Jorge Martinez
Dimension in algebraic frames

Czechoslovak Mathematical Journal, Vol. 56 (2006), No. 2, 437–474

Persistent URL: `http://dml.cz/dmlcz/128078`

Czechoslovak Mathematical Journal, 56 (131) (2006), 437–474

DIMENSION IN ALGEBRAIC FRAMES

Jorge Martínez, Gainesville

(Received August 22, 2003)

Abstract. In an algebraic frame L the dimension, $\dim(L)$, is defined, as in classical ideal theory, to be the maximum of the lengths n of chains of primes $p_0 < p_1 < \ldots < p_n$, if such a maximum exists, and ∞ otherwise. A notion of "dominance" is then defined among the compact elements of L, which affords one a primefree way to compute dimension.

Various subordinate dimensions are considered on a number of frame quotients of L, including the frames dL and zL of d-elements and z-elements, respectively. The more concrete illustrations regarding the frame convex ℓ-subgroups of a lattice-ordered group and its various natural frame quotients occupy the second half of this exposition.

For example, it is shown that if A is a commutative semiprime f-ring with finite ℓ-dimension then A must be hyperarchimedean. The d-dimension of an ℓ-group is invariant under formation of direct products, whereas ℓ-dimension is not. r-dimension of a commutative semiprime f-ring is either 0 or infinite, but this fails if nilpotent elements are present. sp-dimension coincides with classical Krull dimension in commutative semiprime f-rings with bounded inversion.

Keywords: algebraic frame, dimension, d-elements, z-elements, lattice-ordered group, f-ring

MSC 2000: 06D22, 06F15

1. Introduction

This article considers the notion of a "Krull" dimension in an algebraic frame, which is the structure of discourse throughout. We assume the Axiom of Choice from the outset, all the while realizing that one can get by with less to produce "points", that is to say, the primes of the frame.

In spirit, this paper picks up where [23] leaves off. Following a general review of the relevant frame-theoretic principles, we will recall from [23] the notions of the d-element and z-element. These two concepts are motivated by ideas that form the basis for the discussion in [11], [12].

The reader familiar with commutative algebra will presumably recall that Krull dimension is a number which records the length of the longest chain of prime ideals (relative to inclusion). We consider the corresponding notion in algebraic frames. It is fair to say that in [23] the "dimension zero" case was thoroughly examined, with ample illustration. For rings of continuous functions, [10] studies the topological properties of a space X such that, for the associated ring $C(X)$ of all continuous real valued functions on X, the frame of z-ideals has dimension not exceeding one.

During the remainder of this introduction we review the basic information which will be relevant the rest of the way.

Definition & Remarks 1.1. A *frame* is a complete lattice L in which the following distributive law holds:

$$a \wedge \left(\bigvee S \right) = \bigvee \{a \wedge s\colon\ s \in S\}$$

for each $a \in L$ and $S \subseteq L$.

Let us also fix some notation: a complete lattice has a largest and a least element, denoted 1 and 0, respectively. For each $a \in L$ we denote $\uparrow a = \{x \in L\colon\ x \geqslant a\}$ and $\downarrow a = \{x \in L\colon\ x \leqslant a\}$.

A small directory of terms now follows; it is assumed throughout that L is a complete lattice. The terminology discussed here is that of [23]; we concede that it differs in places from usage elsewhere.

(i) An element $a \in L$ is *compact* if $a \leqslant \bigvee X$ implies that $a \leqslant \bigvee X_o$ for some finite subset X_o of X. L is *algebraic* if every element of L is a supremum of compact elements. We denote the set of compact elements of L by $\mathfrak{k}(L)$. $\mathfrak{k}(L)$ is always closed under finite joins.

   L is said to be *compact* if 1 is compact.

   It is well known that if L is algebraic, then it is a frame if and only if it is distributive; this is Exercise 9, p. 189, in [4].

(ii) Call $p < 1$ *prime* if $x \wedge y \leqslant p$ implies that $x \leqslant p$ or $y \leqslant p$. Note that if L is distributive then p is prime if and only if it is *meet-irreducible*; that is, $x \wedge y = p$ implies that $x = p$ or $y = p$. $\operatorname{Spec}(L)$ denotes the set of all primes of L; this is the *spectrum* of the frame. Observe that if L is algebraic and $p < 1$ satisfies that $a \wedge b \leqslant p$ implies that $a \leqslant p$ or $b \leqslant p$, for all a and b compact, then p is prime.

Now assume that L is a frame.

(iii) L is said to be *coherent* if it is algebraic and $\mathfrak{k}(L)$ is closed under finite meets. This includes the empty meet; thus, it is built into this definition that 1 is compact.

(iv) When $\mathfrak{k}(L)$ has the feature that $a, b \in \mathfrak{k}(L)$ implies that $a \wedge b \in \mathfrak{k}(L)$ we say that L has the *finite intersection property on compact elements*, abbreviated FIP. Thus, a frame is coherent if and only if it is compact and has the FIP.

(v) For each $a \in L$, let

$$a^{\perp} = \bigvee\{x \in L\colon\ x \wedge a = 0\}.$$

We also denote $(a^{\perp})^{\perp} = a^{\perp\perp}$. The elements of the form $x^{\perp}$ are referred to in most of the literature as the *pseudo-complemented elements* of L; coming from the theory of lattice-ordered groups, we prefer the term *polar* for these elements.

An element $a \in L$ is *complemented* if $a \vee a^{\perp} = 1$.

(vi) Finally, in this presentation of introductory material, if L is any frame and $x, y \in L$, then

$$x \to y \equiv \bigvee\{a \in L\colon\ x \wedge a \leqslant y\}.$$

Evidently, $x \wedge (x \to y) \leqslant y$, and $x \to y$ is the maximum with this property. Observe as well that $x^{\perp} = x \to 0$ for each $x \in L$.

For general lattice theory we refer the reader to [4]; for the background on frame theory, the most comprehensive reference is still [14].

The work in [23] was motivated by our interest in lattice-ordered groups and f-rings. The relevant aspects of these theories will be presented as the circumstances and the development below warrant it. For now it will suffice to remark that for all the applications we have in mind there is an underlying algebraic frame of substructures and, in many applications, there are several that will interest us.

This concludes our general introduction.

2. Primes and dimension

This section records the background on primes of algebraic frames, as well as the definition of dimension.

Let $\mathrm{Min}(L)$ denote the set of minimal primes of the frame L. An application of Zorn's Lemma easily shows that in an algebraic frame each prime element exceeds a minimal prime. We begin by recalling a well known characterization of the minimal primes. A proof of the following lemma appears in [17].

Lemma 2.1. *Suppose L is an algebraic frame possessing the FIP. Then $p \in \mathrm{Spec}(L)$ is minimal if and only if*

$$p = \bigvee\{c^{\perp}\colon\ c \in \mathfrak{k}(L), c \nleqslant p\}.$$

Remark 2.2. It is a routine matter to verify that, in any algebraic frame, each polar is an infimum of minimal primes.

Definition & Remarks 2.3. Here is a recapitulation of the main result in [17, Theorem 2.4]. We say that an algebraic lattice L has the *compact splitting property*, abbreviated *CSP*, if every compact element of L is complemented. [17, Theorem 2.4] asserts that an algebraic frame L has the CSP if and only if L has the FIP and every prime is minimal.

[23, Theorem 2.4] shows that the above two conditions are, in turn, equivalent to the regularity of the frame. We will review the concept of a regular frame in 2.5 below.

In [17] and again in [23, 3.5] the reader may find examples showing that both the conditions listed are needed in the theorem cited above.

Now here are the main definitions of the paper.

Definition & Remarks 2.4. Let L be an algebraic frame. The *length* of a chain of primes $p_0 < p_1 < \ldots < p_n$ is the number n. The *dimension* of L, written $\dim(L)$, is the maximum of the lengths of chains of primes, if such a maximum exists, and ∞ otherwise.

This is in the spirit of Krull dimension in the ring theory, indeed. However, since this dimension will be applied to a number of different structures associated with a given algebraic frame, one ought to resist calling it Krull dimension.

Definition & Remarks 2.5. (a) Suppose that L is a frame. Define $a \in L$ to be *well below* $b \in L$ if $a^{\perp} \vee b = 1$; if so we write $a \preceq b$. $a \in L$ is *regular* if

$$a = \bigvee \{x \in L\colon\ x \preceq a\}.$$

L is *regular* if each $a \in L$ is regular. This terminology is adapted from topology. Let X be a topological space, and suppose that $\mathfrak{O}(X)$ denotes the frame of open sets, with respect to union and infimum defined as

$$\bigwedge \mathscr{S} = \operatorname*{int}_X \left(\bigcap \mathscr{S}\right).$$

Then $\mathfrak{O}(X)$ is regular if and only if X is regular in the familiar sense.

(b) [23, Theorem 2.4(a)] shows that an algebraic frame is regular if and only if it satisfies the CSP.

(c) Finally, note that if $a \preceq b$ then $a^{\perp\perp} \leqslant b$.

The following simple proposition defines $\mathfrak{h}(L)$; in an algebraic frame with the FIP it is the largest element x such that $\downarrow x$ is regular. The proofs of Proposition 2.6 and Theorem 2.8 may be found in [21].

Proposition 2.6. *Suppose that L is an algebraic frame. Define*

$$\mathfrak{h}(L) = \bigwedge_{c \in \mathfrak{k}(L)} c \vee c^{\perp}.$$

Then $\downarrow \mathfrak{h}(L)$ is a regular frame. If L also has the FIP and $x \in L$ is such that $\downarrow x$ is regular, then $x \leqslant \mathfrak{h}(L)$.

Definition & Remarks 2.7. Throughout this commentary L denotes a fixed algebraic frame with the FIP. The element $\mathfrak{h}(L)$ defined in Proposition 2.6 will be referred to as *the regular top of L*.

(a) We sketch now the construction of a transfinite sequence in L. $\mathfrak{h}^1(L) \equiv \mathfrak{h}(L)$; assuming that β is an ordinal and that $\mathfrak{h}^{\alpha}(L)$ is defined for each ordinal $\alpha < \beta$, let

$$\mathfrak{h}^{\beta}(L) = \bigvee_{\alpha < \beta} \mathfrak{h}^{\alpha}(L)$$

if β is a limit ordinal. Otherwise (if β_0 precedes β), let

$$\mathfrak{h}^{\beta}(L) = \mathfrak{h}(\uparrow \mathfrak{h}^{\beta_0}(L)).$$

This produces a sequence $0 = \mathfrak{h}^0(L) \leqslant \mathfrak{h}^1(L) \leqslant \ldots \leqslant \mathfrak{h}^{\beta}(L) \leqslant \ldots$ of elements of L such that each interval

$$(\downarrow \mathfrak{h}^{\beta+1}(L)) \cap (\uparrow \mathfrak{h}^{\beta}(L))$$

is a regular frame.

We shall call the above sequence the *canonical regular interval series* of L (or cris(L)).

(b) The cris(L) terminates; that is, there is an ordinal τ such that $\mathfrak{h}^{\tau}(L) = \mathfrak{h}^{\tau+1}(L) = \ldots$. If $1 = \mathfrak{h}^{\tau}(L)$ (for some τ) we shall say that L is a *limit-regular* frame.

Next, we have a characterization of limit-regular frames with FIP. Recall that a frame homomorphism is a map between frames which preserves all suprema and all finite infima. The proof hinges on the well known fact that the image under any frame homomorphism of a regular frame is again regular.

Theorem 2.8. *Suppose that L is an algebraic frame with the FIP. Then L is limit-regular if and only if there is a transfinite sequence*

$$(\dagger) \qquad x_1 \leqslant x_2 \leqslant \ldots \leqslant x_\beta \leqslant \ldots$$

such that $x_\tau = 1$ for some τ, $x_\beta = \vee_{\alpha<\beta} x_\alpha$ for each limit ordinal β, and

$$(\downarrow x_{\beta+1}) \cap (\uparrow x_\beta)$$

is regular for each ordinal β.

Remark 2.9. Assume that L is a limit-regular algebraic frame with the FIP.

(a) A sequence (x_β) such as in ($\dagger$) of Theorem 2.8 is referred to as a *regular interval series*. The theorem may be reasonably interpreted as stating that the cris(L) is the maximum among regular interval series.

Now the *index* of a regular interval series (x_β) is the least ordinal τ for which $1 = x_\tau$. The *regular index* of L, denoted $\mathrm{Reg}(L)$, is the least of the indices among all regular interval series. Theorem 2.8 implies that $\mathrm{Reg}(L)$ is the index of the cris(L).

To underscore, to say that $\mathrm{Reg}(L) = \tau$ is to say that $\mathfrak{h}^\tau(L) = 1$, but $x_\alpha < 1$ for all $\alpha < \tau$ and all regular interval series (x_β).

(b) $\mathrm{Reg}(L)$ could be a limit ordinal. However, if L is compact (and therefore coherent) then it is a successor ordinal.

Finally, in this section we have a connection with dimension. We sketch the easy induction proof.

Corollary 2.10. *Let L be an algebraic frame with the FIP such that* $\mathrm{Reg}(L) = n+1$ *(with n a nonnegative integer). Then* $\dim(L) \leqslant n$.

Proof. For $n = 0$ we have a regular frame in L. Thus $\dim(L) = 0$ by 2.5(b). Assume that $n > 0$ and $\mathrm{Reg}(L) = n+1$; denote $h = \mathfrak{h}^n(L)$. By assumption we have that $\mathrm{Reg}(\downarrow h) = n$ and $\uparrow h$ is regular. By the inductive hypothesis, $\dim(\downarrow h) \leqslant n-1$. Now, if $p_0 < p_1 < \ldots < p_k$ is a sequence of primes, we must have $p_i \geqslant h$ for each $i \geqslant n$, and then it follows that $k \leqslant n$. Thus, $\dim(L) \leqslant n$. $\square$

Remark 2.11. (a) One would like to conclude in Corollary 2.10 that $\dim(L) = n$ when $\mathrm{Reg}(L) = n+1$. In general this is false. We shall give an example of an algebraic frame of dimension 1 for which $\mathrm{Reg}(L) = 3$; see Example 5.12. For now, we have the corollary following these comments. The proof is easy, and an imitation of that of [18, Proposition 1.11]; we therefore omit it.

(b) A word of caution though: there are examples of algebraic frames L of dimension 1 for which $\mathfrak{h}(L) = 0$; in particular, these frames are not even limit-regular. In

[7, §52] one may find an account of the free abelian lattice-ordered groups, and the relevance of that information here lies in [7, Corollary 52.17], which states that for the algebraic frame $\mathscr{C}(F)$ of the free abelian ℓ-group on two generators the dimension is 1. On the other hand, it is well known that $\mathfrak{h}(\mathscr{C}(F)) = 0$; see [19, Proposition 2.4].

Corollary 2.12. *Suppose that L is an algebraic frame with the FIP. If* $\mathrm{Reg}(L) = 2$ *then* $\dim(L) = 1$.

3. Good vs. bad supplements

Our aim in this section is to calculate the dimension of an algebraic frame without reference to primes. As a companion to dimension it is convenient to consider the following measuring standard on compact elements.

Definition & Remarks 3.1. Suppose that L is an algebraic frame and $a_0 < a_1 < \ldots < a_k$ is a chain of compact elements of L. We say that it is a *dominance chain of length k* if there is a prime p of L such that, in $\uparrow p$,

$$p < a_0 \vee p < \ldots < a_k \vee p.$$

The *dominance* of L, denoted $\mathrm{dom}(L)$, is the supremum of the lengths of dominance chains of L.

A companion definition: a chain $a_0 < a_1 < \ldots < a_n < \ldots$ is an *ascending dominance chain* if there is a prime p such that

$$p < a_0 \vee p < \ldots < a_n \vee p < \ldots.$$

If $p_0 < p_1 < \ldots < p_k$ is a chain of primes, we may find, for each $i = 0, 1, \ldots, k$, a compact element a_i such that $a_i \leqslant p_{i+1}$ for each $i = 0, 1, \ldots, k-1$, and $a_i \nleqslant p_i$ for each $i = 0, 1, \ldots, k$. Without loss of generality we may assume that $a_0 < a_1 < \ldots < a_k$. It is then easy to see that $p_0 < a_0 \vee p_0 < \ldots < a_k \vee p_0$, so that $a_0 < a_1 < \ldots < a_k$ is a dominance chain. Thus, $\dim(L) \leqslant \mathrm{dom}(L)$.

In Proposition 3.4 we provide a reasonable sufficient condition for the reverse inequality to hold. Let us first record a definition.

Definition & Remarks 3.2. Let L be an algebraic frame. We say that L has the *disjointification property* (or, simply, that L is a frame *with disjointification*) if for each pair of compact elements $a, b \in L$ there exist disjoint $c, d \in \mathfrak{k}(L)$ such that

1. $c \leqslant a$ and $d \leqslant b$, and
2. $a \vee b = a \vee d = c \vee b$.

In Section 5 and subsequent sections we shall encounter some natural examples of algebraic frames with disjointification.

The reader who knows the frame-theoretic terminology in this regard will recognize the hand of "normality" in the above. Recall that a frame L is *normal* if for each pair $x, y \in L$ such that $x \vee y = 1$ there exist disjoint a and b such that $a \vee x = 1 = b \vee y$. Further, L is *coherently normal* if $\downarrow a$ is normal for each $a \in \mathfrak{k}(F)$. It is obvious that the disjointification in an algebraic frame L implies coherent normality, and if L possesses the FIP then the converse is also true. We emphasize that compactness (of 1) is not assumed here.

For the proof of Proposition 3.4 the following lemma is a must. When the condition in the lemma is satisfied it is said that $\mathfrak{k}(L)$ is *relatively normal*. One also says that $\operatorname{Spec}(L)$ is a *root system*. This lemma is apparently due to Monteiro ([24]); see also [25, Lemma 2.1], where a proof is given.

Lemma 3.3. *Suppose that L is an algebraic frame with disjointification. Then, for any $p \in \operatorname{Spec}(L)$, $\uparrow p$ is a chain. The converse is true if L has the FIP.*

Proposition 3.4. *Suppose that L is an algebraic frame with disjointification. Then $\dim(L) = \operatorname{dom}(L)$.*

P r o o f. What remains to be shown is that $\operatorname{dom}(L) \leqslant \dim(L)$. To this end suppose that $a_0 < a_1 < \ldots < a_k$ is a dominance chain. As $p < a_i \vee p$, we may select a prime $p_i \geqslant p$ which is maximal with respect to $a_{i-1} \vee p \leqslant p_i$ and $a_i \vee p \nleqslant p_i$ (for each $i = 1, \ldots, k$) and $a_0 \vee p \nleqslant p_0$. Since $\uparrow p$ is a chain, we have

$$p \leqslant p_0 < a_0 \vee p \leqslant p_1 < \ldots \leqslant p_k < a_k \vee p.$$

In particular, $p_0 < p_1 < \ldots p_k$, which proves that $\operatorname{dom}(L) \leqslant \dim(L)$, as claimed. $\square$

The following is a corollary of the proof of Proposition 3.4.

Corollary 3.5. *Suppose that L is an algebraic frame with disjointification. Then $\operatorname{Spec}(L)$ satisfies the ascending chain condition if and only if there are no ascending dominance chains in L.*

Next, a comment which also introduces the framework for a primefree version of dominance. First a general definition.

Definition 3.6. Suppose that L is an algebraic frame and $a \leqslant c$ are compact elements. We say that $b \in \mathfrak{k}(L)$ *supplements a for c* if $a \vee b = c$.

Definition & Remarks 3.7. Throughout this commentary L denotes a fixed algebraic frame which possesses the FIP.

Suppose that we have compact elements $0 < a_0 < \ldots < a_k$ in L and $F_1, \ldots, F_k$ is a collection of finite subsets of $\mathfrak{k}(L)$. We say that $F_1, \ldots, F_k$ is a *supplementing array for the* a_i if for each $i = 1, \ldots, k$, each $c \in F_i$ supplements a_{i-1} for a_i.

The supplementing array $F_1, \ldots, F_k$ for the a_i is *good* if

$$a_0 \wedge \Big(\bigwedge F_1\Big) \wedge \ldots \wedge \Big(\bigwedge F_k\Big) > 0.$$

Otherwise, $F_1, \ldots, F_k$ is called a *bad* supplementing array.

We make a number of observations about supplementing arrays:

1. Some of the F_i above may be empty.
2. Given $0 \leqslant a \leqslant b \in \mathfrak{k}(L)$ and compact elements $0 \leqslant c_1, c_2 \leqslant b$ which supplement a for b, then $c_1 \wedge c_2$ also supplements a for b. This implies that if there is a bad supplementing array for the a_i, then there is one for which $|F_i| \leqslant 1$ for each $i = 1, \ldots, k$.
3. If $F_1, \ldots, F_k$ and $G_1, \ldots, G_k$ are supplementing arrays for the a_i, then so is $F_1 \cup G_1, \ldots, F_k \cup G_k$.
4. Finally, if $F_1, \ldots, F_k$ is a supplementing array for the a_i with $|F_i| \leqslant 1$, then by defining G_i to be F_i if $F_i \neq \emptyset$, and $G_i = \{a_i\}$ otherwise, we have a supplementing array $G_1, \ldots, G_k$, for which $|G_i| = 1$ for each $i = 1, \ldots, k$, and which is good if and only if the original one is.

   Concluding these remarks, if there is a bad supplementing array for the a_i, then there is a bad supplementing array $F_1, \ldots, F_k$ for the a_i for which $|F_i| = 1$ for each $i = 1, \ldots, k$.

The connection between supplementing arrays and the dimension of L is the following.

Theorem 3.8. *Suppose that L is an algebraic frame with the FIP and disjointification. Then* $\dim(L) = k$ *if and only if*

(a) *for each chain of compact elements $a_0 < a_1 < \ldots < a_m$ with the property that every supplementing array for it is good we have $m \leqslant k$, and*

(b) *there exists a chain of compact elements of length k for which every supplementing array is good.*

Proof. It suffices, by Proposition 3.4, to show that a chain $0 < a_0 < \ldots < a_k$ of compact elements is a dominance chain if and only if every supplementing array for the a_i is good.

On the one hand, suppose that $0 < a_0 < \ldots < a_k$ is a dominance chain in L. There is a minimal prime p such that $p < a_0 \vee p < \ldots < a_k \vee p$. This means that

$a_i \not\leqslant a_{i-1} \vee p$ for each $i = 1, \ldots, k$, which, in turn, implies that $a_i \not\leqslant a_{i-1} \vee c$ for each compact element $c \leqslant p$. Putting it differently, if $a_i \leqslant a_{i-1} \vee c$ with $c \in \mathfrak{k}(L)$, then $c \not\leqslant p$. Thus, if $F_1, \ldots, F_k$ is a supplementing array for the a_i, we have that

$$a_0 \wedge \left(\bigwedge F_1\right) \wedge \ldots \wedge \left(\bigwedge F_k\right) \not\leqslant p,$$

whence it is clear that the array is good.

Conversely, suppose that every supplementing array for the a_i is good. Consider the set S defined as follows:

$$S = \left\{\left(\bigwedge F_1\right) \wedge \ldots \wedge \left(\bigwedge F_k\right) :\ F_1, \ldots, F_k \text{ is a supplementing array for the } a_i\right\}.$$

Since all supplementing arrays for the a_i are good, $0 \notin S$. Moreover, in view of the comment in 3.7.3, it is clear that S is closed under finite meets. Thus, S is a filter base of compact elements which meet a_0 nontrivially. Applying [17, Lemma 2.5], there is a minimal prime q such that $a_0 \not\leqslant q$ and each $c \in S \Rightarrow c \not\leqslant q$. We leave it to the reader that the prime q witnesses that $0 < a_0 < a_1 \ldots < a_k$ is a dominance chain in L. □

4. d-Elements and z-Elements

In this section we interpret Proposition 3.4 and Theorem 3.8 for the frames of d-elements and z-elements.

We begin with a review of the basics on d-elements, and, in the second part of the section, follow with a similar review of z-elements; for additional information we refer the reader to §5 and §6 of [23]. Throughout it is assumed that L is an algebraic frame.

If j is any closure operator on L we denote by $\mathrm{fix}(j)$ the set of all $x \in L$ for which $j(x) = x$.

Definition & Remarks 4.1. $a \in L$ is a *d-element* if

$$a = \bigvee\{c^{\perp\perp} :\ c \leqslant a,\ c \in \mathfrak{k}(L)\}, \tag{$\dagger^*$}$$

dL denotes the subset of all d-elements of L. It is easy to see that $a \in dL$ if and only if $c \leqslant a$, with c compact, implies that $c^{\perp\perp} \leqslant a$. There is an associated closure operator, given by

$$d(x) = \bigvee\{c^{\perp\perp} :\ c \leqslant x,\ c \in \mathfrak{k}(L)\},$$

for each $x \in L$.

Let us summarize the principal features of d; for amplification the reader is referred to §4 and §5 in [23]:

(i) d is a closure operator; if L also has the FIP then, as a consequence of the identity

$$(x \wedge y)^{\perp\perp} = x^{\perp\perp} \wedge y^{\perp\perp},$$

one also has $d(x \wedge y) = d(x) \wedge d(y)$. Since $\mathrm{fix}(d) = dL$, the upshot is that dL is an algebraic frame; note that

$$\mathfrak{k}(dL) = \{a^{\perp\perp} \colon a \in \mathfrak{k}(L)\}.$$

(ii) For each $c \in \mathfrak{k}(L)$ we have $d(c) = c^{\perp\perp}$.

For the remainder of this commentary, assume that L has the FIP.

(iii) The term "prime d-element" is unambiguous; it refers to a d-element which is prime in L ([23, 5.1(iii)]). Note that a minimal prime element of L is a d-element.

(iv) Using 2.5(b) and [23, Proposition 5.2] we conclude that the following are equivalent.

(a) dL is regular.

(b) For each pair of compact elements $a \leqslant c$ in L there is a compact $b \in L$ such that $a \wedge b = 0$ and $(a \vee b)^{\perp\perp} = c^{\perp\perp}$.

(c) Every prime d-element is a minimal prime.

(d) For any d-element x, any prime p which is minimal in $\uparrow x$ is a minimal prime.
If dL is a regular frame then we call L *d-regular*, following the usage in [11], [12].

Theorem 3.8 reads as follows in dL. Since the closure operator d preserves disjointness, (a) follows. We leave the translation of the rest to the reader.

Theorem 4.2. *Suppose that L is an algebraic frame with the FIP and disjointification.*

(a) *dL has disjointification.*

(b) *$\dim(dL) \leqslant k$ if and only if for each chain $a_0 < a_1 < \ldots < a_{k+1}$ of nonzero compact elements of L, there exist $b_1, \ldots, b_{k+1} \in \mathfrak{k}(L)$ such that $(a_i \vee b_{i+1})^{\perp\perp} = a_{i+1}^{\perp\perp}$ for each $i = 0, 1, \ldots, k$, and*

$$a_0 \wedge b_1 \wedge \ldots \wedge b_{k+1} = 0.$$

Next, we present a brief review of z-elements, once again following [11], [12]. We do it in two parts, discussing archimedean lattices first.

Definition & Remarks 4.3. (a) With $x \in L$ we say that $m < x$ is *maximal under* x if m is maximal in $\downarrow x$. Denote the set of elements which are maximal under x by $\mathrm{Max}(x)$. There should be no confusion issuing from the convention that $\mathrm{Max}(L) \equiv \mathrm{Max}(1)$. In the sequel we will also be interested in such relatively maximal elements for an interval $\uparrow y$ (with $y \leqslant x \in L$). $\mathrm{Max}_y(x)$ will denote the set of elements which are maximal under x in the lattice $\uparrow y$.

Owing to [23, Lemma 4.6], the elements of $\mathrm{Max}(x)$ are in a one-to-one correspondence with

$$\{p \in \mathrm{Spec}(L)\colon\ x \nleqslant p \text{ and } p \text{ is maximal with this property}\}.$$

(b) L is an *archimedean lattice* if, for each $c \in \mathfrak{k}(L)$, $\bigwedge \mathrm{Max}(c) = 0$. This concept first appeared in [17].

We say that $x \in L$ is *upper-archimedean* if $\uparrow x$ is archimedean. Denote the set of all upper-archimedean elements of L by $\mathbf{a}^{\uparrow}(L)$. Observe that if L is compact then L is archimedean if and only if $\bigwedge \mathrm{Max}(L) = 0$. Thus, if L is compact then $x \in \mathbf{a}^{\uparrow}(L)$ precisely when x is an infimum of maximal elements of L.

Definition & Remarks 4.4. [23, Lemma 6.2] guarantees that there is a closure operator ar on L which preserves finite infima, such that $\mathrm{fix}(ar) = \mathbf{a}^{\uparrow}(L)$. Note that $ar(0) = 0$ if and only if L is archimedean. Also, if L is archimedean then $\mathbf{a}^{\uparrow}(L)$ contains all polars of L ([23, 6.3(a)]). If L is compact then, in view of the comments in 4.3(c),

$$ar(x) = \bigwedge\{m \in \mathrm{Max}(L)\colon\ x \leqslant m\}.$$

Now define $x \in L$ to be a *z-element* if $c \leqslant x$, with $c \in \mathfrak{k}(L)$, implies that $ar(c) \leqslant x$.

The following features of z-elements are taken from [23, 6.3]:

(a) Assume L is archimedean. Then $ar(x) \leqslant x^{\perp\perp}$ for each $x \in L$, and we have that every d-element is necessarily a z-element.

(b) For any algebraic frame L we have

$$z(x) = \bigvee\{ar(c)\colon\ c \leqslant x,\ c \in \mathfrak{k}(L)\}.$$

The lattice zL of all z-elements, is an algebraic lattice. If L satisfies the FIP, then zL is a frame, also with the FIP.

(c) If L is an archimedean frame, then $z(x) \leqslant d(x)$ for each $x \in L$.

(d) It follows from (a) that, for any archimedean frame L,

$$\dim(dL) \leqslant \dim(zL) \leqslant \dim(L).$$

In the upcoming sections we investigate the concept of dimension in a number of contexts involving lattice-ordered algebraic structures. However, with regard to z-dimension—the dimension of zL—we postpone any applications of Theorem 3.8 until the exposition in [20]. Any substantial calculations involving the closure operators ar and z, in an archimedean lattice-ordered structure, turn on the concept of uniform convergence. A discussion of that does not really fit in the present exposition and, with the exception of the comments in 9.1, where the subject can't be avoided in preparation for Theorem 9.2, we will not have anything further to say in this article about uniform convergence.

5. The ℓ-dimension of an ℓ-group

Here we consider the dimension of the algebraic frame of all convex ℓ-subgroups of a lattice-ordered group. Our standard references for the theory of lattice-ordered groups are [3] and [7].

Definition & Remarks 5.1. All lattice-ordered groups in this paper will be written additively, though they are not necessarily abelian. For the record, $(G, +, 0, -(\cdot), \vee, \wedge)$ is a *lattice-ordered group* (abbreviated *ℓ-group*) if $(G, +, 0, -(\cdot))$ is a group with $(G, \vee, \wedge)$ as an underlying lattice, and the following distributive law holds:

$$a + (b \vee c) = (a + b) \vee (a + c),$$

together with the left-right dual of the above. These then imply the corresponding distributive laws for sum over infimum.

The elements of G for which $g \geqslant 0$ are said to be *positive*; the set of positive elements of G is denoted by G^+. For each $g \in G$, $g^+ = g \vee 0$ and $g^- = (-g) \vee 0$. Then also $|g| = g \vee (-g) = g^+ + g^-$.

A group homomorphism between two ℓ-groups which is simultaneously a lattice homomorphism is called an *ℓ-homomorphism.*

We recite some of the basic information about these structures. In the sequel G stands for an ℓ-group.

1. The underlying lattice of an ℓ-group is distributive ([7, Corollary 3.17]), and the group structure is torsion free ([7, Propositions 3.15 & 3.16]).
2. G is *archimedean* if $a, b \in G^+$ and $na \leqslant b$ for each $n \in \mathbb{N}$ imply that $a = 0$. In general, if $na \leqslant b$ for each positive integer n, we say that a is *infinitesimal* with respect to b and write $a \ll b$.
3. A subgroup of G is called an *ℓ-subgroup* if it is a sublattice as well. The ℓ-subgroup C is *convex* if $a \leqslant g \leqslant b$ with $a, b \in C$ implies that $g \in C$. Let $\mathscr{C}(G)$

denote the lattice of all convex ℓ-subgroups of G. $\mathscr{C}(G)$ is a complete sublattice of the lattice of all subgroups of G ([7, Theorem 7.5]), and an algebraic frame; the latter is due to G. Birkhoff ([7, Proposition 7.10]). $\mathscr{C}(G)$ possesses the FIP ([7, Proposition 7.15]) but, in general, fails to be coherent.

In $\mathscr{C}(G)$ the convex ℓ-subgroup generated by $a \in G$ is denoted by $G(a)$. It is well known that each compact element of $\mathscr{C}(G)$ is of this form; this follows from the fact that every finitely generated convex ℓ-subgroup is principal ([7, Proposition 7.16]).

Note that G is archimedean if and only if $\mathscr{C}(G)$ is archimedean.

4. The polars of $\mathscr{C}(G)$ are also called polars in this context. We also adopt the conventions that $a^{\perp} \equiv G(a)^{\perp}$ for each $a \in G$; note that $a^{\perp\perp} \equiv G(a)^{\perp\perp}$.
5. We will need this fact about the elements of $\mathrm{Min}(\mathscr{C}(G))$: there is a one-to-one correspondence between $\mathrm{Min}(\mathscr{C}(G))$ and the set of all ultrafilters of positive elements. To each $P \in \mathrm{Min}(\mathscr{C}(G))$ one assigns $G^{+} \setminus P$. The reverse correspondence is
$$U \mapsto \bigcup\{a^{\perp} :\ a \in U\}.$$
(See [7, pp. 77–79].)
6. It is well known that, for every ℓ-group G, $\mathscr{C}(G)$ is a frame with disjointification. Indeed, if $a, b \geqslant 0$ in G, let $c = a - (a \wedge b)$ and $d = b - (a \wedge b)$; then $G(c)$ and $G(d)$ witness the disjointification of $G(a)$ and $G(b)$. Lemma 3.3 then guarantees that $\mathrm{Spec}(\mathscr{C}(G))$ is a root system.
7. Finally, in this series of remarks, observe that if $C \in \mathscr{C}(G)$, a natural lattice ordering is induced on the set G/C of cosets of the form $C + g$ $(g \in G)$ by setting
$$C + a \leqslant C + b \quad \text{iff} \quad a \leqslant g + b \text{ for some } g \in C.$$
The above lattice ordering is a total ordering if and only if $C \in \mathrm{Spec}(\mathscr{C}(G))$. If C is a normal subgroup then G/C is an ℓ-group and the natural map $g \mapsto C + g$ is an ℓ-homomorphism. G/C is a totally ordered group if and only if C is a normal prime in $\mathscr{C}(G)$.

For purposes of this paper we will assume that each ℓ-group is *representable***; that is, that G is a subdirect product of totally ordered groups.** It is well known that G is representable if and only if each polar G is a normal subgroup, or, alternatively, if and only if each minimal prime convex ℓ-subgroup of G is normal. (See [7, Proposition 47.1].)

This is the appropriate place to define ℓ-dimension.

Definition & Remarks 5.2. Let G be an ℓ-group. We define $\dim_{\ell}(G) \equiv \dim(\mathscr{C}(G))$, and call it the *$\ell$-dimension* of G.

Note that $\dim_\ell(G) = 0$ means that G is *hyperarchimedean*; that is, every ℓ-homomorphic image of G is archimedean. (See [23, 3.6].) The study of hyperarchimedean ℓ-groups originated with Conrad in [5]; [7, Theorem 55.1] gives some of the principal conditions which characterize a hyperarchimedean ℓ-group; (d) of this theorem tells us that G is hyperarchimedean if and only if $\mathscr{C}(G)$ has the CSP.

The goal of this section is to give an elementwise characterization of the ℓ-groups of finite ℓ-dimension. Instead of adapting the language of supplementing arrays here, we opt for a direct application of Proposition 3.4, which is less cumbersome and more transparent.

Definition 5.3. Let G be an ℓ-group and $a_0 < a_1 < \ldots < a_k$ a sequence of positive elements. It is an *ℓ-dominance chain of length k* if

$$a_0 \wedge (a_1 - n_1 a_0)^+ \wedge \ldots \wedge (a_k - n_k a_{k-1})^+ > 0$$

for all positive integers $n_1, \ldots, n_k$.

$a_0 < a_1 < \ldots < a_n < \ldots$ is an *ascending ℓ-dominance chain* if

$$a_0 \wedge (a_1 - n_1 a_0)^+ \wedge \ldots \wedge (a_k - n_k a_{k-1})^+ > 0,$$

for all positive integers $n_1, \ldots, n_k$ and every positive integer k.

Theorem 5.4. *Let G be an ℓ-group. Then $\dim_\ell(G) \leqslant k$ if and only if every ℓ-dominance chain of G has length $\leqslant k$.*

Proof. What we actually show is that if $a_0 < a_1 < \ldots < a_l$ is an ℓ-dominance chain, then $G(a_0) \subset G(a_1) \subset \ldots \subset G(a_l)$ is a dominance chain in $\mathscr{C}(G)$, and vice-versa.

Indeed, suppose that $a_0 < a_1 < \ldots < a_l$ is an ℓ-dominance chain. Then the set

$$S = \{a_0 \wedge (a_1 - n_1 a_0)^+ \wedge \ldots \wedge (a_l - n_l a_{l-1})^+ \colon\ n_1, \ldots, n_l \in \mathbb{N}\}$$

is a filter base of strictly positive elements of G, which may then be embedded in an ultrafilter. In view of 5.1.5, there is a minimal prime of $\mathscr{C}(G)$, say P, that excludes each member of S. Then, in the totally ordered group G/P, we have

$$P < P + a_0 \ll \ldots \ll P + a_l.$$

Clearly, $P \subset P + G(a_0) \subset \ldots \subset P + G(a_l)$, proving that $G(a_0) \subset \ldots \subset G(a_l)$ is a dominance chain of $\mathscr{C}(G)$.

Conversely, suppose that $G(a_0) \subset \ldots \subset G(a_l)$ is a dominance chain of $\mathscr{C}(G)$. First, we may assume that $0 < a_0 < \ldots < a_l$. Next, pick a prime $P \in \mathscr{C}(G)$ such that $P \subset P+G(a_0) \subset \ldots \subset P+G(a_l)$. Without loss of generality we may assume that P is minimal. Then, since G/P is totally ordered, we have that $P < P+a_0 \ll \ldots \ll P+a_l$. We leave it to the reader to check that this implies that $0 < a_0 < \ldots < a_l$ is an ℓ-dominance chain. $\square$

The preceding theorem has a number of consequences. The first two are easy to prove.

Corollary 5.5. *If G is an ℓ-group and $\dim_\ell(G) < \infty$, then the same is true of any ℓ-subgroup and any ℓ-homomorphic image of G.*

Indeed, if A is an ℓ-subgroup of G, then $\dim_\ell(A) \leqslant \dim_\ell(G)$, and if $\varphi\colon G \to H$ is any surjective ℓ-homomorphism, then $\dim_\ell(H) \leqslant \dim_\ell(G)$.

P r o o f. For ℓ-subgroups the conclusion is an immediate consequence of Theorem 5.4. If $\varphi\colon G \to H$ is a surjective ℓ-homomorphism, then the map $N \mapsto \varphi^{-1}(N)$ embeds $\mathrm{Spec}(\mathscr{C}(H))$ as a partially ordered subset of $\mathrm{Spec}(\mathscr{C}(G))$. This makes it clear that if $\dim_\ell(G)$ is finite then so is $\dim_\ell(H)$.

The inequalities claimed in the corollary are now also clear. $\square$

Recall that an ℓ-group G is *laterally σ-complete* if each countable set of pairwise disjoint elements has a supremum. We will say that G is *properly laterally σ-complete* if it is laterally σ-complete and contains an infinite pairwise disjoint set.

Next we have the following.

Corollary 5.6. *If $\dim_\ell(G) < \infty$ then G contains no ℓ-subgroup which is properly laterally σ-complete.*

P r o o f. Suppose that $\{g_1, g_2, \ldots\}$ is an infinite pairwise disjoint set, lying in the laterally σ-complete ℓ-subgroup H. Put

$$a_n = \bigvee_{m \in \mathbb{N}} m^n g_m, \ \forall\, n \geqslant 0.$$

(Interpret the supremum as being calculated in H.) We leave it to the reader to verify that, for each integer k, $a_0 < a_1 < \ldots < a_k$ is an ℓ-dominance chain. This contradicts the assumption that $\dim_\ell(G) < \infty$. $\square$

Since any infinite direct product of nontrivial ℓ-groups contains a copy of the group of all integer-valued sequences, $\mathbb{Z}^{\mathbb{N}}$, and the latter is properly laterally σ-complete, we have, in particular:

Corollary 5.7. *Suppose that G is an ℓ-group of finite ℓ-dimension. Then there is no ℓ-subgroup of G which is ℓ-isomorphic to an infinite product of nontrivial ℓ-groups.*

The counterpart of Corollary 3.5 and an easy adaptation of the proof of Theorem 5.4 yield the following.

Corollary 5.8. *Let G be an ℓ-group. Then* $\mathrm{Spec}(\mathscr{C}(G))$ *satisfies the ascending chain condition precisely when there are no ascending ℓ-dominance chains. If this is the case then no ℓ-subgroup of G is properly laterally σ-complete.*

Proof. (Sketch) If $a_0 < a_1 < \ldots < a_n < \ldots$ is an ascending ℓ-dominance chain, then

$$S = \{a_0 \wedge (a_1 - n_1 a_0)^+ \wedge \ldots \wedge (a_l - n_l a_{l-1})^+ \colon\ n_1, \ldots, n_l \in \mathbb{N} \text{ and each } l \in \mathbb{N}\}$$

is a filter of strictly positive elements. Embed that in an ultrafilter; then proceed as in the proof of Theorem 5.4. Apply Corollary 3.5. □

Remark 5.9. Any attempt at a converse of these results on subgroups which are properly laterally σ-complete is hopeless. If G is any totally ordered group then it has no ℓ-subgroup which is properly laterally σ-complete, but there are such groups with arbitarily long ascending chains of prime subgroups.

One can even manufacture archimedean ℓ-groups with these properties. For example, let G be the ℓ-group of all sequences of integers which are eventually polynomial. In G the subgroup S of sequences which are eventually zero is in $\mathrm{Spec}(\mathscr{C}(G))$, and G/S has an infinite ascending chain of convex subgroups. Yet G contains no properly laterally σ-complete ℓ-subgroups, as such a subgroup is necessarily uncountable, whereas G is countable.

The other application we have in mind concerns f-rings. For this reason and for later use, we review here the basic information on f-rings.

Definition & Remarks 5.10. In this commentary A stands for a commutative ring with identity. A is an *ℓ-ring* if it has a lattice structure such that $(A, +, 0, -(\cdot), \vee, \wedge)$ is an ℓ-group and A^+ is closed under multiplication.

1. If A is an ℓ-ring such that $a \wedge b = 0$ and $c \geqslant 0$ imply that $ac \wedge b = 0$ then we say that A is an *f-ring*. Assuming the Axiom of Choice, A is an f-ring if and only if it is a subdirect product of totally ordered rings; see [3, Theorem 9.1.2]. (In the product one assumes coordinatewise operations, and the substructure is both an ℓ-subgroup and a subring.)

2. The ℓ-ring A is an f-ring if and only if each polar is a ring ideal or, equivalently, if and only if each minimal prime of $\mathscr{C}(A)$ is a ring ideal [3, Theorem 9.1.2].
3. Recall that a ring A is *semiprime* if there are no nonzero nilpotent elements in A. Suppose that A is an f-ring. If A is semiprime then $ab = 0$ if and only if $|a| \wedge |b| = 0$ [3, Theorem 9.3.1]. Thus, in a semiprime f-ring the polars of $(A, +, 0, -(\cdot), \vee, \wedge)$ are annihilator ideals and conversely.
4. In a semiprime f-ring A, $\mathrm{Min}(\mathscr{C}(A))$ consists of all the minimal prime ring ideals [3, Theorem 9.3.2].
5. Any f-ring A for which $(A, +, 0, -(\cdot), \vee, \wedge)$ is archimedean, is necessarily semiprime. We say that an f-ring is *hyperarchimedean* if the additive structure is a hyperarchimedean ℓ-group.

   It is well known, and an application of [7, Theorem 55.1], that if A is an f-ring then it is hyperarchimedean if and only if it can be represented as a ring of real valued functions on a set X such that the identity is associated to the constant function 1, and each $0 < f \in A$ is represented as a bounded function which is also bounded away from zero; that is, if $0 < f \in A$, then there is a positive real number r such that $f(x) \neq 0 \Rightarrow f(x) \geqslant r$.
6. Suppose that $f \in A$; if $1 < f \vee 1$ is an ℓ-dominance chain we say that f is *unbounded*; otherwise, we say that f is *bounded*. The reader will easily determine that $f \in A^+$ is bounded if and only if $f \leqslant n{\cdot}1$ for some positive integer n, which conforms to the most reasonable—i.e., intuitive—interpretations of boundedness.

   Likewise, if $0 < f \wedge 1 < 1$ is an ℓ-dominance chain we call f *unbounded away from zero*; otherwise, f is *bounded away from zero*. In the proof of Theorem 5.11 it will be claimed that if each $f \in A$ is both bounded and bounded away from zero, then A is hyperarchimedean. This follows from [9, Theorem 2.3], and shows what is claimed in the second paragraph of 5, without appealing to the representation.
7. If A is an f-ring we may consider convex ℓ-subgroups which are simultaneously ring ideals. These are called the *ℓ-ideals* of A; the lattice $\mathscr{I}_\ell(A)$ of all ℓ-ideals of A is a subframe of $\mathscr{C}(A)$. In fact, $\mathscr{I}_\ell(A)$ is a coherent frame; see [2, Proposition 2.2] and also the discussion comprised by Lemma 4.2 of [23] and its consequences.

Now here is the application of Theorem 5.4 we had in mind.

Theorem 5.11. *Suppose that A is a semprime, commutative f-ring with identity. Then*

(a) *if $\mathrm{Spec}(\mathscr{C}(A))$ satisfies the ascending chain condition then every $0 < f \in A$ is bounded;*

(b) *if* $\operatorname{Spec}(\mathscr{C}(A))$ *satisfies the descending chain condition then every* $0 < f \in A$ *is bounded away from zero;*
(c) *if* $\dim_\ell(A)$ *is finite then* A *is hyperarchimedean.*

Proof. (a) It suffices to show that each $f \geqslant 1$ in A is bounded. By way of contradiction, suppose that $1 < f$ is an ℓ-dominance chain. Then, as in the proof of Theorem 5.4, there is a $P \in \operatorname{Min}(\mathscr{C}(A))$—which is also a minimal prime ideal of the ring A—such that $P + 1 \ll P + f$ in the totally ordered domain A/P. But then also $P + 1 \ll P + f \ll \ldots \ll P + f^n \ll \ldots$, which contradicts the assumption that $\operatorname{Spec}(\mathscr{C}(A))$ satisfies the ascending chain condition.

The proof of (b) is similar, and is therefore omitted. As to (c), if $\dim_\ell(A) < \infty$, then both the ascending and descending chain conditions hold for $\operatorname{Spec}(\mathscr{C}(A))$, so that each $f \in A^+$ is both bounded and bounded away from zero; this implies that A is hyperarchimedean, as has already been explained. □

Finally, in this section, we have the example promised in 2.11.

Example 5.12. An archimedean ℓ-group G such that $\dim_\ell(G) = 1$, yet for which $\operatorname{Reg}(\mathscr{C}(G)) = 3$.

Let $\mathscr{E}$ be a partition of $\mathbb{N}$ into a countably infinite number of infinite subsets. For each $E \in \mathscr{E}$ we enumerate $E = \{s_1^E, s_2^E, \ldots\}$, with $s_1^E < s_2^E < \ldots$, in the natural ordering. Next, we consider the real vector space of real valued sequences generated by

- the finitely nonzero sequences;
- the characteristic functions χ_E for each $E \in \mathscr{E}$; and
- the functions g_E $(E \in \mathscr{E})$, g and g^2, where, for each $E \in \mathscr{E}$,

$$g_E(m) = \begin{cases} n & \text{if } m = s_n^E, \\ 0 & \text{otherwise,} \end{cases}$$

  and $g(s_n^E) = n$ for each natural number n and each $E \in \mathscr{E}$, and $g(m) = 0$ elsewhere.

The following facts are then easily verified:

1. G is an archimedean ℓ-group for which G/S is ℓ-isomorphic to the vector lattice H generated by the eventually constant sequences and a single unbounded sequence of integers, where S denotes the subgroup of bounded sequences in G.
2. $\operatorname{Reg}(\mathscr{C}(H)) = 2$, proving that $\operatorname{Reg}(\mathscr{C}(G)) = 3$.
3. S is the regular top of $\mathscr{C}(G)$.
4. All but one of the minimal primes of $\mathscr{C}(G)$ fail to contain S; that singular prime is the subgroup of all sequences in G generated by S and the g_E.
5. Putting the above together, one concludes that $\dim_\ell(G) = 1$.

6. The d-dimension of an ℓ-group

We retain the assumption that the ℓ-groups be representable.

Recall that if L is an algebraic frame with the FIP, then dL denotes the algebraic frame of all d-elements. If G is an ℓ-group we denote $d(\mathscr{C}(G)) \equiv \mathscr{C}_d(G)$; the *$d$-dimension* of G is defined as

$$\dim_d(G) = \dim(\mathscr{C}_d(G)).$$

Observe that since $\mathscr{C}(G)$ has the FIP and disjointification, so does $\mathscr{C}_d(G)$.

Remark 6.1. The elements of $\mathscr{C}_d(G)$ will be called *d-subgroups.* They have been given different names in literature; for a discussion of this see [23, Remark 5.6].

We have already recalled in 4.1(iv) a characterization of the algebraic frames L with the FIP for which $\dim(dL) = 0$. If G is an ℓ-group and $\dim_d(G) = 0$—that is to say, $\mathscr{C}_d(G)$ is a regular frame—we say that G is *d-regular.* Also in [23, Remark 5.6] a reference is made to [6, Theorem 3.1], showing that G is d-regular precisely when every prime d-subgroup is minimal.

Recall that an ℓ-group G is *complemented* if for each $0 < a \in G$ there is a $b \in G^+$ such that $a \wedge b = 0$ and $a \vee b$ is a weak order unit. G is *locally complemented* if each principal convex ℓ-subgroup is complemented. We shall express this here as follows (leaving it to the reader to check that this condition is equivalent to local complementation): for each pair $0 \leqslant a \leqslant g$ in G there exists a $0 \leqslant b \leqslant g$ such that $a \wedge b = 0$ and $(a \vee b)^{\perp\perp} = g^{\perp\perp}$.

It is also shown in [6] that G is d-regular if and only if G is locally complemented.

In the following we will apply Theorem 4.2 to $\mathscr{C}_d(G)$ to obtain information about the d-dimension of an ℓ-group.

Among finite valued ℓ-groups, $\dim_d(G) \leqslant k$ has an interesting interpretation in the root system of values. Prior to stating the result, let us review some information about values in ℓ-groups and set up some terminology.

Definition & Remarks 6.2. In this commentary G stands for a fixed ℓ-group.

1. A convex ℓ-subgroup V of G which is maximal with respect to not containing some $g \in G$ is called a *value* of G; we say that g has a *value at V*. It is well known that values are prime subgroups. In fact, $V \in \mathscr{C}(G)$ is a value precisely when it is a *meet-irreducible* element of the frame $\mathscr{C}(G)$; that is, $V = \cap_{i\in I} C_i$ in $\mathscr{C}(G)$ implies that $V = C_j$ for some $j \in I$.

   In addition, for $a, b \in G^+$ we have $G(a) \subseteq G(b)$ if and only if each value of a is contained in a value of b. This is a consequence of the fact that in any algebraic frame L, each compact element is the infimum of meet-irreducible elements ([17, Lemma 1.3]; for ℓ-groups, see [7, Proposition 10.7]).

Let $\mathrm{Val}(G)$ denote the set of values of G.

2. A value V is said to be *special* if it is the only value of some $g \in G$. An element having a single value is also called *special*. By a theorem of Conrad ([7, §46]), the following are equivalent.

(a) G is finite valued; that is, every nonzero element of G has at most finitely many values.

(b) Each $g \in G^+$ can be written uniquely as

$$g = g_1 + \ldots + g_n,$$

where $g_i \wedge g_j = 0$ for all $i \neq j$, with each g_i special.

(c) Every value of G is special.

3. Let P be a prime convex ℓ-subgroup of G. P is a *branch point* of $\mathrm{Spec}(\mathscr{C}(G))$ if there exist prime convex ℓ-subgroups Q_1 and Q_2, both properly contained in P such that $P = Q_1 \vee Q_2$. If P is a branch point, then $Q_1, Q_2, \ldots, Q_m$ is a *branching set* for P if each $Q_i \subset P$ and $Q_i \vee Q_j = P$ for each $i \neq j$. The *branching index* of P is the size of the largest branching set for P if there is some maximum size, or ∞ otherwise. We denote the branching index of P by $\mathrm{br}(V)$.

4. Last, suppose that $V_0 \subset V_1 \subset \ldots \subset V_m$ is a chain of values in G. We will call it *adequately branched* if for each $i = 1, \ldots, m$, V_{i-1} does not lie in any maximal finite antichain of $\mathrm{Val}(G) \cap (\downarrow V_i)$. The reader will easily verify that if each V_i $(i = 1, \ldots, m)$ in this chain has infinite branching index, then the chain is adequately branched.

Here is the theorem we had in mind.

Theorem 6.3. *Suppose that G is a finite valued ℓ-group. Then $\dim_d(G) \leqslant k$ if and only if for each adequately branched chain of values $V_0 \subset V_1 \subset \ldots \subset V_m$, it follows that $m \leqslant k$.*

Proof. Let us first assume that there is an adequately branched chain of values $V_0 \subset V_1 \subset \ldots \subset V_m$. Choose, for each $j = 0, 1, \ldots, m$, a positive special element a_j having its value at V_j. Let P be any minimal prime convex ℓ-subgroup contained in V_0. Observe that $a_0 \ll \ldots \ll a_m$. We now establish that

$$a_0^{\perp\perp} \subset \ldots \subset a_m^{\perp\perp}$$

is a dominance chain in $\mathscr{C}_d(G)$ and appeal to Proposition 3.4 directly. To that end, suppose that $g \in P^+$ and write $g = g_1 + \ldots + g_n$ as a sum of pairwise disjoint special elements; let W_i be the value of g_i. Since $g \in P$ it follows that $V_j \not\subseteq W_i$ for each $j = 0, 1, \dots, m$ and $i = 1, \dots, n$. Now choose $s \in G^+$ to be special with value W, such that $W \subset V_j$ and $W \parallel W_i$ for each $i = 1, \dots, n$; since the chain of V_j is adequately branched, this can be done. Note that then $s \wedge g = s \wedge a_{j-1} = 0$, whence $s \wedge (a_{j-1} + g) = 0$, while $s \ll a_j$. All this implies that $a_j \notin (a_{j-1} + g)^{\perp\perp}$, and hence

$$d(G(a_{j-1}) \vee P) \subset d(G(a_j) \vee P).$$

This proves that $a_0^{\perp\perp} \subset \dots \subset a_m^{\perp\perp}$ is a dominance chain in $\mathscr{C}_d(G)$, as claimed.

Conversely, suppose that $a_0^{\perp\perp} \subset \dots \subset a_m^{\perp\perp}$ is a dominance chain. We shall exhibit an adequately branched chain of values of length m. It is easily checked that, without loss of generality, each of the a_i may be assumed to be special, and then $a_0 \ll \dots \ll a_m$. We denote the value of a_i by V_i. Now we show that $V_0 \subset V_1 \subset \dots \subset V_m$ is adequately branched. We shall apply Theorem 4.2.

By way of contradiction, suppose that V_{j-1} lies in the maximal antichain of values $\{V_{j-1} = W_0, W_1, \dots, W_n\}$ in $\mathrm{Val}(G) \cap (\downarrow V_j)$. Pick a positive special element b_i having its value at W_i; do this for $i \geqslant 1$. Then it is easy to see that $a_j \in (a_{j-1} + b_1 + \dots + b_n)^{\perp\perp}$, which establishes $\{\{b_1 + \dots + b_n\}^{\perp\perp}\}$ as a bad supplementing array in $\mathscr{C}_d(G)$ for $a_j^{\perp\perp}$, contradicting the dominance. This shows that $V_0 \subset V_1 \subset \dots \subset V_m$ is adequately branched, as asserted. $\square$

By the comment in 6.2.4 we have the following immediate consequence of the preceding theorem.

Corollary 6.4. *Suppose that G is a finite valued ℓ-group. If $\dim_d(G) \leqslant k$ then, for each chain of values $V_1 \subset \dots \subset V_m$ such that $\mathrm{br}(V_j) = \infty$ for each $j = 1, \dots, m$, it follows that $m \leqslant k$.*

The following comment is for the reader who is expert in the theory of ℓ-groups.

Remark 6.5. One might be tempted to generalize Theorem 6.3 to special valued ℓ-groups. (For the definition of this concept the reader is referred to [7, p. 276]. It suffices to say here that in a special valued ℓ-group the special values also play a pivotal role, although not every value is necessarily special.) To discourage any attempts at generalization, we observe that all Hahn groups are special valued, no matter how complicated the root system on which they are defined, yet they are all d-regular, that is to say, locally complemented.

Still in the realm of finite valued ℓ-groups the following example is telling, we believe, although how typical it is is not clear.

Example 6.6. A finite valued abelian ℓ-group G of d-dimension 1 having no primes in $\mathscr{C}(G)$ with infinite branching index.

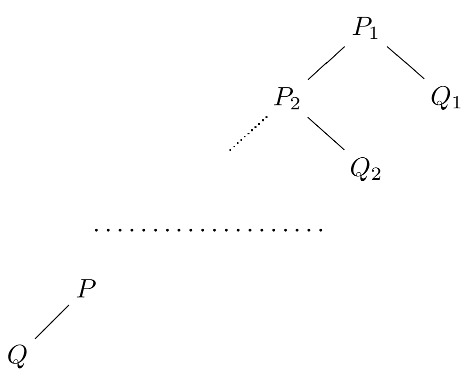
G is the direct sum of copies of $\mathbb{R}$, indexed over the set $\{m_1, n_1, m_2, n_2, \dots, p\}$, where $g = (g_{m_1}, g_{n_1}, \dots; g_p)$ is positive if there is an m_i for which $g_{m_i} \neq 0$, and $g_{m_j} > 0$ for the first such index, and if so then $g_{n_k} \geqslant 0$ for each $k < j$; or else each $g_{m_i} = 0$, and then each $g_{n_i} \geqslant 0$ and also $g_p \geqslant 0$. It is easily verified that G is a finite valued ℓ-group.

Now $\mathrm{Spec}(\mathscr{C}(G))$ consists of the following subgroups:

- $P = \{g \in G\colon\ g_{m_i} = 0, \forall\, i \in \mathbb{N}\}$ and $Q = \{g \in G\colon\ g_{m_i} = 0, \forall\, i \in \mathbb{N}$ and $g_p = 0\}$, with $Q \subset P$.
- For each $i \in \mathbb{N}$, $P_i = \{g \in G\colon\ g_{m_j} = 0, \forall\, j \leqslant i\}$ and $Q_i = \{g \in G\colon\ g_{n_i} = 0$ and $g_{m_j} = 0, \forall\, j \leqslant i\}$.

The diagram below depicts $\mathrm{Spec}(\mathscr{C}(G))$. Note that $\mathrm{Min}(G) = \{Q\} \cup \{Q_i\colon\ i \in \mathbb{N}\}$, while each prime subgroup is a value, except P. On the other hand, P is the only nonminimal prime d-subgroup; thus $\dim_d(G) = 1$.

It is also interesting to note that any chain of values which is adequately branched must begin (at the bottom) with Q, and clearly, from the picture, cannot include more than one of the P_i. This is what Theorem 6.3 predicts. However, each prime in $\mathscr{C}(G)$ has finite branching index.



Since an infinite product of copies of $\mathbb{R}$ is not hyperarchimedean, we have that the class of ℓ-groups G for which $\dim_\ell(G) = 0$ is not closed under products. d-dimension behaves differently.

Theorem 6.7. *Suppose that $\{G_\lambda\colon\ \lambda \in \Lambda\}$ is a family of ℓ-groups. Let $G = \prod_{\lambda\in\Lambda} G_\lambda$, the direct product with coordinatewise operations. Then*

$$\dim_d(G) = \sup\{\dim_d(G_\lambda)\colon\ \lambda \in \Lambda\}.$$

P r o o f. Suppose that $0 < a_0 < a_1 < \ldots < a_k$ is a sequence in G_λ such that $0 \subset a_0^{\perp\perp} \subset \ldots \subset a_k^{\perp\perp}$ is a dominance chain in $\mathscr{C}_d(G_\lambda)$. In G define b_i by

$$(b_i)_\mu = \begin{cases} 0 & \text{if } \mu \neq \lambda, \\ a_i & \text{otherwise.} \end{cases}$$

We show, using Theorem 4.2, that $0 \subset b_0^{\perp\perp} \subset \ldots \subset b_k^{\perp\perp}$ defines a dominance chain in $\mathscr{C}_d(G)$. Suppose, to the contrary, that $g_1, g_2, \ldots, g_k \in G^+$ are such that $(b_i \vee g_{i+1})^{\perp\perp} = b_{i+1}^{\perp\perp}$ for each $i = 0, \ldots, k-1$, yet $b_0 \wedge g_1 \wedge \ldots \wedge g_k = 0$. Then, as the reader will easily be able to check, $((b_i)_\lambda \vee (g_{i+1})_\lambda)^{\perp\perp} = (b_{i+1})_\lambda^{\perp\perp}$ for each $i = 0, \ldots, k-1$, with $(b_0)_\lambda \wedge (g_1)_\lambda \wedge \ldots \wedge (g_k)_\lambda = 0$, which is a contradiction.

The preceding argument shows that $\dim_d(G) \geqslant \sup\limits_{\lambda \in \Lambda} \dim_d(G_\lambda)$.

As to the reverse inequality, it suffices to show that if $0 < x_0 < \ldots < x_k$ in G is such that $0 \subset x_0^{\perp\perp} \subset \ldots \subset x_k^{\perp\perp}$ is a dominance chain in $\mathscr{C}_d(G)$, then, for some $\lambda \in \Lambda$, the polars in G_λ of the coordinates

$$(\Delta_\lambda) \qquad 0 \subset (x_0)_\lambda^{\perp\perp} \subset \ldots \subset (x_k)_\lambda^{\perp\perp},$$

define a dominance chain.

To the contrary, suppose that for each $\lambda \in \Lambda$ there exists a subset of G_λ^+, $\{g_1^\lambda, \ldots, g_k^\lambda\}$, such that $(g_1^\lambda)^{\perp\perp}, \ldots, (g_k^\lambda)^{\perp\perp}$ is a bad supplementing array in $\mathscr{C}_d(G_\lambda)$ for the chain (Δ_λ). Now form elements $g_i \in G$ defined by $(g_i)_\lambda = g_i^\lambda$ for each $i = 1, \ldots, k$. We shall leave it to the reader to verify that $g_1^{\perp\perp}, \ldots, g_k^{\perp\perp}$ is a bad supplementing array in $\mathscr{C}_d(G)$ for the original chain of x_i's. This contradicts our assumption about the x_i, and it follows that there is an index $\lambda \in \Lambda$ such that

$$0 \subset (x_0)_\lambda^{\perp\perp} \subset \ldots \subset (x_k)_\lambda^{\perp\perp}$$

defines a dominance chain. This also completes the proof. □

What follows is an immediate consequence. The second claim can be proved directly; it is mentioned in [23, 5.6(b)].

Corollary 6.8. *The direct product of ℓ-groups of d-dimension k has d-dimension k. In particular, the direct product of d-regular ℓ-groups is d-regular.*

7. The r-dimension of an f-ring

There are at least two obvious dimensions associated with an f-ring. In this section we consider the r-dimension, arising from the frame of ℓ-ideals of a commutative f-ring. We begin with basic definitions; the reader might also refer to 5.10.7 and [23, 7.2].

Definition & Remarks 7.1. In this discussion A stands for a commutative f-ring with identity.

(a) Recall that a convex ℓ-subgroup of A which is also a ring ideal is called an *ℓ-ideal*. $\mathscr{I}_\ell(A)$ stands for the algebraic frame of all ℓ-ideals of A; recall that it is a subframe of $\mathscr{C}(A)$.

Now $\mathscr{I}_\ell(A) = \mathrm{fix}(\varrho)$ for the closure operator ϱ defined on $\mathscr{C}(A)$ by letting $\varrho(K)$ denote the ℓ-ideal of A generated by K for $K \in \mathscr{C}(A)$; see [2, Lemma 2.2]. As explained in [23, 7.2],, the prime elements of $\mathscr{I}_\ell(A)$ are the ℓ-ideals $\mathfrak{r}$ which are prime in $\mathscr{C}(A)$; that is, those for which $A/\mathfrak{r}$ is a totally ordered ring; see also [2, Remark 2.3].

As has already been noted, $\mathscr{I}_\ell(A)$ is a coherent frame and, in particular, has the FIP. It is also easy to see that any finitely generated ℓ-ideal is principal. Moreover, it is well known that ϱ preserves finite intersections, which implies that $\mathscr{I}_\ell(A)$ has disjointification.

(b) The *r-dimension* of A, written $\dim_r(A)$, *is* $\dim(\mathscr{I}_\ell(A))$. Applying 2.5(b), we have the following characterization of the f-rings with r-dimension 0. See [23, Proposition 7.3]. An f-ring with these properties is said to be *ℓ-regular*. For an f-ring A, the following are equivalent.

(i) $\dim_r(A) = 0$.

(ii) For each $a \geqslant 0$ in A there exist $d \geqslant 0$ and an idempotent $e \leqslant ad$ such that $a = ea$.

(iii) Each ℓ-ideal of A is an intersection of minimal prime convex ℓ-subgroups of A.

(iv) If $\mathfrak{r} \in \mathscr{I}_\ell(A)$ and P is any prime convex ℓ-subgroup which is minimal over $\mathfrak{r}$, then P is a minimal prime convex ℓ-subgroup and, in particular, an ℓ-ideal.

(v) Each ℓ-ideal of A is an intersection of minimal prime ideals of A.

The reader will readily observe that (ii) above implies that ℓ-regular f-rings are semiprime; it will follow from Corollary 7.6, in any event. However, as noted in [23, §7], such f-rings need not be von Neumann regular, nor hyperarchimedean. In fact, every singular f-ring and, indeed, every bounded away f-ring is ℓ-regular; the reader should refer to [23, §7] and [9, Theorem 2.3] for details.

Condition (ii) also implies that the class of ℓ-regular f-rings is closed under formation of products, as observed in [23, 7.4].

To study the r-dimension of an f-ring we need the f-ring-theoretic version of dominance chains.

Definition 7.2. Once again, A stands for a commutative f-ring with 1.

The sequence $a_0 < a_1 < \ldots < a_k$ of positive elements of A is an *r-dominance chain* if

$$a_0 \wedge \left(\bigwedge_{i=1}^{k} (a_i - r_i a_{i-1})^+ \right) > 0,$$

for each $r_1, r_2, \ldots, r_k \in A^+$. The *length* of the chain is k.

Here is the current analogue of Theorem 5.4. The proof is hardly surprising by now, and so a sketch will suffice.

Theorem 7.3. *Suppose A is a commutative f-ring with identity. Then* $\dim_r(A) \leqslant k$ *if and only if every r-dominance chain of A has length $\leqslant k$.*

Proof. Suppose that an r-dominance chain $a_0 < \ldots < a_k$ of length k exists. Then there is a minimal prime convex ℓ-subgroup P—which is a minimal prime in $\mathscr{I}_\ell(A)$—such that

$$a_0 \wedge \left(\bigwedge_{i=1}^{k} (a_i - r_i a_{i-1})^+ \right) > 0$$

for each $r_1, r_2, \ldots, r_k \in A^+$. The reader will readily check that this produces a chain of ℓ-ideals in the totally ordered ring A/P,

$$0 \subset \varrho(\{P + a_0\}) \subset \ldots \subset \varrho(\{P + a_k\}),$$

and then one may choose primes $P_1 \subset \ldots \subset P_k$ in $\mathscr{I}_\ell(A)$ containing P such that

$$\varrho(\{P + a_0\}) \subseteq P_1/P_0 \subset \varrho(\{P + a_1\}) \subseteq \ldots \subseteq P_k/P_0 \subset \varrho(\{P + a_k\}).$$

Thus it is clear that $\dim_r(A) \geqslant k$.

Conversely, suppose that a chain $Q_0 \subset \ldots \subset Q_k$ of primes in $\mathscr{I}_\ell(A)$ exists. Choose positive elements $b_i \in Q_{i+1} \setminus Q_i$ for each $i = 0, 1, \ldots, k$ (with $A = P_{k+1}$); without loss of generality the b_i may be chosen such that $b_0 < b_1 < \ldots < b_k$. Then, for each choice $r_1, \ldots, r_k \geqslant 0$ in A, we have $b_0 \notin Q_0$ and $b_i - r_i b_{i-1} > 0 \bmod Q_0$. Then it is easy to verify that $b_0 < b_1 < \ldots < b_k$ is an r-dominance chain. $\square$

We have already observed that $\dim_r(A) = 0$ forces the ring to be semiprime. Here is an example of a totally ordered ring with nonzero nilpotent elements and r-dimension 1.

Example 7.4. For starters, let B be any ℓ-group, endowed with the zero multiplication. Form $A = B \times \mathbb{Z}$, with coordinatewise addition and multiplication given by the rule

$$(b_1, m_1)(b_2, m_2) = (m_2 b_1 + m_1 b_2, m_1 m_2).$$

Order A lexicographically: $(b, m) \geqslant 0$ if $m > 0$, or else $m = 0$ and $b \geqslant 0$. This defines an f-ring structure on A, in which $B \times \{0\}$ is the set of nilpotent elements. (Note: the reader familiar with general algebra will recognize the product above as that which serves to adjoin an identity to a ring, whether it already possesses one or not.)

Next, observe that, as B bears the zero multiplication, every convex ℓ-subgroup of B, and hence also of A, is an ℓ-ideal. Thus, $\dim_\ell(A) = \dim_r(A) = \dim_\ell(B) + 1 = \dim_r(B) + 1$. So, letting $B = \mathbb{Z}$, we have that A is totally ordered, and $\dim_r(A) = 1$.

A good deal of information about r-dimension comes out of an examination of when the chain $a < 1$ is an r-dominance chain. First, let us establish an easy lemma.

Lemma 7.5. *Suppose that A is a commutative f-ring with identity. Then, for $a \in A^+$, $a < 1$ is not an r-dominance chain precisely when $\varrho(\{a\})$ is a summand of A.*

P r o o f. If $a < 1$ is not an r-dominance chain, then there is a positive $r \in A$ such that

$$a \wedge (1 - (ra \wedge 1)) = a \wedge (1 - ra)^+ = 0.$$

This means that $1 = (ra \wedge 1) + (1 - (ra \wedge 1))$ is the desired decomposition of 1 into a sum of components in $\varrho(\{a\})$ and $a^\perp$, respectively, whence it easily follows that $A = \varrho(\{a\}) + a^\perp$.

Conversely, suppose that $A = \varrho(\{a\}) + a^\perp$ with $0 < a < 1$. Then there exists an e, necessarily idempotent, such that $a \wedge (1 - e) = 0$ and $e \leqslant ra$ for a suitable positive $r \in A$. Thus, $a \wedge (1 - ra)^+ \leqslant a \wedge (1 - e) = 0$, proving that $a < 1$ is not an r-dominance chain. $\square$

Here is a corollary of the lemma, which substantiates our earlier observation that an ℓ-regular f-ring is necessarily semiprime.

Corollary 7.6. *Let A be a commutative f-ring with 1. If $a \in A^+$ is nilpotent, then $a \wedge 1 < 1$ is an r-dominance chain.*

P r o o f. If $a \geqslant 0$ is nilpotent then so is $a \wedge 1$. To simplify notation, let $b = a \wedge 1$. If $b < 1$ is not r-dominant, then $e \leqslant rb$ for a suitable choice of $e, r \in A^+$, with $be = b$. Then if k is the least integer for which $b^k = 0$, we have $b^{k-1} \leqslant rb^k = 0$, a contradiction. $\square$

The next result reveals the consequences of having $a < 1$ r-dominant, at least in a semiprime f-ring: in this context, finite dimension is an all-or-nothing proposition.

Theorem 7.7. *Suppose that A is a semiprime commutative f-ring with 1. Then if $a \in A^+$ with $a < 1$ r-dominant, it follows that*

$$a^k < a^{k-1} < \ldots < a < 1$$

is an r-dominance chain for each positive integer k. Thus, if $\dim_r(A) < \infty$ then A is ℓ-regular.

Proof. All we need to prove is the first claim, as the second clearly follows from the first. Now if $a < 1$ is r-dominant, then $a \wedge (1 - ra)^+ > 0$ for each $r \in A^+$. Owing to the semiprimeness of A, we have the following string of estimates for each $k \in \mathbb{N}$ and all $r_1, r_2, \ldots, r_k \in A^+$, the first inequality being a consequence of the observation that if $0 \leqslant x, y \leqslant 1$ in any f-ring, then $xy \leqslant x \wedge y$:

$$\begin{aligned} a^k \wedge \left(\bigwedge_{i=0}^{k-1} (a^i - r_{i+1}a^{i+1})^+ \right) &\geqslant a^k (a \ldots a^{k-1})(1 - r_1 a)^+ \ldots (1 - r_k a)^+ \\ &\geqslant a^{(k(k+1)/2)}(1 - (r_1 \vee \ldots \vee r_k)a)^+ > 0. \end{aligned}$$

This proves that $a^k < \ldots < a < 1$ is an r-dominance chain, as claimed. □

Combining the preceding theorem with the proof of Theorem 7.3, one gets the following corollary.

Corollary 7.8. *Suppose that A is a semiprime commutative f-ring with identity. If $\dim_r(A) > 0$ then $\operatorname{Spec}(\mathscr{I}_\ell(A))$ fails the descending chain condition.*

We now turn to the dimension of a semiprime f-ring defined by its frame of radical ℓ-ideals.

8. The *sp*-dimension of a semiprime f-ring

Definition & Remarks 8.1. In this discussion A stands for a semiprime commutative f-ring with identity. Define the closure operator on $\mathscr{C}(A)$

$$\begin{aligned} \sqrt{K} &= \{a \in A \colon\ a^n \in \varrho(K) \text{ for a suitable } n \in \mathbb{N}\} \\ &= \{a \in A \colon\ |a|^n \leqslant rb \ \text{ for suitable } n \in \mathbb{N},\ r \in A^+,\ b \in K^+\}. \end{aligned}$$

From [23, Lemma 4.2] it follows that $\operatorname{Rad}_\ell(A) \equiv \operatorname{fix}(\sqrt{(\cdot)})$ is a coherent frame. For every semiprime f-ring A, $\operatorname{Rad}_\ell(A)$ has disjointification. Recall—see 5.10.4—that in a semiprime f-ring the minimal prime ideals and the minimal prime convex ℓ-subgroups coincide. Thus,

$$\mathrm{Spec}(\mathrm{Rad}_\ell(A)) = \mathrm{Rad}_\ell(A) \cap \mathrm{Spec}(\mathscr{C}(A)),$$

and it contains all the minimal prime ideals of A.

The *sp-dimension* of A is $\dim_{sp}(A) \equiv \dim(\mathrm{Rad}_\ell(A))$. It should be clear that $\dim_{sp}(A) \leqslant \dim_r(A)$ for every A. In brief we shall have more to say about the comparison of the various dimensions introduced thus far.

Next up is a discussion of the dominance feature in the context of $\mathrm{Rad}_\ell(A)$.

Definition 8.2. Suppose that A is a semiprime commutative f-ring with 1, and $a_0 < \ldots < a_k$ is a chain of positive elements. We say that it is an *sp-dominance chain* if, for each choice of $r_1, r_2, \ldots, r_k \in A^+$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$,

$$a_0 \wedge \left(\bigwedge_{i=1}^{k} (a_i^{n_i} - r_i a_{i-1})^+ \right) > 0.$$

k is the *length* of the chain.

The following lemma simplifies calculations involving *sp*-dominance. Recall that the commutative f-ring A with 1 is *bounded* if $A(1)$, the convex ℓ-subgroup of A generated by 1, is A.

Lemma 8.3. *Let A be a semiprime commutative f-ring with 1. Suppose $0 < a_0 < \ldots < a_k$ in A.*

(a) *If $a_0 < \ldots < a_k$ is an sp-dominance chain, then so is $a_0 \wedge 1 < \ldots < a_k \wedge 1$.*

(b) *$a_0 < \ldots < a_k$ is an sp-dominance chain if and only if*

$$a_0 \wedge \left(\bigwedge_{i=1}^{k} (a_i^{n_i} - r a_{i-1})^+ \right) > 0$$

for each $0 \leqslant r \in A$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$.

(c) *If $a_0 < \ldots < a_k \leqslant 1$ and*

$$a_0 \wedge \left(\bigwedge_{i=1}^{k} (a_i^{n} - r a_{i-1})^+ \right) > 0$$

for all $0 \leqslant r \in A$ and all $n \in \mathbb{N}$, then $a_0 < \ldots < a_k$ is an sp-dominance chain.

(d) *Assume that A is bounded. Then $a_0 < \ldots < a_k$ is an sp-dominance chain if and only if*

$$a_0 \wedge \Big(\bigwedge_{i=1}^{k} (a_i^{n_i} - m a_{i-1})^+ \Big) > 0$$

for each $m, n_1, n_2, \ldots, n_k \in \mathbb{N}$.

Proof. Note that (c) easily follows from (a), and that (b) and (d) are routine.

We prove (a). Suppose, to the contrary, that $r_1, \ldots, r_k \in A^+$ and $n_1, \ldots, n_k \in \mathbb{N}$ exist such that

$$(a_0 \wedge 1) \wedge \Big(\bigwedge_{i=1}^{k} ((a_i \wedge 1)^{n_i} - r_i(a_{i-1} \wedge 1))^+ \Big) = 0.$$

Now repeatedly use the f-ring identity $f = (f \vee 1)(f \wedge 1)$ for $f \in A^+$, multiplying the above identity successively by $(a_0 \vee 1), (a_1 \vee 1)^{n_1}, \ldots, (a_k \vee 1)^{n_k}$ and distributing appropriately, to obtain

$$\begin{aligned} 0 &= a_0 \wedge (a_0 \vee 1)\Big(\bigwedge_{i=1}^{k} ((a_i \wedge 1)^{n_i} - r_i(a_{i-1} \wedge 1))^+ \Big) \\ &\geqslant a_0 \wedge \Big(\bigwedge_{i=1}^{k} ((a_i \wedge 1)^{n_i} - r_i(a_{i-1} \wedge 1))^+ \Big) \ \geqslant \ 0, \end{aligned}$$

followed by

$$\begin{aligned} 0 &= (a_1 \vee 1)a_0 \wedge (a_1^{n_1} - (a_1 \vee 1)^{n_1} r_1 a_0)^+ \wedge \ldots \wedge (a_1 \vee 1)^{n_1}(a_k^{n_k} - r_k a_{k-1})^+ \\ &\geqslant a_0 \wedge (a_1^{n_1} - (a_1 \vee 1)^{n_1} r_1 a_0)^+ \wedge \ldots \wedge (a_k^{n_k} - r_k a_{k-1})^+ \ \geqslant \ 0, \end{aligned}$$

and so on to the identity

$$a_0 \wedge \Big(\bigwedge_{i=1}^{k} (a_i^{n_i} - (a_i \vee 1)^{n_i} r_i a_{i-1})^+ \Big) = 0,$$

which contradicts that $a_0 < \ldots < a_k$ is an *sp*-dominance chain. □

The reader should now expect what follows next; the proof is closely patterned on that of Theorem 7.3, and we shall leave it as an exercise.

Proposition 8.4. *Let A be a semiprime commutative f-ring with identity. Then $\dim_{sp}(A) \leqslant k$ if and only if every sp-dominance chain in A has length $\leqslant k$.*

Now we combine the two descriptions in this section, of dimension via dominance. The second assertion in the next corollary easily follows from the first.

Corollary 8.5. *Suppose that A is a semiprime commutative f-ring with 1. Then if* $\dim_{sp}(A) = 0$ *then A is ℓ-regular. Thus,* $\dim_r(A) = 0$ *if and only if* $\dim_{sp}(A) = 0$.

Proof. By Lemma 7.5, it suffices to show that no $a \in A^+$ with $a < 1$ is r-dominant. Now, since such a chain is not sp-dominant, there is a positive $r \in A$ and an exponent $n \in \mathbb{N}$ such that $a \wedge (1 - ra)^+ = a \wedge (1^n - ra)^+ = 0$; that is, $a < 1$ is not r-dominant. □

It might appear that Proposition 8.4 could be used to show that products preserve sp-dimension. This, however, is not the case. What is true is stated next; the proof closely resembles the argument in the corresponding part of the proof of Theorem 6.7, and is left to the reader.

Proposition 8.6. *Suppose that $\{A_\lambda\colon\ \lambda \in \Lambda\}$ is a family of semiprime commutative f-rings with 1; put $A = \prod_{\lambda\in\Lambda} A_\lambda$. Then*

$$\dim_{sp}(A) \geqslant \sup\{\dim_{sp}(A_\lambda)\colon\ \lambda \in \Lambda\}.$$

Strict inequality is the norm in the preceding proposition; dramatically so, as we shall presently see. Equality is achieved for 0 sp-dimension, because in that event it coincides with r-dimension, according to Corollary 8.5.

Before proceeding to examine the sp-dimension of archimedean f-rings, we shall give a number of examples, which we believe will assist the intuition of the reader.

Examples 8.7. (a) Let $A = \mathbb{R}[[T]]$, the ring of formal power series in one variable with real coefficients. This is a totally ordered integral domain when ordered as follows:

$$f = \sum_{n=0}^{\infty} r_n T^n > 0 \Leftrightarrow r_k > 0, \text{ for the least } k \text{ such that } r_k \neq 0.$$

It is well known that A is a discrete valuation ring—[1, p. 94]; thus, A is local. In fact, its unique maximal ideal is convex. Thus, $\dim_{sp}(A) = 1$, and this is also the Krull dimension. (We return to consider the relationship between Krull dimension and the sp-dimension in the next section.)

Note that $\dim_r(A) = \infty$, while $\dim_d(A) = 0$.

(b) The result of (a) can be obtained with the following archimedean f-ring. Consider the ℓ-subring A of the ring of all bounded real-valued sequences, generated by the eventually constant sequences and the sequence $j(n) = 1/n$. We leave it to the reader to verify the following:

1. A is a bounded archimedean f-ring with identity—the constant 1—in which the only minimal prime ideal which is not maximal is P_∞, the ideal of all sequences which are eventually zero.
2. A/P_∞ is a totally ordered integral domain with *sp*-dimension 1 and infinite r-dimension.
3. $\dim_{sp}(A) = 1$, $\dim_r(A) = \infty$, while $\dim_d(A) = 0$, as A is complemented. (Recall that an ℓ-group G is *complemented* if for each $0 < a \in G$ there is a $b \in G^+$ such that $a \wedge b = 0$ and $a \vee b$ is a weak order unit.)

(c) We construct a countably infinite product A of f-rings whose *sp*-dimensions are all 1, yet for which $\dim_{sp}(A) = \infty$. This example, though it is not archimedean, will serve as a model for the arguments to follow in the archimedean case.

For each $n \in \mathbb{N}$, A_n denotes the ring of formal power series, $\mathbb{R}[[T]]$, from (a). Let $A = \prod_{n=1}^{\infty} A_n$. Denote the variable in A_n by T_n. For each nonnegative integer k, define $f_0 = (1, 1, \ldots)$ and

$$f_k = (T_1, T_2^{2^k}, \ldots, T_n^{n^k}, \ldots).$$

We verify that, for each k, $f_k < \ldots < f_1 < f_0$ is *sp*-dominant. For each coordinate m, each $g \in A^+$ and each exponent n, we have

$$\begin{aligned}\left(f_k \wedge \left(\bigwedge_{i=1}^{k}(f_{k-i}^n - g f_{k-i+1})^+\right)\right)_m &\geqslant T_m^{m^k} \wedge \left(\bigwedge_{i=1}^{k}(T_m^{m^{k-i}n} - g_m T_m^{m^{k-i+1}})^+\right)\\ &\geqslant T_m^{m^k} \wedge \left(\bigwedge_{i=1}^{k}(T_m^{m^{k-i}n} - r_m T_m^{m^{k-i+1}})^+\right),\end{aligned}$$

each r_m being a suitable positive real number such that $r_m \geqslant g_m$. The reader will now observe that, for all $m > n$, the last entry in the above array is strictly positive. This shows that $f_k < \ldots < f_1 < f_0$ is an *sp*-dominance chain; it also follows that $\dim_{sp}(A) = \infty$. (d) To get an archimedean example with the same behavior as the example in (c), replace $\mathbb{R}[[T]]$ in each coordinate by the example in (b), denote the function j in the n-th coordinate by j_n, and again define $f_0 = 1$, while

$$f_k = (j_1, j_2^{2^k}, \ldots, j_n^{n^k}, \ldots).$$

The same argument as in (c) shows that the product of such archimedean f-rings, each of *sp*-dimension 1, has infinite *sp*-dimension.

Our next goal is the analogue of Corollary 5.6 for *sp*-dimension. In order to decipher as exactly as possible why this kind of result works, we state it in a fairly technical way. There is a simple consequence of $a \in A^+$, with $a < 1$ not idempotent, which we highlight in a lemma prior to the theorem. The lemma is doubtless known, but in this context we find it helpful to provide at least a sketch of a proof.

Lemma 8.8. *Suppose that A is a semiprime commutative f-ring with identity and $0 < a < 1$ in A. If a is not idempotent, then for all $n, q \in \mathbb{N}$ there is a $p \in \mathbb{N}$ such that $(a^q - na^m)^+ > 0$ for all $m \geqslant p$.*

Proof. Since a is not idempotent there is a minimal prime ideal $\mathfrak{p}$ of A such that, mod $\mathfrak{p}$, $1 > a > \ldots > a^m > \ldots$. Now if $a \ll 1$ mod $\mathfrak{p}$, we may take $p = 1$ in the claim of the lemma. If, on the other hand, some $ka > 1$ mod $\mathfrak{p}$, then in the field of quotients of $A/\mathfrak{p}$ (with the natural ordering, which restricts to the given one on $A/\mathfrak{p}$), $\mathfrak{p} + 1$ and $(\mathfrak{p} + a)^{-m}$ generate the same convex subgroup for every $m \in \mathbb{N}$. This is enough to ensure that, mod $\mathfrak{p}$ and for any $n \in \mathbb{N}$, $1 > na^p$ for a sufficiently high p, and then for all $m \geqslant p$. The lemma now follows easily. $\square$

We say that an ℓ-group G is *boundedly laterally σ-complete* if every countable set of pairwise disjoint elements which has an upper bound in G has a supremum in G.

Theorem 8.9. *Suppose that A is a commutative semiprime f-ring with identity, and that A is bounded. Assume that there is a countably infinite subset $\{f_1, f_2, \ldots\} \subseteq A^+$ of pairwise disjoint elements such that*

1. *no $f_i < 1$ is idempotent;*
2. *$\{f_1, f_2, \ldots\}$ is contained in an ℓ-subring B of A which is boundedly laterally σ-complete.*

Then $\dim_{sp}(A) = \infty$.

Proof. Define, for each nonnegative integer n,

$$g_n = f_1 \vee f_2^{2^n} \vee f_3^{3^n} \vee \ldots,$$

with the suprema being computed in B. Now, for every $k \in \mathbb{N}$ and $r \in A^+$ we calculate: first,

$$g_k \wedge (g_{k-1}^m - rg_k)^+ \wedge \ldots \wedge (g_0^m - rg_1)^+ \geqslant g_k \wedge (g_{k-1}^m - tg_k)^+ \wedge \ldots \wedge (g_0^m - tg_1)^+$$

for some positive integer $t \geqslant r$, which exists since A is bounded. Call the latter expression a, for brevity; next, write a as a disjoint supremum:

$$a = 0 \vee \left[\bigvee_{n=2}^{\infty} f_n^{n^k} \wedge (f_n^{n^{k-1}m} - tf_n^{n^k})^+ \wedge \ldots \wedge (f_n^m - tf_n^n)^+ \right].$$

Now, invoking Lemma 8.8, we note that

$$f_n^{n^k} \wedge (f_n^{n^{k-1}m} - tf_n^{n^k})^+ \wedge \ldots \wedge (f_n^m - tf_n^n)^+ > 0$$

for sufficiently large n. Thus, $a > 0$, proving that, for any natural number k, $g_k < \ldots < g_1 < g_0$ is an sp-dominance chain, and hence $\dim_{sp}(A) = \infty$. $\square$

Here are some immediate consequences of Theorem 8.9.

Corollary 8.10. *Suppose that A is a semiprime commutative f-ring with identity which is also bounded. If $\dim_{sp}(A) < \infty$, then A contains no ℓ-subring which is isomorphic to the ring of bounded rational valued sequences.*

Corollary 8.11. *With the hypotheses of Theorem 8.9, the descending chain condition fails in $\mathrm{Spec}(\mathrm{Rad}_\ell(A))$.*

Remark 8.12. One might wonder whether the hypothesis that A be bounded is really needed in Theorem 8.9. It might be possible to weaken that assumption, but not drop it entirely. The f-ring of all real sequences $\mathbb{R}^{\mathbb{N}}$ is von Neumann regular and therefore ℓ-regular, which means that its sp-dimension is also zero (by Corollary 8.5).

9. Comparison of dimensions

We turn now to a comparison of the various dimensions hitherto introduced. The result is stated in the next theorem, which, for the most part, collects remarks made earlier. It is here that we have to invoke some background on uniform closure.

Definition & Remarks 9.1. For the concepts of uniformly Cauchy and uniformly convergent sequences in an ℓ-group we refer the reader to [16], in general, and more specifically, regarding z-subgroups, to [11] and [12].

Let G be an abelian ℓ-group. It is shown in [16, Theorem 60.2] that $K \in \mathbf{a}^{\uparrow}(\mathscr{C}(G))$ if and only if K is uniformly closed, provided that G is divisible. A reading of the proof reveals that for the necessity divisibility is not required.

In particular, as is pointed out in [11], in a divisible ℓ-group, our notion of "z-subgroup" coincides with the concept of "z-ideal" presented there. We will not use any of that here. Nonetheless, it seems reasonable to point out that—see [11, Theorem 3.3]—that "z-subgroup" means the same thing as "z-ideal" in a ring of continuous functions.

Further, and this is used in the upcoming proof, if A is an archimedean f-ring with identity and $K \in \mathbf{a}^{\uparrow}(\mathscr{C}(A))$, then K is a ring ideal, by [13, Proposition 3.1]. We note that, once more, in the proof of this fact, the real vector lattice structure and, indeed, divisibility is not needed.

In the theorem, $\dim_z(G) \equiv \dim(z\mathscr{C}(G))$ for any ℓ-group G.

Theorem 9.2. *Suppose that A is a semprime f-ring. Then*

$$\dim_d(A) \leqslant \dim_{sp}(A) \leqslant \dim_r(A) \leqslant \dim_\ell(A);$$

if, in addition, A is archimedean, then we have

$$\dim_d(A) \leqslant \dim_z(A) \leqslant \dim_{sp}(A) \leqslant \dim_r(A) \leqslant \dim_\ell(A).$$

P r o o f. Recall that any d-subgroup of A is a union over a directed supremum of polars. Since every polar is an intersection of minimal primes in $\mathscr{C}(A)$, and each of those is a prime ideal of A, we have that every polar, and hence every d-subgroup is a semiprime ideal. That explains the first inequality in the initial claim. The others are clear.

If A is archimedean, then, as has just been noted, $K \in \mathbf{a}^{\uparrow}(\mathscr{C}(A))$ is a ring ideal; since A/K is an archimedean f-ring it is necessarily semiprime, which means that K is a semiprime ideal. It is then clear that every z-subgroup is a semiprime ideal and, thus, that the insertion of $\dim_z(A)$ in the second string of inequalities is justified. $\square$

The following corollary is intriguing; it is an immediate consequence of Theorems 7.7 and 9.2.

Corollary 9.3. *Suppose that A is an abelian ℓ-group which admits a structure making it into a commutative semiprime f-ring with identity. If $\dim_\ell(A)$ is finite, then A is ℓ-regular and, consequently, also d-regular. Thus, A is complemented.*

Let us now link these various dimensions to Krull dimension. Recall that the *Krull dimension* of a commutative ring with identity A, written $\dim_K(A)$, is the supremum of the lengths of prime ideals.

Proposition 9.4. *Suppose A is a commutative semiprime f-ring with identity. Then $\dim_{sp}(A) \leqslant \dim_K(A)$. If every prime ideal of A is also convex, then equality holds.*

P r o o f. Simply observe that if $\mathfrak{p} \in \operatorname{Spec}(\operatorname{Rad}_\ell(A))$ then $\mathfrak{p}$ is a prime ideal. $\square$

Remark 9.5. A comment is in order concerning when prime ideals are convex in a semiprime commutative f-ring A with 1. It is easy to check that this condition is equivalent to

$$0 \leqslant a \leqslant b \Rightarrow a^n = rb$$

for a suitable $r \in A$ and $n \in \mathbb{N}$. This is implied by any of the n-convexity conditions introduced by S. Larson in [15]. Recall that A is *n-convex* if

$$0 \leqslant a \leqslant b^n \Rightarrow a = rb$$

for suitable $r \in A$. It is well known that every ring of continuous functions is 2-convex. The 1-convex f-rings are reasonably well understood; we refer the reader to [8, Chapter 14] and [22].

Finally, and in spite of the length of this article, the following summary seems appropriate. First, we give the following example, a variation on the one in 8.7(b).

Example 9.6. An archimedean f-ring which is complemented has z-dimension 1, and *sp*-dimension $m < \infty$.

We say that a sequence $s(n)$ of real numbers is *eventually k-logarithmic* if there is an integer m such that

$$s(n) = \frac{1}{\ln(\ln \ldots (n))},$$

k-fold, for each $n \geqslant m$.

Fix a positive integer m and let A be the ℓ-subring of all bounded sequences of real numbers generated by the eventually constant plus

$$j_0(n) = \frac{1}{n}, \quad j_1, \ldots, j_{m-1},$$

with j_k eventually k-logarithmic. With the aid of a little elementary calculus, the reader should readily verify all but the last of the following facts:

1. A is a bounded archimedean f-ring with identity, in which the only minimal prime ideal which is not maximal is P_∞ the ideal of all sequences which are eventually zero.
2. A/P_∞ is a totally ordered integral domain with *sp*-dimension m.
3. $\dim_{sp}(A) = m$, while $\dim_d(A) = 0$, as A is complemented.
4. $\dim_z(A) = 1$.

As to (4) above, let us first label M_∞, the maximal ideal of sequences in A which converge to zero. This is the only nonminimal prime z-ideal of A; it is, in fact, the z-closure of any of the j_k.

Remark 9.7. Let us consider a commutative semiprime f-ring A in light of the foregoing discussion.

On the one hand, Theorem 7.7 surely says that r-dimension is not a suitable tool to gauge the complexity of the ring: either the ring is ℓ-regular, in which case all of the smaller dimensions considered in these pages are zero, or else $\dim_r(A) = \infty$, which tells us very little. Thus, unless we are prepared to entertain the notion of a dimension attached to a measure of ordinal complexity of spectra, r-dimension is disappointing.

On the other hand, as the preceding remark points out, there are large and important classes for which the hypothesis of Proposition 9.4 is satisfied. So, if one is classically inclined, it can be reasonably argued that the sp-dimension is the most interesting.

Regardless, as Example 9.6 already hints at, it seems likewise reasonable to propose—for subsequent investigations—that, regarding now an archimedean f-ring A, it is the comparison of $\dim_d(A)$ and $\dim_z(A)$ and $\dim_{sp}(A)$ which holds out the most promise for purposes of classification.

References

[1] *M. F. Atiyah and I. G. MacDonald*: Introduction to Commutative Algebra. Addison-Wesley, 1969. Zbl 0175.03601

[2] *B. Banaschewski*: Pointfree topology and the spectrum of f-rings. Ordered Algebraic Structures. W. C. Holland and J. Martínez, Eds., Kluwer Acad. Publ., 1997, pp. 123–148. Zbl 0870.06017

[3] *A. Bigard, K. Keimel and S. Wolfenstein*: Groupes et Anneaux Réticulés. Lecture Notes in Mathematics 608, Springer-Verlag, Berlin-Heidelberg-New York, 1977. Zbl 0384.06022

[4] *G. Birkhoff*: Lattice Theory (3rd Ed.). AMS Colloq. Publ. XXV, Providence, 1967. Zbl 0153.02501

[5] *P. F. Conrad*: Epi-archimedean groups. Czechoslovak Math. J. *24* (1974), 192–218. Zbl 0319.06009

[6] *P. Conrad and J. Martínez*: Complemented lattice-ordered groups. Indag. Math. (N.S.) *1* (1990), 281–297. Zbl 0735.06006

[7] *M. Darnel*: Theory of Lattice-Ordered Groups. Marcel Dekker, New York, 1995. Zbl 0810.06016

[8] *L. Gillman and M. Jerison*: Rings of Continuous Functions. Graduate Texts Math. 43, Springer Verlag, Berlin-Heidelberg-New York, 1976. Zbl 0327.46040

[9] *A. W. Hager, C. M. Kimber and W. Wm. McGovern*: Least-integer closed groups. Proc. Conf. Ord. Alg. Struc. (Univ. of Florida, March 2001); J. Martínez, Ed.; Kluwer Acad. Publ. (2002), 245–260. Zbl 1074.06005

[10] *M. Henriksen, J. Martínez and R. G. Woods*: Spaces X in which all prime z-ideals of $C(X)$ are either minimal or maximal. Comment. Math. Univ. Carolinae. *44* (2003), 261–294. Zbl 1067.54015

[11] *C. B. Huijsmans and B. de Pagter*: On z-ideals and d-ideals in Riesz spaces, I. Indag. Math. 42, Fasc. 2 (1980), 183–195. Zbl 0442.46022

[12] *C. B. Huijsmans, B. de Pagter*: On z-ideals and d-ideals in Riesz spaces, II. Indag. Math. 42, Fasc. 4 (1980), 391–408. Zbl 0451.46003

[13] *C. B. Huijsmans and de Pagter*: Ideal theory in f-algebras. Trans AMS *269* (January, 1982), 225–245. Zbl 0483.06009

[14] *P. J. Johnstone*: Stone Spaces. Cambridge Studies in Adv. Math, Cambridge Univ. Press. *3* (1982). Zbl 0499.54001
[15] *S. Larson*: Convexity conditions on f-rings. Canad. J. Math. *XXXVIII* (1986), 48–64. Zbl 0588.06011
[16] *W. A. J. Luxemburg and A. C. Zaanen*: Riesz Spaces, I. North Holland, Amsterdam-London, 1971. Zbl 0231.46014
[17] *J. Martínez*: Archimedean lattices. Alg. Universalis *3* (1973), 247–260. Zbl 0272.06013
[18] *J. Martínez*: Archimedean-like classes of lattice-ordered groups. Trans. AMS *186* (1973), 33–49. Zbl 0298.06022
[19] *J. Martínez*: The hyper-archimedean kernel sequence of a lattice-ordered group. Bull. Austral. Math. Soc. *10* (1974), 337–349. Zbl 0275.06026
[20] *J. Martínez*: The z-dimension of an archimedean f-ring. Work in progress.
[21] *J. Martínez*: The regular top of an algebraic frame. Work in progress.
[22] *J. Martínez and S. D. Woodward*: Bézout and Prüfer f-rings. Comm. in Alg. *20* (1992), 2975–2989. Zbl 0766.06018
[23] *J. Martínez and E. R. Zenk*: When an algebraic frame is regular. Alg. Universalis *50* (2003), 231–257.
[24] *A. Monteiro*: L'Arithmétique des filtres et les espaces topologiques. Segundo Symposium de Matemática; Villavicencio (Mendoza). 1954, pp. 129–162. Zbl 0058.38503
[25] *J. T. Snodgrass and C. Tsinakis*: Finite-valued algebraic lattices. Alg. Universalis *30* (1993), 311–318. Zbl 0806.06011

Author's address: Department of Mathematics, University of Florida, P. O. Box 118105, Gainesville, FL 32611-8105, USA, e-mail: `jmartine@math.ufl.edu`.