2.2}$$

To pass from the matrix J to operators, we introduce the Hilbert space $\ell_w^2(\mathbb{N})$ $(w\colon = \{w_n\},\ n \in \mathbb{N})$ consisting of all complex sequences $y = \{y_n\}$ $(n \in \mathbb{N})$ such that $\sum\limits_{n=0}^{\infty} w_n |y_n|^2 < \infty$, with the inner product $(y, z) = \sum\limits_{n=0}^{\infty} w_n y_n \overline{z}_n$. Next, denote by D the linear set of all elements $y \in \ell_w^2(\mathbb{N})$ such that $\ell y \in \ell_w^2(\mathbb{N})$. We define the operator L on D by the equality $Ly = \ell y$.

It follows from (2.2) that for all $y, z \in D$ the limit $[y, z]_\infty = \lim\limits_{n \to \infty} [y, z]_n$ exists and is finite. Therefore, passing to the limit as $n \to \infty$ in (2.2), we get for two arbitrary vectors y and z of D

$$(Ly, z) - (y, Lz) = -[y, z]_\infty. \tag{2.3}$$

In $\ell^2_w(\mathbb{N})$ we consider the linear set D_0' consisting of finite vectors (i.e. vectors having only finite many nonzero components). Denote by L_0' the restriction of the operator L to D_0'. It follows from (2.3) that L_0' is symmetric. Consequently, it admits closure. Its closure is denoted by L_0. The domain D_0 of L_0 consists of precisely those vectors $y \in D$ satisfying the condition

$$[y, z]_\infty = 0, \quad \forall\, z \in D. \tag{2.4}$$

The operator L_0 is a closed symmetric operator with defect index $(0, 0)$ or $(1, 1)$ and $L = L_0^*$ [1]–[4], [6], [16]. The operators L_0 and L are called, respectively, the *minimal* and *maximal operators*. For defect index $(0, 0)$ the operator L_0 is selfadjoint, that is, $L_0^* = L_0 = L$.

Denote by $P(\lambda) = \{P_n(\lambda)\}$ and $Q(\lambda) = \{Q_n(\lambda)\}$ $(n \in \mathbb{N})$ the solutions of the equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda w_n y_n \quad (n = 1, 2, \ldots) \tag{2.5}$$

satisfying the initial conditions

$$P_0(\lambda) = 1, \quad P_1(\lambda) = \frac{\lambda w_0 - b_0}{a_0}, \quad Q_0(\lambda) = 0, \quad Q_1(\lambda) = \frac{1}{a_0}. \tag{2.6}$$

The function $P_n(\lambda)$ is a polynomial of degree n in λ and is called a *polynomial of the first kind*, while $Q_n(\lambda)$ is a polynomial of degree $n - 1$ in λ and is called a *polynomial of the second kind*.

Note that $P(\lambda)$ is a solution of the equation $(Jy)_n = \lambda w_n y_n$, but $Q(\lambda)$ is not: $(JQ)_n = \lambda w_n Q_n$ for $1 \leqslant n$, but $(JQ)_0 = 1 \neq 0 = \lambda Q_0$. The equation $(Jy)_n = \lambda w_n y_n$ is equivalent to (2.5) for $n \in \mathbb{N}$ and under the boundary condition $y_{-1} = 0$. The *Wronskian* of two solutions $y = \{y_n\}$ and $z = \{z_n\}$ $(n \in \mathbb{N})$ of (2.5) is defined to be $W_n(y, z) := a_n(y_n z_{n+1} - y_{n+1} z_n)$, so that $W_n(y, z) = [y, \overline{z}]_n$ $(n \in \mathbb{N})$. The Wronskian of two solutions of (2.5) does not depend on n, and two solutions of this equation are linearly independent if and only if their Wronskian is nonzero. It follows from the conditions (2.6) and the constancy of the Wronskian that $W_n(P, Q) = 1$ $(n \in \mathbb{N})$. Consequently, $P(\lambda)$ and $Q(\lambda)$ form a fundamental system of solutions of (2.5).

We assume that the minimal symmetric operator L_0 has defect index $(1, 1)$, so that the Weyl limit-circle case holds for the expression ℓy (see [1]–[4], [6], [16]). Since L_0 has defect index $(1, 1)$, $P(\lambda)$ and $Q(\lambda)$ belong to $\ell^2_w(\mathbb{N})$ for all $\lambda \in \mathbb{C}$.

Let $u = P(0)$ and $v = Q(0)$, so that $u = \{u_n\}$ and $v = \{v_n\}$ $(n \in \mathbb{N})$ are solutions of (2.5) with $\lambda = 0$ that satisfy the initial conditions

$$u_0 = 1, \quad u_1 = -\frac{b_0}{a_0}, \quad v_0 = 0, \quad v_1 = \frac{1}{a_0}. \tag{2.7}$$

We have $u, v \in \ell^2_w(\mathbb{N})$; what is more, $u, v \in D$, and

$$(Ju)_n = 0 \ \ (n \in \mathbb{N}), \quad (Jv)_0 = 1, \quad (Jv)_n = 0, \ \ n \geqslant 1. \tag{2.8}$$

Lemma 2.1. *For arbitrary $\alpha, \beta \in \mathbb{C}$, there exists a vector $y \in D$ satisfying*

$$[y, u]_\infty = \alpha, \quad [y, v]_\infty = \beta. \tag{2.9}$$

Proof. Let f be an arbitrary vector in $\ell^2_w(\mathbb{N})$ satisfying

$$(f, u) = -\alpha, \quad (f, v) = -\beta. \tag{2.10}$$

Such vector f exists, even among linear combinations of u and v. Indeed, if $f = c_1u + c_2v$, then the conditions (2.10) are a system of equations in the constants c_1 and c_2 whose determinant is the Gram determinant of the linearly independent vectors u and v, and hence is nonzero. □

Denote by $y = \{y_n\}$ the solution of $(Jy)_n = w_n f_n$ $(n \in \mathbb{N})$ satisfying $y_0 = 0$. This solution can be expressed by

$$y_n = \sum_{j=0}^{n} (u_j v_n - u_n v_j) w_j f_j \quad (n \in \mathbb{N}),$$

and hence belongs to $\ell^2_w(\mathbb{N})$.

We show that y is the desired vector. Indeed, applying (2.3) to f and u, v we obtain

$$\begin{aligned} (f, u) &= (Ly, u) = -[y, u]_\infty + (y, Lu), \\ (f, v) &= (Ly, v) = -[y, v]_\infty + (y, Lv). \end{aligned} \tag{2.11}$$

But $Lu = 0$, and thus $(y, Lu) = 0$; moreover, in view of the condition $y_0 = 0$ and the third relation in (2.8) we also have $(y, Lv) = 0$. Therefore, (2.9) follows from (2.10) and (2.11). The lemma is proved. □

Since the vectors $u = \{u_n\}$ and $v = \{v_n\}$ $(n \in \mathbb{N})$ are real and since $[u, v]_n = 1$ $(n \in \mathbb{N})$, the following assertion can be verified immediately on the basis of the (2.1).

Lemma 2.2. *For arbitrary vectors $y = \{y_n\} \in D$ and $z = \{z_n\} \in D$*

$$[y, z]_n = [y, u]_n [\overline{z}, v]_n - [y, v]_n [\overline{z}, y]_n \quad (n \in \mathbb{N} \cup \{\infty\}). \tag{2.12}$$

Then we have

Theorem 2.3. *The domain D_0 of the operator L_0 consists precisely of those vectors $y \in D$ satisfying the following boundary conditions*

$$[y, u]_\infty = [y, v]_\infty = 0. \tag{2.13}$$

Proof. As noted above, the domain D_0 of L_0 coincides with the set of all vectors $y \in D$ satisfying (2.4). By virtue of Lemma 2.2, (2.4) is equivalent to

$$[y, u]_\infty [\bar{z}, v]_\infty - [y, v]_\infty [\bar{z}, u]_\infty = 0. \tag{2.14}$$

Further, by Lemma 2.1, the numbers $[v, z]_\infty$ and $[u, z]_\infty$ $(z \in D)$ can be arbitrary. Therefore, the equality (2.14) for all $z \in D$ is possible if and only if the conditions (2.13) hold. The theorem is proved. □

The triple $(\mathscr{H}, \Gamma_1, \Gamma_2)$, where $\mathscr{H}$ is a Hilbert space and Γ_1 and Γ_2 are linear mappings of $D(A^*)$ into $\mathscr{H}$, is called ([5], [8], [9]) a *space of boundary values* of a closed symmetric operator A, acting in a Hilbert space H with equal (finite or infinite) defect index if

1) for any $f, g \in D(A^*)$,

$$(A^* f, g)_H - (f, A^* g)_H = (\Gamma_1 f, \Gamma_2 g)_{\mathscr{H}} - (\Gamma_2 f, \Gamma_1 g)_{\mathscr{H}};$$

2) for any $F_1, F_2 \in \mathscr{H}$ there exists a vector $f \in D(A^*)$ such that $\Gamma_1 f = F_1$, $\Gamma_2 f = F_2$.

In our case, we denote by Γ_1 and Γ_2 the linear mappings of D into $\mathbb{C}$ defined by

$$\Gamma_1 y = [y, v]_\infty, \quad \Gamma_2 f = [y, u]_\infty \quad (y \in D). \tag{2.15}$$

Then we have

Theorem 2.4. *The triple $(\mathbb{C}, \Gamma_1, \Gamma_2)$ defined by (2.15) is the space of boundary values of the operator L_0.*

Proof. By Lemma 2.1, for arbitrary $y, z \in D$ we have

$$\begin{aligned}(Ly, z) - (y, Lz) = -[y, z]_\infty &= [y, v]_\infty [\bar{z}, u]_\infty - [y, u]_\infty [\bar{z}, v]_\infty \\ &= (\Gamma_1 y, \Gamma_2 z) - (\Gamma_2 y, \Gamma_1 z),\end{aligned}$$

i.e., the first requirement of the definition of the space of boundary values is valid. Its second requirement is valid due to Lemma 2.1. The theorem is proved. □

By [5], [8, Theorem 1.6, p. 156], [9], Theorem 2.4 implies the following

Theorem 2.5. *Every maximal dissipative (accretive) extension $\mathbf{L}_h$ of L_0 is determined by the equality $\mathbf{L}_h y = Ly$ on the vectors y in D satisfying the boundary condition*

$$[y,v]_\infty - h[y,u]_\infty = 0, \tag{2.16}$$

where $\operatorname{Im} h \geqslant 0$ or $h=\infty$ ($\operatorname{Im} h \leqslant 0$ or $h=\infty$). Conversely, for an arbitrary h with $\operatorname{Im} h \geqslant 0$ or $h=\infty$ ($\operatorname{Im} h \leqslant 0$ or $h=\infty$), the boundary condition (2.16) *determines a maximal dissipative (accretive) extension on L_0. The selfadjoint extensions of L_0 are obtained precisely when h is a real number or infinity. For $h=\infty$ the condition* (2.16) *should be replaced by $[y,u]_\infty = 0$.*

3. Selfadjoint dilation of the dissipative operator

In the sequel we shall study the maximal dissipative operators $\mathbf{L}_h$ ($\operatorname{Im} h > 0$) generated by the expression $\ell(y)$ and the boundary condition (2.16).

Let us add to the space $H := \ell^2_w(\mathbb{N})$ the 'incoming' and 'outgoing' channels $D_- := L^2(-\infty,0)$ and $D_+ := L^2(0,\infty)$. We form the *main Hilbert space of the dilation* $\mathscr{H} = L^2(-\infty,0)\oplus H \oplus L^2(0,\infty)$, and in $\mathscr{H}$ we consider the operator $\mathscr{L}_h$ generated by the expression

$$\mathscr{L}\langle \varphi_-, y, \varphi_+\rangle = \left\langle \mathrm{i}\frac{\mathrm{d}\varphi_-}{\mathrm{d}\xi}, \ell(y), \mathrm{i}\frac{\mathrm{d}\varphi_+}{\mathrm{d}\varsigma}\right\rangle \tag{3.1}$$

on the set $D(\mathscr{L}_h)$ of vectors $\langle \varphi_-, y, \varphi_+\rangle$ satisfying the conditions: $\varphi_- \in W^1_2(-\infty,0)$, $\varphi_+ \in W^1_2(0,\infty)$, $y\in D$,

$$[y,v]_\infty - h[y,u]_\infty = \alpha\varphi_-(0), \quad [y,v]_\infty - \overline{h}[y,u]_\infty = \alpha\varphi_+(0), \tag{3.2}$$

where $\alpha^2 := \operatorname{Im} h$, $\alpha>0$ and W^1_2 is the Sobolev space. Then we have

Theorem 3.1. *The operator $\mathscr{L}_h$ is selfadjoint in $\mathscr{H}$ and is a selfadjoint dilation of the operator $\mathbf{L}_h$.*

P r o o f. Suppose that $f,g\in D(\mathscr{L}_h)$, $f=\langle \varphi_-, y, \varphi_+\rangle$ and $g=\langle \psi_-, z, \psi_+\rangle$. Then, integrating by parts and using (2.13), we get

$$\begin{aligned}(\mathscr{L}_h f, g)_{\mathscr{H}} &= \int_{-\infty}^0 \mathrm{i}\varphi_-'\overline{\psi}_-\,\mathrm{d}\xi + (\ell y, z)_H + \int_0^\infty \mathrm{i}\varphi_+'\overline{\psi}_+\,\mathrm{d}\xi \\ &= \mathrm{i}\varphi_-(0)\overline{\psi}_-(0) - \mathrm{i}\varphi_+(0)\overline{\psi}_+(0) - [y,z]_\infty + (f,\mathscr{L}_h g)_{\mathscr{H}}.\end{aligned} \tag{3.3}$$

Next, using the boundary conditions (3.2) for the components of the vectors f, g and the relation (2.12), we see by direct computation that

$$\mathrm{i}\varphi_-(0)\overline{\psi}_-(0) - \mathrm{i}\varphi_+(0)\overline{\psi}_+(0) - [y, z]_\infty = 0.$$

Thus, $\mathscr{L}_h$ is symmetric. Therefore, to prove that $\mathscr{L}_h$ is selfadjoint, it suffices to show that $\mathscr{L}_h^* \subseteq \mathscr{L}_h$.

Take $g = \langle \psi_-, z, \psi_+ \rangle \in D(\mathscr{L}_h^*)$. Let $\mathscr{L}_h^* g = g^* = \left\langle \psi_-^*, z^*, \psi_+^* \right\rangle \in \mathscr{H}$, so that

$$(\mathscr{L}_h f, g)_{\mathscr{H}} = (f, g^*)_{\mathscr{H}}, \quad \forall\, f \in D(\mathscr{L}_h). \tag{3.4}$$

By choosing vectors with suitable components as the $f \in D(\mathscr{L}_h)$ in (3.4), it is not difficult to show that $\psi_- \in W_2^1(-\infty, 0)$, $\psi_+ \in W_2^1(0, \infty)$, $z \in D$ and $g^* = \mathscr{L} g$, with the operation $\mathscr{L}$ defined by (3.1). Consequently, (3.4) takes the form

$$(\mathscr{L} f, g)_{\mathscr{H}} = (f, \mathscr{L} g)_{\mathscr{H}}, \quad \forall\, f \in D(\mathscr{L}_h).$$

Therefore, the sum of the integral terms in the bilinear form $(\mathscr{L} f, g)_{\mathscr{H}}$ must be equal to zero:

$$\mathrm{i}\varphi_-(0)\overline{\psi}_-(0) - \mathrm{i}\varphi_+(0)\overline{\psi}_+(0) - [y, z]_\infty = 0 \tag{3.5}$$

for all $f = \langle \varphi_-, y, \varphi_+ \rangle \in D(\mathscr{L}_h)$. Further, solving the boundary conditions (3.2) for $[y, u]_\infty$ and $[y, v]_\infty$ we find that

$$\begin{aligned} [y, u]_\infty &= \frac{1}{\mathrm{i}\alpha}(\varphi_+(0) - \varphi_-(0)), \\ [y, v]_\infty &= \alpha\varphi_-(0) + \frac{h}{\mathrm{i}\alpha}(\varphi_+(0) - \varphi_-(0)). \end{aligned}$$

Therefore, using (2.12), we find that (3.5) is equivalent to the equality

$$\begin{aligned} \mathrm{i}\varphi_-(0)\overline{\psi}_-(0) - \mathrm{i}\varphi_+(0)\overline{\psi}_+(0) &= -[y, z]_\infty \\ &= \frac{1}{\mathrm{i}\alpha}(\varphi_+(0) - \varphi_-(0))[\overline{z}, v]_\infty \\ &\quad - \left[\alpha\varphi_-(0) + \frac{h}{\mathrm{i}\alpha}(\varphi_+(0) - \varphi_-(0))\right][\overline{z}, u]_\infty. \end{aligned}$$

Since the values $\varphi_\pm(0)$ can be arbitrary complex numbers, a comparison of the coefficients of $\varphi_\pm(0)$ on both sides of the last equality gives us that the vector $g = \langle \psi_-, z, \psi_+ \rangle$ satisfies the boundary conditions

$$[z, v]_\infty - h[z, u]_\infty = \alpha\psi_-(0), \quad [z, v]_\infty - \overline{h}[z, u]_\infty = \alpha\psi_+(0).$$

Consequently, the inclusion $\mathscr{L}_h^* \subseteq \mathscr{L}_h$ is established, and hence $\mathscr{L}_h = \mathscr{L}_h^*$.

The selfadjoint operator $\mathscr{L}_h$ generates in $\mathscr{H}$ a unitary group $U_t = \exp[\mathrm{i}\mathscr{L}_h t]$ ($t \in \mathbb{R} := (-\infty, \infty)$). Denote by $P\colon \mathscr{H} \to H$ and $P_1\colon H \to \mathscr{H}$ the mappings acting according to the formulae $P\colon \langle \varphi_-, y, \varphi_+ \rangle \to y$ and $P_1\colon y \to \langle 0, y, 0 \rangle$. Let $Z_t = PU_tP_1$ ($t \geqslant 0$). The family $\{Z_t\}$ ($t \geqslant 0$) of operators is a strongly continuous semigroup of completely nonunitary contractions on H. Denote by A_h the generator of this semigroup: $A_h y = \lim_{t\to+0} (\mathrm{i}t)^{-1}(Z_t y - y)$. The domain of A_h consists of all the vectors for which the limit exists. The operator A_h is a maximal dissipative. The operator $\mathscr{L}_h$ is called the *selfadjoint dilation* of A_h [10], [12]. We show that $A_h = \mathbf{L}_h$, and hence $\mathscr{L}_h$ is a selfadjoint dilation of $\mathbf{L}_h$. To do this, we first verify the equality [10], [12]

$$P(\mathscr{L}_h - \lambda I)^{-1} P_1 y = (\mathbf{L}_h - \lambda I)^{-1} y, \quad y \in H, \quad \operatorname{Im}\lambda < 0. \tag{3.6}$$

With this goal, we set $(\mathscr{L}_h - \lambda I)^{-1} P_1 y = g = \langle \psi_-, z, \psi_+ \rangle$. Then $(\mathscr{L}_h - \lambda I)g = P_1 y$, and hence $Lz - \lambda z = y$, $\psi_-(\xi) = \psi_-(0)\mathrm{e}^{-\mathrm{i}\lambda\xi}$ and $\psi_+(\varsigma) = \psi_+(0)\mathrm{e}^{-\mathrm{i}\lambda\varsigma}$. Since $g \in D(\mathscr{L}_h)$, and hence $\psi_- \in L^2(-\infty, 0)$; it follows that $\psi_-(0) = 0$, and consequently, z satisfies the boundary condition $[z, v]_\infty - h[z, u]_\infty = 0$. Therefore, $z \in D(\mathbf{L}_h)$, and since a point λ with $\operatorname{Im}\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $z = (\mathbf{L}_h - \lambda I)^{-1} y$. We remark that $\psi_+(0)$ is found from the formula

$$\psi_+(0) = \alpha^{-1}([z, v]_\infty - \overline{h}[z, u]_\infty).$$

Thus,

$$(\mathscr{L}_h - \lambda I)^{-1} P_1 y = \left\langle 0, (\mathbf{L}_h - \lambda I)^{-1} y, \alpha^{-1}([z, v]_\infty - \overline{h}[z, u]_\infty)\mathrm{e}^{-\mathrm{i}\lambda\varsigma} \right\rangle$$

for $y \in H$ and $\operatorname{Im}\lambda < 0$. Upon applying the mapping P, we obtain (3.6).

It is now easy to show that $A_h = \mathbf{L}_h$. Indeed by (3.6),

$$\begin{aligned}(\mathbf{L}_h - \lambda I)^{-1} &= P(\mathscr{L}_h - \lambda I)^{-1} P_1 = -\mathrm{i}P \int_0^\infty U_t \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{d}t P_1 \\ &= -\mathrm{i} \int_0^\infty Z_t \mathrm{e}^{-\mathrm{i}\lambda t}\, \mathrm{d}t = (A_h - \lambda I)^{-1}, \quad \operatorname{Im}\lambda < 0,\end{aligned}$$

from which it is clear that $\mathbf{L}_h = A_h$. Theorem 3.1 it is proved. □

4. Scattering theory of dilation and the functional model of dissipative operator

The unitary group $U_t = \exp[\mathrm{i}\mathscr{L}_h t]$ $(t \in \mathbb{R})$ has an important property which makes it possible to apply to it the Lax-Phillips scheme [11]. Namely, it has incoming and outgoing subspaces $D_- = \langle L^2(-\infty, 0), 0, 0\rangle$ and $D_+ = \langle 0, 0, L^2(0,\infty)\rangle$ possessing the following properties

(1) $U_t D_- \subset D_-$, $t \leqslant 0$ and $U_t D_+ \subset D_+$, $t \geqslant 0$;

(2) $\bigcap\limits_{t\geqslant 0} U_t D_- = \bigcap\limits_{t\geqslant 0} U_t D_+ = \{0\}$;

(3) $\overline{\bigcup\limits_{t\geqslant 0} U_t D_-} = \overline{\bigcup\limits_{t\leqslant 0} U_t D_+} = \mathscr{H}$;

(4) $D_- \perp D_+$.

Property (4) is obvious. To prove property (1) for D_+ (the proof for D_- is similar), we set $R_\lambda = (\mathscr{L}_h - \lambda I)^{-1}$, then for all λ with $\operatorname{Im}\lambda < 0$ and for any $f = \langle 0, 0, \varphi_+\rangle \in D_+$, we have

$$R_\lambda f = \left\langle 0, 0, -\mathrm{i}\mathrm{e}^{-\mathrm{i}\lambda\xi}\int_0^\xi \mathrm{e}^{-\mathrm{i}\lambda s}\varphi_+(s)\,\mathrm{d}s\right\rangle;$$

from this it follows that $R_\lambda f \in D_+$, therefore, if $g \perp D_+$, then

$$0 = (R_\lambda f, g)_{\mathscr{H}} = -\mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}\lambda t}(U_t f, g)_{\mathscr{H}}\,\mathrm{d}\lambda, \quad \operatorname{Im}\lambda < 0.$$

From this it follows that $(U_t f, g)_{\mathscr{H}} = 0$ for all $t \geqslant 0$. Hence $U_t D_+ \subset D_+$ for $t \geqslant 0$, and property (1) has thus been proved.

To prove property (2), we denote by $P^+\colon \mathscr{H} \to L^2(0,\infty)$ and $P_1^+\colon L^2(0,\infty) \to D_+$ the mappings acting according to the formulae $P^+\colon \langle\varphi_-, u, \varphi_+\rangle \to \varphi_+$ and $P_1^+\colon \varphi \to \langle 0, 0, \varphi\rangle$, respectively. We note that the semigroup of isometries $V_t = P^+ U_t P_1^+$, $t \geqslant 0$ is a one-sided shift in $L^2(0,\infty)$. Indeed, the generator of the semigroup of the one-sided shift V_t in $L^2(0,\infty)$ is the differential operator $\mathrm{i}(\mathrm{d}/\mathrm{d}\xi)$ with the boundary condition $\varphi(0) = 0$. On the other hand, the generator A of the semigroup of isometries U_t^+, $t \geqslant 0$, is the operator

$$A\varphi = P^+\mathscr{L}_h P_1^+ f = P^+\mathscr{L}_h\langle 0, 0, \varphi\rangle = P^+\left\langle 0, 0, \mathrm{i}\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right\rangle = \mathrm{i}\frac{\mathrm{d}\varphi}{\mathrm{d}\xi},$$

where $\varphi \in W_2^1(0,\infty)$ and $\varphi(0) = 0$. But since a semigroup is uniquely determined by its generator, it follows that $U_t^+ = V_t$, and hence

$$\bigcap_{t\geqslant 0} U_t D_+ = \left\langle 0, 0, \bigcap_{t\geqslant 0} V_t L^2(0,\infty)\right\rangle = t\{0\},$$

i.e. property (2) is proved. $\square$

In this scheme of the Lax-Phillips scattering theory, the scattering matrix is defined in terms of the theory of spectral representations. We proceed to their construction. Along the way, we also prove property (3) of the incoming and outgoing subspaces.

Lemma 4.1. *The operator* $\mathbf{L}_h$ *is totally nonselfadjoint (simple).*

Proof. Let $H' \subset H$ be a nontrivial subspace in which $\mathbf{L}_h$ induces a selfadjoint operator $\mathbf{L}'_h$ with domain $D(\mathbf{L}'_h) = H' \cap D(\mathbf{L}_h)$. If $y \in D(\mathbf{L}'_h)$, then $y \in D(\mathbf{L}'^{*}_h)$ and $\operatorname{Im}(\mathbf{L}_h y, y) = 0$. Since we get from $\operatorname{Im}(\mathbf{L}_h y, y) = (\operatorname{Im} h)|[y,u]_\infty|^2 = 0$ that $[y,u]_\infty = 0$. This and the boundary condition (2.16) also imply the equality $[y,v]_\infty = 0$. Thus,

$$[y,u]_\infty = [y,v]_\infty = 0, \quad y \in D(\mathbf{L}'_h). \tag{4.1}$$

Denote by $\mathbf{L}_0$ and $\mathbf{L}_\infty$ the selfadjoint extensions of L_0 determined by the boundary conditions $[y,v]_\infty = 0$ and $[y,u]_\infty = 0$, respectively. By (4.1) $D(\mathbf{L}'_h)$ is contained in each of $D(\mathbf{L}_0)$ and $D(\mathbf{L}_\infty)$. Suppose that λ belongs to the spectrum of $\mathbf{L}'_h$. Then λ is real, and there exists a sequence of vectors $f_n \in D(\mathbf{L}'_h)$ such that $\|f_n\| = 1$ and $\|\mathbf{L}_h f_n - \lambda f_n\| \to 0$ as $n \to \infty$. This implies that λ belongs also to the spectra of the operators $\mathbf{L}_0$ and $\mathbf{L}_\infty$. Since the spectra of $\mathbf{L}_0$ and $\mathbf{L}_\infty$ are purely discrete, λ is an eigenvalue of these operators. The corresponding eigenvectors differ from $P(\lambda)$ only by a scalar factor, because $P(\lambda)$ is the only linearly independent solution of the equations $(Jy)_n = \lambda w_n y_n$ $(n \in \mathbb{N})$. Consequently, $[P(\lambda),u]_\infty = [P(\lambda),v]_\infty = 0$. Further, from (2.12) with $y = P(\lambda)$ and $z = Q(\lambda)$ we have

$$[P(\lambda),Q(\lambda)]_\infty = [P(\lambda),u]_\infty[Q(\lambda),v]_\infty - [P(\lambda),v]_\infty[Q(\lambda),u]_\infty. \tag{4.2}$$

The right-hand side is equal to 0 in view of (4.2), while the left-hand side, being the value of the Wronskian of the solutions $P(\lambda)$ and $Q(\lambda)$ of (2.5), is equal to 1. This contradiction shows that $H' = \{0\}$. The lemma is proved. □

We set $\mathscr{H}_- = \overline{\bigcup_{t \geqslant 0} U_t D_-}$, $\mathscr{H}_+ = \overline{\bigcup_{t \leqslant 0} U_t D_+}$.

Lemma 4.2. $\mathscr{H}_- + \mathscr{H}_+ = \mathscr{H}$.

Proof. Taking into account property (1) of the subspace D_+, it is easy to show that the subspace $\mathscr{H}' = \mathscr{H} \ominus (\mathscr{H}_- + \mathscr{H}_+)$ is invariant relative to the group $\{U_t\}$ and has the form $\mathscr{H}' = \langle 0, H', 0\rangle$, where H' is a subspace in H. Therefore, if the subspace $\mathscr{H}'$ (and hence also H') were nontrivial, then the unitary group $\{U'_t\}$, restricted to this subspace, would be a unitary part of the group $\{U_t\}$, and hence the restriction $\mathbf{L}'_h$ of $\mathbf{L}_h$ to H' would be a selfadjoint operator in H'. From the simplicity of the operator $\mathbf{L}_h$ it follows that $H' = \{0\}$, i.e. $\mathscr{H}' = \{0\}$. The lemma is proved. □

Let us adopt the following notation $n_h(\lambda) := [P(\lambda), u]_\infty - h[P(\lambda), v]_\infty$,

$$n(\lambda) := \frac{[P(\lambda), u]_\infty}{[P(\lambda), v]_\infty}, \tag{4.3}$$

$$S_h(\lambda) := \frac{n_h(\lambda)}{n_{\overline{h}}(\lambda)} = \frac{n(\lambda) - h}{n(\lambda) - \overline{h}}. \tag{4.4}$$

From (4.3), it follows that $n(\lambda)$ is a meromorphic function on the complex plane $\mathbb{C}$ with a countable number of poles on the real axis. Further, it is possible to show that the function $n(\lambda)$ possesses the following properties: $\operatorname{Im}\lambda \cdot \operatorname{Im} n(\lambda) > 0$ for $\operatorname{Im}\lambda \neq 0$ and $n(\overline{\lambda}) = \overline{n(\lambda)}$ for $\lambda \in \mathbb{C}$ with the exception of the real poles of $n(\lambda)$.

Let

$$\mathscr{U}_\lambda^-(\xi, \zeta) = \Big\langle \mathrm{e}^{-\mathrm{i}\lambda\xi}, \frac{\alpha}{n_h(\lambda)} P(\lambda), \overline{S}_h(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\zeta} \Big\rangle.$$

We note that the vectors $U_\lambda^-(\xi, \zeta)$ for real λ do not belong to the space $\mathscr{H}$. However, $U_\lambda^-(\xi, \zeta)$ satisfy the equation $\mathscr{L}U = \lambda U$ ($\lambda \in \mathbb{R}$) and the corresponding boundary conditions for the operator $\mathscr{L}_h$.

With the help of the vector $U_\lambda^-(\xi, \zeta)$, we define the transformation $\mathscr{F}_-\colon\ f \to \tilde{f}_-(\lambda)$ by $(\mathscr{F}_- f)(\lambda) := \tilde{f}_-(\lambda) := 1/\sqrt{2\pi}\,(f, U_\lambda^-)_{\mathscr{H}}$ on the vector $f = \langle \varphi_-, y, \varphi_+ \rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\zeta)$ are smooth, compactly supported functions, and $y = \{y_n\}$ ($n \in \mathbb{N}$) is a finite nonzero sequence.

Lemma 4.3. *The transformation $\mathscr{F}_-$ maps H_- isometrically onto $L^2(\mathbb{R})$. For all vectors $f, g \in \mathscr{H}_-$ the Parseval equality and the inversion formula hold:*

$$(f, g)_{\mathscr{H}} = (\tilde{f}_-, \tilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-^-(\lambda)}\,\mathrm{d}\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}_-(\lambda) U_\lambda^-\,\mathrm{d}\lambda,$$

where $\tilde{f}_-(\lambda) = (\mathscr{F}_- f)(\lambda)$ and $\tilde{g}_-(\lambda) = (\mathscr{F}_- g)(\lambda)$.

P r o o f. For $f, g \in D_-$, $f = \langle \varphi_-, 0, 0 \rangle$, $g = \langle \psi_-, 0, 0 \rangle$, we have

$$\tilde{f}_-(\lambda) := \frac{1}{\sqrt{2\pi}}(f, U_\lambda^-)_{\mathscr{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \varphi_-(\xi)\mathrm{e}^{\mathrm{i}\lambda\xi}\,\mathrm{d}\xi \in H_-^2,$$

and in view of the usual Parseval equality for Fourier integrals

$$(f, g)_{\mathscr{H}} = \int_{-\infty}^{0} \varphi_-(\xi)\overline{\psi}_-(\xi)\,\mathrm{d}\xi = \int_{-\infty}^{\infty} \tilde{f}_-(\lambda)\overline{\tilde{g}_-(\lambda)}\,\mathrm{d}\lambda = (\mathscr{F}_- f, \mathscr{F}_- g)_{L^2}.$$

Here and below, $H^2_\pm$ denote the Hardy classes in $L^2(\mathbb{R})$ consisting of the functions analytically extendable to the upper and lower half-planes, respectively.

We now extend the Parseval equality to the whole of $\mathscr{H}_-$. With this goal, we consider in $\mathscr{H}_-$ the dense set $\mathscr{H}'_-$ of vectors obtained as follows from the smooth, compactly supported functions in D_-: $f \in \mathscr{H}'_-$ if $f = U_T f_0$, $f_0 = \langle \varphi_-, 0, 0\rangle$, $\varphi_- \in C_0^\infty(-\infty, 0)$, where $t = T_f$ is a non-negative number (depending on f). In this case, if $f, g \in \mathscr{H}'_-$, then for $T > T_f$ and $T > T_g$ we have $U_{-T}f, U_{-T}g \in D_-$ and, moreover, the first components of these vectors belong to $C_0^\infty(-\infty, 0)$. Therefore, since the operators U_t $(t \in \mathbb{R})$ are unitary, by the equality

$$\mathscr{F}_- U_t f = (U_t f, U_\lambda^-)_{\mathscr{H}} = \mathrm{e}^{\mathrm{i}\lambda t}(f, U_\lambda^-)_{\mathscr{H}} = \mathrm{e}^{\mathrm{i}\lambda t}\mathscr{F}_- f,$$

we have

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (U_{-T}f, U_{-T}g)_{\mathscr{H}} = (\mathscr{F}_- U_{-T}f, \mathscr{F}_- U_{-T}g)_{L^2} \\ &= (\mathrm{e}^{-\mathrm{i}\lambda T}\mathscr{F}_- f, \mathrm{e}^{-\mathrm{i}\lambda T}\mathscr{F}_- g)_{L^2} = (\mathscr{F}_- f, \mathscr{F}_- g)_{L^2}.\end{aligned} \tag{4.5}$$

Taking closure in (4.5), we obtain the Parseval equality for the whole space $\mathscr{H}_-$. The inversion formula follows from the Parseval equality if all integrals in it are understood as limits in the mean of integrals over finite intervals. Finally,

$$\mathscr{F}_-\mathscr{H}_- = \overline{\bigcup_{t \geqslant 0} \mathscr{F}_- U_t D_-} = \overline{\bigcup_{t\geqslant 0} \mathrm{e}^{-\mathrm{i}\lambda t} H^2_-} = L^2(\mathbb{R}),$$

i.e. $\mathscr{F}_-$ maps $\mathscr{H}_-$ onto the whole of $L^2(\mathbb{R})$. The lemma is proved. □

We set

$$U_\lambda^+(\xi, \varsigma) = \left\langle S_h(\lambda)\mathrm{e}^{-\mathrm{i}\lambda\xi}, \frac{\alpha}{n_{\bar{h}}(\lambda)}P(\lambda), \mathrm{e}^{-\mathrm{i}\lambda\varsigma}\right\rangle.$$

We note that the vectors $U_\lambda^+(\xi, \varsigma)$ for real λ do not belong to the space $\mathscr{H}$. However, $U_\lambda^+(\xi, \varsigma)$ satisfy the equation $\mathscr{L}U = \lambda U$ $(\lambda \in \mathbb{R})$ and the corresponding boundary conditions for the operator $\mathscr{L}_h$. With the help of the vector $U_\lambda^+(\xi, \varsigma)$, we define the transformation $\mathscr{F}_+ \colon f \to \tilde{f}_+(\lambda)$ on vectors $f = \langle \varphi_-, y, \varphi_+\rangle$ in which $\varphi_-(\xi)$, $\varphi_+(\varsigma)$ are smooth, compactly supported functions, and $y = \{y_n\}$ $(n \in \mathbb{N})$ is a finite nonzero sequence, by setting $(\mathscr{F}_+ f)(\lambda) := \tilde{f}_+(\lambda) := 1/\sqrt{2\pi}\,(f, U_\lambda^+)_{\mathscr{H}}$. The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. *The transformation $\mathscr{F}_+$ maps $\mathscr{H}_+$ isometrically onto $L^2(\mathbb{R})$, and for all vectors $f, g \in \mathscr{H}_+$, the Parseval equality and the inversion formula hold:*

$$(f,g)_{\mathscr{H}} = (\tilde{f}_+, \tilde{g}_+)_{L^2} = \int_{-\infty}^{\infty} \tilde{f}_+(\lambda)\overline{\tilde{g}_+(\lambda)}\,\mathrm{d}\lambda, \qquad f = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}_+(\lambda)U_\lambda^+\,\mathrm{d}\lambda,$$

where $\tilde{f}_+(\lambda) = (\mathscr{F}_+ f)(\lambda)$ and $\tilde{g}_+(\lambda) = (\mathscr{F}_+ g)(\lambda)$.

According to (4.3), the function $S_h(\lambda)$ satisfies $|S_h(\lambda)| = 1$ for $\lambda \in \mathbb{R}$. Therefore, it follows from the explicit formula for the vectors U_λ^+ and U_λ^- that

$$U_\lambda^- = \overline{S}_h(\lambda)U_\lambda^+ \quad (\lambda \in \mathbb{R}). \tag{4.6}$$

It follows from Lemmas 3.3 and 3.4 that $\mathscr{H}_- = \mathscr{H}_+$. Together with Lemma 3.2 this shows that $\mathscr{H} = \mathscr{H}_- = \mathscr{H}_+$, and property (3) for the U_t above has been established for the incoming and outgoing subspaces.

Thus, the transformation $\mathscr{F}_-$ maps $\mathscr{H}$ isometrically onto $L^2(\mathbb{R})$ with the subspace D_- mapped onto H_-^2 and the operators U_t passing into the operators of multiplication by $\mathrm{e}^{\mathrm{i}\lambda t}$. In other words, $\mathscr{F}_-$ is the incoming spectral representation for the group $\{U_t\}$. Similarly, $\mathscr{F}_+$ is the outgoing spectral representation for $\{U_t\}$. It follows from (4.6) that the passage from the $\mathscr{F}_+$-representation of a vector $f \in \mathscr{H}$ to its $\mathscr{F}_-$-representation is realized by multiplication of the function $S_h(\lambda)$: $\tilde{f}_-(\lambda) = S_h(\lambda)\tilde{f}_+(\lambda)$. According to [11], the scattering matrix (function) of the group $\{U_t\}$ with respect to the subspaces D_- and D_+, is the coefficient by which the $\mathscr{F}_-$-representation of a vector $f \in \mathscr{H}$ must be multiplied in order to get the corresponding $\mathscr{F}_+$-representation: $\tilde{f}_+(\lambda) = \overline{S}_h(\lambda)\tilde{f}_-(\lambda)$. According to [11], we have now proved the following theorem.

Theorem 4.5. *The function $\overline{S}_h(\lambda)$ is the scattering matrix of the group $\{U_t\}$ (of the selfadjoint operator $\mathscr{L}_h$).*

Let $S(\lambda)$ be an arbitrary inner function [12] on the upper half-plane. Define $\mathscr{K} = H_+^2 \ominus SH_+^2$. Then $\mathscr{K} \neq \{0\}$ is a subspace of the Hilbert space H_+^2. We consider the semigroup of the operators Z_t ($t \geqslant 0$) acting in $\mathscr{K}$ according to the formula $Z_t\varphi = P[\mathrm{e}^{\mathrm{i}\lambda t}\varphi]$, $\varphi := \varphi(\lambda) \in \mathscr{K}$, where P is the orthogonal projection from H_+^2 onto $\mathscr{K}$. The generator of the semigroup $\{Z_t\}$ is denoted by T: $T\varphi = \lim_{t\to+0}(\mathrm{i}t)^{-1}(Z_t\varphi - \varphi)$, which is a dissipative operator acting in $\mathscr{K}$ and having domain $D(T)$ consisting of all functions $\varphi \in \mathscr{K}$, such that the limit exists. The operator T is called a *model dissipative operator*. (We remark that this model dissipative operator, which is associated with the names of Lax and Phillips [11], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [12].) The basic assertion is that $S(\lambda)$ is the characteristic function of the operator T.

Let $K = \langle 0, H, 0\rangle$ so that $\mathscr{H} = D_- \oplus K \oplus D_+$. It follows from the explicit form of the unitary transformation $\mathscr{F}_-$ that under the mapping $\mathscr{F}_-$,

$$\begin{gathered}\mathscr{H} \to L^2(\mathbb{R}), \quad f \to \tilde{f}_-(\lambda) = (\mathscr{F}_- f)(\lambda), \quad D_- \to H^2_-, \quad D_+ \to S_h^- H^2_+, \\ K \to H^2_+ \ominus S_h^- H^2_+, \quad U_t f \to (\mathscr{F}_- U_t \mathscr{F}_-^{-1} \tilde{f}_-)(\lambda) = \mathrm{e}^{\mathrm{i}\lambda t}\tilde{f}_-(\lambda).\end{gathered} \tag{4.7}$$

The formulas (4.7) show that the operator $\mathbf{L}_h$ is a unitarily equivalent to the model dissipative operator with characteristic function $S_h(\lambda)$. Since the characteristic functions of unitarily equivalent dissipative operators coincide [12], we have proved the following theorem.

Theorem 4.6. *The characteristic function of the dissipative operator* $\mathbf{L}_h$ *coincides with the function* $S_h(\lambda)$ *defined by* (4.4).

5. Completeness theorem for the system of eigenvectors and associated vectors of the dissipative operator

It is known that the characteristic function of a dissipative operator $\mathbf{L}_h$ carries complete information about the spectral properties of this operator [10], [12]. For example, the absence of the singular factor of the characteristic function $S_h(\lambda)$ in the factorization $S_h(\lambda) = s(\lambda)\mathscr{B}(\lambda)$ (where $\mathscr{B}(\lambda)$ is a Blaschke product) guarantees the completeness of the system of eigenvectors and associated vectors of the dissipative operator $\mathbf{L}_h$.

Theorem 5.1. *For all values of* h *with* $\operatorname{Im} h > 0$, *except possibly for a single value* $h = h^0$, *the characteristic function* $S_h(\lambda)$ *of the dissipative operator* $\mathbf{L}_h$ *is a Blaschke product and the spectrum of* $\mathbf{L}_h$ *is purely discrete and belongs to the open upper half plane. The operator* $\mathbf{L}_h$ $(h \neq h^0)$ *has a countable number of isolated eigenvalues with finite multiplicity and limit point at the infinity, and the system of eigenvectors and associated vectors of this operator is complete in* $\ell^2_w(\mathbb{N})$.

Proof. It is clear from the explicit formula (4.4) that the $S_h(\lambda)$ is an inner function in the upper half-plane and, moreover, meromorphic in the whole λ-plane. Therefore, it can be factored in the form

$$S_h(\lambda) = \mathrm{e}^{\mathrm{i}\lambda c}\mathscr{B}_h(\lambda) \quad c = c(h) > 0, \tag{5.1}$$

where $\mathscr{B}_h(\lambda)$ is a Blaschke product. It follows from (5.1) that

$$|S_h(\lambda)| \leqslant e^{-c(h)\,\mathrm{Im}\,\lambda}, \quad \mathrm{Im}\,\lambda \geqslant 0. \tag{5.2}$$

Further, expressing $n(\lambda)$ in terms of $S_h(\lambda)$, we find from (4.4) that

$$n(\lambda) = \frac{\overline{h}S_h(\lambda) - h}{S_h(\lambda) - 1}. \tag{5.3}$$

If $c(h) > 0$ for a given value of h ($\mathrm{Im}\,h > 0$), then (5.2) implies that $\lim_{t\to+\infty} S_h(it) = 0$, and then (5.3) gives us that $\lim_{t\to+\infty} n(it) = h$. Since $n(\lambda)$ does not depend on h, this implies that that $c(h)$ can be nonzero at most at a single point $h = h^0$ (and, further, $h^0 = \lim_{t\to+\infty} n(it)$). The theorem is proved. □

References

[1] *N. I. Akhiezer*: The Classical Moment Problem and Some Related Questions in Analysis. Fizmatgiz, Moscow, 1961; English transl. Oliver and Boyd, Hafner, London-New York, 1965.

[2] *F. V. Atkinson*: Discrete and Continuous Boundary Problems. Academic Press, New York, 1964.

[3] *M. Benammar, W. D. Evans*: On the Friedrichs extension of semi-bounded difference operators. Math. Proc. Camb. Philos. Soc. *116* (1994), 167–177.

[4] *Yu. M. Berezanskij*: Expansion in Eigenfunctions of Selfadjoint Operators. Naukova Dumka, Kiev, 1965; English transl. Amer. Math. Soc., Providence, 1968.

[5] *V. M. Bruk*: On a class of boundary-value problems with a spectral parameter in the boundary conditions. Mat. Sb. *100* (1976), 210–216; English transl. Mat USSR Sb. *28* (1976), 186–192.

[6] *L. S. Clark*: A spectral analysis for self-adjoint operators generated by a class of second order difference equations. J. Math. Anal. Appl. *197* (1996), 267–285.

[7] *M. L Gorbachuk, V. I. Gorbachuk and A. N. Kochubei*: The theory of extensions of symmetric operators and boundary-value problems for differential equations. Ukrain. Mat. Zh. *41* (1989), 1299–1312; English transl. Ukrainian Mat. J. *41* (1989), 1117–1129.

[8] *V. I. Gorbachuk and M. L. Gorbachuk*: Boundary Value Problems for Operator Differential Equations. Naukova Dumka, Kiev, 1984; English transl. Kluwer, Dordrecht, 1991.

[9] *A. N. Kochubei*: Extensions of symmetric operators and symmetric binary relations. Mat. Zametki *17* (1975), 41–48; English transl. Math. Notes *17* (1975), 25–28.

[10] *A. Kuzhel*: Characteristic Functions and Models of Nonself-adjoint Operators. Kluwer, Boston-London-Dordrecht, 1996.

[11] *P. D. Lax and R. S. Phillips*: Scattering Theory. Academic Press, New York, 1967.

[12] *B. Sz.-Nagy and C. Foias*: Analyse Harmonique des Operateurs de l'espace de Hilbert. Masson and Akad Kiadó, Paris and Budapest, 1967; English transl. North-Holland and Akad. Kiadó, Amsterdam and Budapest, 1970.

[13] *J. von Neumann*: Allgemeine Eigenwerttheorie Hermitischer Funktionaloperatoren. Math. Ann. *102* (1929), 49–131.

[14] *F. S. Rofe-Beketov*: Self-adjoint extensions of differential operators in space of vector-valued functions. Dokl. Akad. Nauk SSSR *184* (1969), 1034–1037; English transl. Soviet Math. Dokl. *10* (1969), 188–192.
[15] *M. M. Stone*: Linear Transformations in Hilbert Space and Their Applications to Analysis, Vol. 15. Amer. Math. Soc. Coll. Publ., Providence, 1932.
[16] *S. T. Welstead*: Boundary conditions at infinity for difference equations of limit-circle type. J. Math. Anal. Appl. *89* (1982), 442–461.
[17] *H. Weyl*: Über gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Functionen. Math. Ann. *68* (1910), 222–269.

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