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FUNCTIONALS ON FUNCTION AND SEQUENCE SPACES CONNECTED WITH THE EXPONENTIAL STABILITY OF EVOLUTIONARY PROCESSES

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Abstract. The exponential stability property of an evolutionary process is characterized in terms of the existence of some functionals on certain function spaces. Thus are generalized some well-known results obtained by Datko, Rolewicz, Littman and Van Neerven.

Keywords: evolutionary processes, uniform exponential stability

MSC 2000: 34D05, 47D06

1. Introduction

One of the most remarkable results in the theory of stability for a strongly continuous semigroup of linear operators has been obtained by Datko [1] in 1970 and it says that the semigroup $\mathscr{T} = \{T(t)\}_{t\geqslant 0}$ is uniformly exponentially stable if and only if, for each vector x from the Banach space X, the function $t \to \|T(t)x\|$ lies in $L^2(\mathbb{R}_+)$. Later, A. Pazy [see for instance 9] shows that the result remains true even if we replace $L^2(\mathbb{R}_+)$ with $L^p(\mathbb{R}_+)$, where $p \in [1,\infty)$. In 1973, R. Datko [2] generalized the results above, and proved that an evolutionary process $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ with exponential growth is uniformly exponentially stable if and only if there is $p \in [1,\infty)$ such that $\sup\limits_{s\geqslant 0}\int_s^\infty \|U(t,s)x\|^p \,\mathrm{d}t < \infty$, for each $x \in X$. This result was improved by Rolewicz in 1986 (see [10]) when he proved that if $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each strictly positive u, and $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ is an evolutionary process on X with exponential growth such that $\sup\limits_{s\geqslant 0}\int_s^\infty \varphi(\|U(t,s)x\| \,\mathrm{d}t < \infty$ for each $x \in X$ then $\mathscr{U}$ is uniformly exponentially stable. We note here that an analogous result was obtained independently by Littman [4] in 1989, in the case of C_0-semigroup, but without the assumption of continuity for φ. Jan van Neerven generalized the results above in the case of C_0-semigroups and he presented a unified treatment in terms of Banach function spaces as in [6], Theorem 3.1.5. In fact he proved that the semigroup $\mathscr{T} = \{T(t)\}_{t\geqslant 0}$ is exponentially stable if there exists a Banach function space E over R_+ with the property that $\lim\limits_{t\to\infty} \|\chi_{[0,t]}\| = \infty$ such that $\|T(\cdot)x\| \in E$ for all $x \in X$. The Datko-Pazy theorem follows from this by taking $E = L^p(\mathbb{R}_+)$ and Rolewicz's result can be derived as well by taking for E a suitable Orlicz space over $\mathbb{R}_+$. These and related results were extended recently by Jan van Neerven where characterizations of exponential stability for semigroups in terms of lower semi-continuous functionals were obtained. In the spirit of this idea the aim of this paper is to extend this line of results on the general case of evolutionary processes using another type of functionals. A discrete-time variant of the obtained results is also presented.

2. Preliminaries

In the beginning of this section let us recall some standard notations. In all that follows $\mathbb{R}_+$ denotes the set of all positive real numbers, $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$ and $\mathbb{N}^*$ the set of all strictly positive natural numbers. Also, X will be a Banach space, and $B(X)$ the Banach algebra of all bounded linear operators from X into itself. We recall that a family of bounded linear operators $\{U(t,s)\}_{t\geqslant s\geqslant 0}$ is called an evolutionary process if

ep$_1$) $U(t,t) = I$ (where I is the identity operator on X), for all $t \geqslant 0$;
ep$_2$) $U(\cdot,s)x$ is continuous on $[s,\infty)$, for all $s \geqslant 0$, $x \in X$;
$U(t,\cdot)x$ is continuous on $[0,t)$, for all $t \geqslant 0$, $x \in X$;
ep$_3$) $U(t,s) = U(t,r)U(r,s)$, for all $t \geqslant r \geqslant s \geqslant 0$;
ep$_4$) there exist $M, \omega > 0$ such that

$$\|U(t,s)\| \leqslant M\mathrm{e}^{\omega(t-s)} \quad \text{for all} \quad t \geqslant s \geqslant 0.$$

We also remind that an evolutionary process $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ is *uniformly exponentially stable (u.e.s.)* if there exist $N, \nu > 0$ such that $\|U(t+s,s)\| \leqslant N\mathrm{e}^{-\nu t}$, for all $(t,s) \in \mathbb{R}_+^2$.

Proposition 2.1. *The following statements are equivalent*
(i) $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ *is u.e.s.;*
(ii) *there exists a sequence* $a\colon \mathbb{N} \to \mathbb{R}_+$ *with*

$$\inf_{n\in\mathbb{N}^*} a(n) = 0, \quad \|U(n+s,s)\| \leqslant a(n), \quad \textit{for all} \quad (n,s) \in \mathbb{N}\times\mathbb{R}_+;$$

(iii) *there exists a function* $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ *with*

$$\inf_{t>0} b(t) = 0, \quad \|U(t+s,s)\| \leqslant b(t), \quad \textit{for all} \quad (t,s) \in \mathbb{R}_+^2.$$

Proof. (i) ⇒(ii) It is obvious for $a(n) = N\mathrm{e}^{-\nu n}$.

(ii) ⇒ (iii) If $(t,s) \in \mathbb{R}_+^2$ and $n = [t]$ then

$$\|U(t+s,s)\| \leqslant \|U(t+s,n+s)\| \|U(n+s,s)\| \leqslant M\mathrm{e}^{\omega(t-n)} a(n) = b(t)$$

(iii) ⇒ (i). Let $\delta > 0$ be such that $b(\delta) < \mathrm{e}^{-1}$, $(t,s) \in \mathbb{R}_+^2$, $n = [t/\delta]$. It is easy to check that

$$\begin{aligned} \|U(t+s,s)\| &\leqslant \|U(t+s,n\delta+s)\| \|U(n\delta+s,s)\| \\ &\leqslant M\mathrm{e}^{\omega(t-n\delta)} \prod_{k=1}^{n} \|U(k\delta+s,(k-1)\delta+s)\| \leqslant M\mathrm{e}^{\omega\delta} \prod_{k=1}^{n} b(\delta) \\ &\leqslant M\mathrm{e}^{\omega\delta}\mathrm{e}^{-n} \leqslant M\mathrm{e}^{\omega\delta - \frac{t}{\delta}+1} = N\mathrm{e}^{-\nu t}, \end{aligned}$$

where $N = M\mathrm{e}^{\omega\delta+1}$, $\nu = \frac{1}{\delta}$.

For $(\Omega, \mathscr{A}, \mu)$ a measurable space, we will denote by $\mathscr{M}(\Omega,\mathbb{R})$ the space of all measurable functions from Ω to $\mathbb{R}$ and by $\mathscr{M}^+(\Omega,\mathbb{R})$ the set of all $h \in \mathscr{M}(\Omega,\mathbb{R})$ with $h(\xi) \geqslant 0$, for all $\xi \in \Omega$.

Definition 2.1. A function $N\colon \mathscr{M}(\Omega,\mathbb{R}) \to [0,\infty]$ is called a *generalized norm* on $\mathscr{M}(\Omega,\mathbb{R})$ if the following conditions are satisfied:

(n_1) $N(h) = 0$ if and only if $h = 0$ μ-a.e.;

(n_2) $N(h_1 + h_2) \leqslant N(h_1) + N(h_2)$, for all $h_1, h_2 \in \mathscr{M}(\Omega,\mathbb{R})$;

(n_3) $N(ch) = |c|N(h)$, for all $c \in \mathbb{R}$ and all $h \in \mathscr{M}(\Omega,\mathbb{R})$ with $N(h) < \infty$;

(n_4) if $h_1, h_2 \in \mathscr{M}(\Omega,\mathbb{R})$ with $|h_1| \leqslant |h_2|$ then $N(h_1) \leqslant N(h_2)$.

Remark 2.1. If N is a generalized norm on $\mathscr{M}(\Omega,\mathbb{R})$ then $E = \{f \in \mathscr{M}(\Omega,\mathbb{R})\colon N(f) < \infty\}$ is a normed function space with the norm $\|f\|_E = N(f)$.

If we take $\Omega = \mathbb{N}$ with the standard counting measure we have that $\mathscr{M}(\mathbb{N},\mathbb{R}) = \mathscr{S}(\mathbb{R})$, the space of all real sequences, and $\mathscr{M}^+(\mathbb{N},\mathbb{R}) = \mathscr{S}^+(\mathbb{R})$, the set of all positive sequences. Another interesting case is when $\Omega = \mathbb{R}_+$ with the Lebesgue measure. In order to simplify the notations we will use in what follows $\mathscr{M}(\mathbb{R}_+) = \mathscr{M}(\mathbb{R}_+,\mathbb{R})$, $\mathscr{M}^+(\mathbb{R}_+) = \mathscr{M}^+(\mathbb{R}_+,\mathbb{R})$.

Also let $\mathscr{E}(\mathbb{N})$ be the set of all normed sequence spaces with the properties

(e_1) $\chi_{\{0,\dots,m\}} \in E$, for all $m \in \mathbb{N}$;

(e_2) $\lim\limits_{m\to\infty} \|\chi_{\{0,\dots,m\}}\|_E = \infty$;

(e_3) $\inf\limits_{m\in\ N} \|\chi_{\{m\}}\|_E > 0$.

Here χ_A denotes the characteristic function of a set A.

Example 2.1. $l^p \in \mathscr{E}(\mathbb{N})$, for all $p \in [1,\infty)$. Analogously, we will denote by $\mathscr{E}(\mathbb{R}_+)$ the set of all function spaces over $\mathbb{R}_+$ with the properties

(e_1) $\chi_{[0,t]} \in E$, for all $t \geqslant 0$;
(e_2) $\lim\limits_{t\to\infty} \|\chi_{[0,t]}\|_E = \infty$;
(e_3) $\inf\limits_{t\geqslant 0} \|\chi_{[t,t+1)}\|_E > 0$.

Example 2.2. $L^p(\mathbb{R}_+) \in \mathscr{E}(\mathbb{R}_+)$, for all $p \in [1,\infty)$.

Another important set in what follows is $\mathscr{F}$, the set of all functions $F\colon \mathscr{S}^+(\mathbb{R}) \to [0,\infty]$ with the properties
(f_1) if $s_1, s_2 \in \mathscr{S}^+(\mathbb{R})$ with $s_1 \leqslant s_2$ then $F(s_1) \leqslant F(s_2)$;
(f_2) there exist $c > 0$ such that $F(\alpha\chi_{\{n\}}) \geqslant c\alpha$, for all $(\alpha, n) \in \mathbb{R}_+^* \times \mathbb{N}^*$;
(f_3) $\lim\limits_{n\to\infty} F(\alpha\chi_{\{0,\dots,n\}}) = \infty$, for all $\alpha \in \mathbb{R}_+^*$.

Example 2.3. The map $F\colon \mathscr{S}^+(\mathbb{R}) \to [0,\infty]$ defined by

$$F(s) = \sum_{n=0}^{\infty} s(n),$$

belongs to $\mathscr{F}$.

Proposition 2.2. *If $F \in \mathscr{F}, L > 0$ then*

$$\lim_{n\to\infty} \inf_{\alpha\in(0,L]} \frac{F(\alpha\chi_{\{0,\dots,n\}})}{\alpha^2} = \infty.$$

Proof. Let us consider the non-decreasing function $r\colon \mathbb{N} \to [0,\infty]$ given by

$$r(n) = \inf_{\alpha\in(0,L]} \frac{F(\alpha\chi_{\{0,\dots,n\}})}{\alpha^2}$$

and denote by $l = \lim\limits_{n\to\infty} r(n)$. We shall prove that $l = \infty$. Assume for a contradiction that $l < \infty$. Then it is easy to see that for every $n \in \mathbb{N}^*$ there exist $\alpha_n \in (0, L]$ with

$$\frac{c}{\alpha_n} = \frac{c\alpha_n}{\alpha_n^2} \leqslant \frac{F(\alpha_n\chi_{\{0,\dots,n\}})}{\alpha_n^2} \leqslant r(n) + \frac{1}{n},$$

and hence

$$\alpha_n \geqslant \frac{c}{r(n) + 1/n}, \quad \text{for every} \quad n \in \mathbb{N}^*.$$

Using the fact that $l < \infty$ we obtain $\liminf_{n\to\infty} \alpha_n > 0$ which implies that there exist $\alpha > 0$ and $n_0 \in \mathbb{N}^*$ such that $\alpha_n \geqslant \alpha$ for all $n \in \mathbb{N}$ with $n \geqslant n_0$. Then

$$F(\alpha\chi_{\{0,\dots,n\}}) \leqslant \frac{L^2 F(\alpha_n \chi_{\{0,\dots,n\}})}{\alpha_n^2} \leqslant L^2 r(n) + \frac{L^2}{n}$$

and hence $\lim_{n\to\infty} F(\alpha\chi_{\{0,\dots,n\}}) \leqslant L^2 l < \infty$, which is the required contradiction.

It makes sense to consider also the set $\mathscr{G}$ of all functions $G\colon \mathscr{M}^+(\mathbb{R}) \to [0,\infty]$ with the properties:

(g_1) if $u_1, u_2 \in \mathscr{M}^+(\mathbb{R})$ with $u_1 \leqslant u_2$ then $G(u_1) \leqslant G(u_2)$;

(g_2) there exists $c > 0$ such that $G(\alpha\chi_{[t,t+1)}) \geqslant c\alpha$, for all $(\alpha, t) \in \mathbb{R}_+^* \times \mathbb{R}_+$;

(g_3) $\lim_{t\to\infty} G(\alpha\chi_{[0,t]}) = \infty$, for all $\alpha > 0$. □

Example 2.4. The function $G\colon \mathscr{M}^+(\mathbb{R}) \to [0,\infty]$ defined by

$$G(f) = \int_0^\infty f,$$

belongs to $\mathscr{G}$.

Let Φ be the set of all nondecreasing functions $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the property $\varphi(t) > 0$, for all $t > 0$.

Proposition 2.3. *If $\varphi \in \Phi$ then the function $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, defined by $\Psi(t) = \int_0^t \varphi(s)\,\mathrm{d}s$, is a non-decreasing, continuous bijection.*

Proof. It is easy to see that Ψ is continuous, $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$, which implies that Ψ is surjective. If there exists $t_1 < t_2$ such that $\Psi(t_1) = \Psi(t_2)$ then

$$0 \leqslant \frac{t_2 - t_1}{2}\varphi\Big(\frac{t_1 + t_2}{2}\Big) \leqslant \int_{\frac{t_1+t_2}{2}}^{t_2} \varphi(s)\,\mathrm{d}s \leqslant \int_{t_1}^{t_2} \varphi(s)\,\mathrm{d}s = 0$$

and hence $\varphi(\frac{1}{2}(t_1 + t_2)) = 0$, which is a contradiction because $t_1 + t_2 > 0$. □

3. Discrete characterizations for the uniform exponential stability of the evolutionary processes

Let $\mathscr{U} = \{U(t,s)\}_{t\geqslant s\geqslant 0}$ be an evolutionary process. For every $(x, t_0) \in X \times \mathbb{R}_+$ we denote by $s_{x,t_0}\colon \mathbb{N} \to \mathbb{R}_+$ the sequence defined by

$$s_{x,t_0}(n) = \|U(n + t_0, t_0)x\|.$$

Theorem 3.1. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exists $F \in \mathscr{F}$ such that the set*

$$A_F = \Big\{x \in X\colon \sup_{t_0 \geqslant 0} F(s_{x,t_0}) < \infty\Big\}$$

is of the second Baire category.

P r o o f. *Necessity.* This is a simple verification for $F(s) = \sum\limits_{n=0}^{\infty} s(n)$.

Sufficiency. Having in mind that $A_F = \bigcup\limits_{n=1}^{\infty} (A_F \cap \{x \in X\colon \|x\| \leqslant n\})$, it results that there exists $n_0 \in \mathbb{N}^*$ such that the set $A_F \cap \{x \in X\colon \|x\| \leqslant n_0\}$, denoted by A_0, is of the second Baire category. Let $L = \sup\limits_{x \in A_0} \|x\|$ and

$$M_x = \sup_{t_0 \geqslant 0} F(s_{x,t_0}), \quad \text{for} \quad x \in A_0.$$

One can see that

$$s_{x,t_0}(n) \leqslant \frac{1}{c} F(s_{x,t_0}(n)\chi_{\{n\}}) \leqslant \frac{1}{c} F(s_{x,t_0}) \leqslant \frac{1}{c} M_x,$$

for all $(x, t_0, n) \in A_0 \times \mathbb{R}_+ \times \mathbb{N}^*$. It follows that

$$\sup_{(n,t_0) \in \mathbb{N}^* \times \mathbb{R}_+} \|U(n + t_0, t_0)x\| < \infty$$

for all $x \in A_0$. Using the fact that A_0 is of the second Baire category by the Uniform Boundedness Principle (see for instance [3], Theorem 2.5.5, page 26), we obtain

$$(*) \qquad K_1 = \sup_{(n,t_0) \in \mathbb{N} \times \mathbb{R}_+} \|U(n + t_0, t_0)\| < \infty.$$

On the other hand we have

$$\begin{aligned} s_{x,t_0}(n)\chi_{\{0,\ldots,n\}} &= \sum_{k=0}^{n} \|U(n + t_0, t_0)x\|\chi_{\{k\}} \leqslant \sum_{k=0}^{n} \|U(n + t_0, k + t_0)\| s_{x,t_0}(k)\chi_{\{k\}} \\ &\leqslant K_1 \sum_{k=0}^{n} s_{x,t_0}(k)\chi_{\{k\}} = K_1 s_{x,t_0}\chi_{\{0,\ldots,n\}} \leqslant K_1 s_{x,t_0}, \end{aligned}$$

and $s_{x,t_0}(n)/K_1 \leqslant L$, for all $(x, t_0, n) \in A_0 \times \mathbb{R}_+ \times \mathbb{N}$ which implies that

$$\frac{s^2_{x,t_0}(n)}{K_1^2} r(n) \leqslant F\Big(\frac{s_{x,t_0}(n)}{K_1}\chi_{\{0,\ldots,n\}}\Big) \leqslant F(s_{x,t_0}) \leqslant M_x,$$

for all $(x, t_0, n) \in A_0 \times \mathbb{R}_+ \times \mathbb{N}$ and hence

$$\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \sqrt{r(n)}\|U(n+t_0,t_0)x\| < \infty,$$

for all $x \in A_0$. Using again the fact that A_0 is of the second Baire category and the Uniform Boundedness Principle we obtain

$$(**) \qquad K_2 = \sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \sqrt{r(n)}\|U(n+t_0,t_0)\| < \infty.$$

Adding up $(*)$ and $(**)$ we obtain that $\|U(n+t_0,t_0)\| \leqslant (K_1+K_2)/(1+\sqrt{r(n)})$. Denoting by $a\colon \mathbb{N} \to \mathbb{R}_+$, $a(n) = (K_1+K_2)/(1+\sqrt{r(n)})$, by Proposition 2.2. we have $\inf\limits_{n\in\mathbb{N}} a(n) = 0$ and by using Proposition 2.1. we obtain that $\mathscr{U}$ is u.e.s. □

Corollary 3.1. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exists $p \in [1,\infty)$ such that the set*

$$\Big\{x \in X\colon \sup_{t_0\geqslant 0} \sum_{n=0}^{\infty} \|U(n+t_0,t_0)x\|^p < \infty\Big\},$$

is of the second Baire category.

P r o o f. This follows from Theorem 3.1. for

$$F(s) = \Big(\sum_{n=0}^{\infty} s(n)^p\Big)^{1/p}.$$

□

Corollary 3.2. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exists $E \in \mathscr{E}(\mathbb{N})$ and A_0 a set of the second Baire category such that $s_{x,t_0} \in E$, for all $(x,t_0) \in A_0 \times \mathbb{R}_+$ and*

$$\sup_{t_0\geqslant 0} \|s_{x,t_0}\|_E < \infty, \quad \textit{for all} \quad x \in A_0.$$

P r o o f. This follows from Theorem 3.1. for $F(s) = N(s)$, where N is given by Definition 2.1. for $\Omega = \mathbb{N}$. □

Corollary 3.3. *The evolutionary process* $\mathscr{U}$ *is u.e.s. if and only if there exists* $\varphi \in \Phi$ *such that the set*

$$A_{\varphi} = \left\{ x \in X \colon \sup_{t_0 \geqslant 0} \sum_{n=0}^{\infty} \varphi(\|U(n+t_0,t_0)x\|) < \infty \right\}$$

is of the second Baire category.

Proof. *Necessity.* This is a simple verification for $\varphi(t) = t$.

Sufficiency. First we will denote by $M_x = \sup_{t_0 \geqslant 0} \sum_{n=0}^{\infty} \varphi(\|U(n+t_0,t_0)x\|)$ for all $x \in A_{\varphi}$. Next we shall prove that $\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \|U(n+t_0,t_0)x\| < \infty$, for all $x \in A_{\varphi}$. Assume for a contradiction that there exist $x_0 \in A_{\varphi}$ such that $\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \|U(n+t_0,t_0)x_0\| = \infty$.

It is clear that $\sum_{i=0}^{\infty} \varphi(\mathrm{e}^{\omega i}) = \infty$ and so we can consider $n_0 \in \mathbb{N}$ such that

$$\sum_{i=0}^{n_0} \varphi(\mathrm{e}^{\omega i}) \geqslant M_{x_0} + 1.$$

Taking into account that $\sup_{n\leqslant n_0, t_0\geqslant 0} \|U(n+t_0,t_0)x_0\| < \infty$ we have $\sup_{n\geqslant n_0, t_0\geqslant 0} \|U(n+t_0,t_0)x_0\| = \infty$ and so there exist $n_1 \geqslant n_0, t_1 \geqslant 0$ such that

$$\|U(n_1+t_1,t_1)x_0\| \geqslant M\mathrm{e}^{\omega n_0}.$$

However,

$$\begin{aligned} M_{x_0} &\geqslant \sum_{k=0}^{n_1} \varphi(\|U(k+t_1,t_1)x_0\|) \geqslant \sum_{k=0}^{n_1} \varphi\left(\frac{\|U(n_1+t_1,t_1)x_0\|}{M\mathrm{e}^{\omega(n_1-k)}}\right) \\ &= \sum_{j=0}^{n_1} \varphi\left(\frac{\|U(n_1+t_1,t_1)x_0\|}{M\mathrm{e}^{\omega j}}\right) \geqslant \sum_{j=0}^{n_0} \varphi\left(\frac{\|U(n_1+t_1,t_1)x_0\|}{M\mathrm{e}^{\omega j}}\right) \\ &\geqslant \sum_{j=0}^{n_0} \varphi\left(\mathrm{e}^{\omega(n_0-j)}\right) = \sum_{i=0}^{n_0} \varphi(\mathrm{e}^{\omega i}) \geqslant M_{x_0} + 1, \end{aligned}$$

which is a contradiction. Hence, we have

$$\sup_{(n,t_0)\in\mathbb{N}\times\mathbb{R}_+} \|U(n+t_0,t_0)x\| < \infty, \quad \text{for all} \quad x \in A_{\varphi}$$

and so using the fact that $\Psi(t) \leqslant t\varphi(t)$, for all $t \geqslant 0$ we obtain that

$$K_x = \sup_{t_0 \geqslant 0} \sum_{n=0}^{\infty} \Psi(\|U(n+t_0, t_0)x\|) < \infty, \quad \text{for all} \quad x \in A_\varphi.$$

Let $F\colon \mathscr{S}^+(\mathbb{R}) \to [0, \infty]$ the map defined by

$$F(s) = \Psi^{-1}\Bigl(\sum_{n=0}^{\infty} \Psi(s(n))\Bigr).$$

Then $F \in \mathscr{F}$ and $A_\varphi \subset \Bigl\{x \in X\colon \sup_{t_0 \geqslant 0} F(s_{x,t_0}) < \infty\Bigr\}$. By Theorem 3.1. we obtain that $\mathscr{U}$ is u.e.s.

4. Continuous characterizations for the uniform exponential stability of the evolutionary processes

Let $\mathscr{U} = \{U(t,s)\}_{t \geqslant s \geqslant 0}$ be an evolutionary process. For every $(x, t_0) \in X \times \mathbb{R}_+$ we denote by $f_{x,t_0}\colon \mathbb{R}_+ \to \mathbb{R}_+$ the function defined by

$$f_{x,t_0}(t) = \|U(t+t_0, t_0)x\|.$$

Theorem 4.1. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exists $G \in \mathscr{G}$ such that the set*

$$B_G = \Bigl\{x \in X\colon \sup_{t_0 \geqslant 0} G(f_{x,t_0}) < \infty\Bigr\},$$

is of the second Baire category.

Proof. *Necessity.* This is a simple verification for $G(f) = \int_0^\infty f$.

Sufficiency. For every $(x, t_0) \in X \times \mathbb{R}_+$ let us consider the map

$$g_{x,t_0}\colon \mathbb{R}_+ \to \mathbb{R}_+ \quad \text{defined by} \quad g_{x,t_0}(t) = \frac{1}{M\mathrm{e}^{\omega}}\|U([t]+1+t_0, t_0)x\|.$$

Also for $s \in \mathscr{S}^+(\mathbb{R})$ let $g_s\colon \mathbb{R}_+ \to \mathbb{R}_+$ be the map given by $g_s(t) = s([t]+1)/M\mathrm{e}^{\omega}$ and $F_G\colon \mathscr{S}^+(\mathbb{R}) \to [0, \infty]$ the function defined by $F_G(s) = G(g_s)$. Using the fact that $G \in \mathscr{G}$ one can easily verify that $F_G \in \mathscr{F}$ and by observing that

$$F_G(s_{x,t_0}) = G(g_{x,t_0}) \leqslant G(f_{x,t_0}), \quad \text{for all} \quad (x, t_0) \in X \times \mathbb{R}_+$$

it follows that $B_G \subset A_{F_G}$ and hence A_{F_G} is a set of the second Baire category. By Theorem 3.1. we obtain that $\mathscr{U}$ is u.e.s.

Corollary 4.1. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exists $p \in [1,\infty)$ such that*

$$\left\{x \in X\colon \sup_{t_0 \geqslant 0} \int_0^\infty \|U(t+t_0,t_0)x\|^p \,\mathrm{d}t < \infty\right\},$$

is of the second Baire category.

Proof. This follows from Theorem 4.1. for $G\colon \mathscr{M}^+(\mathbb{R}_+) \to [0,\infty]$

$$G(f) = \left(\int_0^\infty f^p\right)^{1/p}.$$

□

Corollary 4.2. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exist $E \in \mathscr{E}(\mathbb{R}_+)$ and a set B_0 of the second Baire category such that $f_{x,t_0} \in E$ for all $(x,t_0) \in B_0 \times \mathbb{R}_+$ and*

$$\sup_{t_0 \geqslant 0} \|f_{x,t_0}\| < \infty, \quad \textit{for all} \quad x \in B_0.$$

Proof. This follows from Theorem 4.1. for $G\colon \mathscr{M}^+(\mathbb{R}_+) \to [0,\infty]$, $G(f) = N(f)$ where N is given by Definition 2.1. for $\Omega = \mathbb{R}_+$. □

Corollary 4.3. *The evolutionary process $\mathscr{U}$ is u.e.s. if and only if there exists $\varphi \in \Phi$ such that the set*

$$B_\varphi = \left\{x \in X\colon \sup_{t_0 \geqslant 0} \int_0^\infty \varphi(\|U(t+t_0,t_0)x\|)\,\mathrm{d}t < \infty\right\},$$

is of the second Baire category.

Proof. *Necessity.* This is a simple verification for $\varphi(t) = t$.

Sufficiency. Let $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$, $\gamma(t) = \varphi(t/M\mathrm{e}^\omega)$. It is obvious that $\gamma \in \Phi$ and

$$\begin{aligned}\sum_{n=0}^\infty \gamma(\|U(n+t_0,t_0)x\|) &= \gamma(\|x\|) + \int_0^\infty \varphi(g_{x,t_0}(t))\,\mathrm{d}t \\ &\leqslant \gamma(\|x\|) + \int_0^\infty \varphi(f_{x,t_0}(t))\,\mathrm{d}t, \quad \text{for all} \quad (x,t_0) \in X \times \mathbb{R}_+,\end{aligned}$$

where g_{x,t_0} was defined in the proof of Theorem 4.1.

It follows that $B_\varphi \subset A_\gamma$ and hence A_γ is of the second Baire category and by Corollary 3.3. we obtain that $\mathscr{U}$ is u.e.s.

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