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THE BOUNDEDNESS OF MULTILINEAR COMMUTATORS OF SINGULAR INTEGRALS ON LEBESGUE SPACES WITH VARIABLE EXPONENT

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Abstract. The boundednees of multilinear commutators of Calderón-Zygmund singular integrals on Lebesgue spaces with variable exponent is obtained. The multilinear commutators of generalized Hardy-Littlewood maximal operator are also considered.

Keywords: commutator, Calderón-Zygmund singular integral, BMO, Lebesgue space with variable exponent, maximal function

MSC 2000: 42B20, 46E30

1. Introduction

For $b \in \mathrm{BMO}(\mathbb{R}^n)$ (for its definition, see Section 2) and T a singular integral operator, the commutator $[b,T]$ is defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

For these commutators, the classical result, which was obtained by Coifman, Rochberg and Weiss in [1], is when T is a standard convolution integral operator of Calderón-Zygmund $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, and conversely, if $[b, R_i]$ is bounded on $L^p(\mathbb{R}^n)$ for every Riesz transform R_i, then $b \in \mathrm{BMO}(\mathbb{R}^n)$. Then Janson [6] pointed that for any singular integral T the boundedness of $[b,T]$ on $L^p(\mathbb{R}^n)$ implies $b \in \mathrm{BMO}(\mathbb{R}^n)$. In 2002, Perez and Trujillo-Gonzalez [15] introduced a generalized commutator, namely multilinear commutator. Let T be a linear

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operator which is initially assumed to be bounded on $L^2(\mathbb{R}^n)$, and suppose

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y$$

whenever f are $L^\infty(\mu)$ functions with compact support and $x \notin \operatorname{supp} f$, where $K(x,y)$ is a standard Calderon-Zygmund kernel, namely, there exist positive and finite constants γ and C such that, for all distinct $x, y \in \mathbb{R}^n$ and all z with $2|x-z| < |x-y$, it verifies (i) $|K(x,y)| \leqslant C|x-y|^{-n}$,

(ii) $|K(x,y) - K(z,y)| + |K(y,x) - K(y,z)| \leqslant C|x-z|^\gamma/|x-y|^{n+\gamma}$.

Let $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_j \in \mathrm{BMO}(\mathbb{R}^n)$ for $j = 1, 2 \ldots, m$, the multilinear commutators $T_{\vec{b}}$ defined by

$$T_{\vec{b}}f(x) = [b_1, [b_2, \ldots [b_n, T] \ldots]]f(x). \tag{1.1}$$

In [15] Perez and Trujillo-Gonzalez proved the $T_{\vec{b}}$ is also bounded on $L^p(\omega)$ for any $\omega \in A_p$, $1 < p < \infty$, where A_p denotes Muckenhoupt's weight class (see [11]). In fact they obtained more and stronger results than that we stated here. For details one can see [15].

Recently, Karlovich and Lerner considered the boundednees of the following commutator

$$[b, K]f(x) = \text{p.v.} \int_{\mathbb{R}^n} (b(x) - b(y))K(x-y)f(y)\,\mathrm{d}y$$

on Lebesgue spaces $L^{p(\cdot)}$ with variable exponent p in [7]. In fact, Karlovich and Lerner obtained analogous results as in [1]. Motivated by [7], we will consider the boundedness of multilinear commutators $T_{\vec{b}}$ in (1.1) on variable exponent Lebesgue spaces $L^{p(\cdot)}$. First, let us recall some definitions and notations.

All functions in the present paper are assumed to be real valued. Let $p\colon \mathbb{R}^n \to [1, \infty)$ be a measurable function. Set the convex modular

$$m(f,p) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\,\mathrm{d}x.$$

Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$. $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0\colon\ m(f/\lambda, p) \leqslant 1\}.$$

It is clear that if $p(\cdot) = p$ is constant, then the space $L^{p(\cdot)}(\mathbb{R}^n)$ is isometrically isomorphic to the Lebesgue space $L^p(\mathbb{R}^n)$. For the properties of the space $L^{p(\cdot)}(\mathbb{R}^n)$, one can see [12]. Recently the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ have attracted a great attention owing to their connection to fluid dynamics discovered by Michael Růžička. For this one can see [2], [3], [4], [8], [13], [17] and the references therein.

If a measurable function $p\colon \mathbb{R}^n \to [1,\infty)$ satisfies

$$1 < p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \qquad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p_+ < \infty, \tag{1.2}$$

then the function

$$p'(x) = p(x)/(p(x)-1).$$

is well defined and satisfies (1.2) itself.

Denote by $\mathcal{M}(\mathbb{R}^n)$ the set of all measurable functions $p\colon \mathbb{R}^n \to [1,\infty)$ such that (1.2) holds and there exists a constant $C > 0$ such that

$$|p(x) - p(y)| \leqslant \frac{C}{-\log|x-y|} \tag{1.3}$$

for every $x, y \in \mathbb{R}^n$, $|x-y| \leqslant \frac{1}{2}$ and

$$|p(x) - p(y)| \leqslant \frac{C}{\log(e+|x|)}$$

for every $x, y \in \mathbb{R}^n$, $|y| \geqslant |x|$.

If $p \in \mathcal{M}(\mathbb{R}^n)$, Cruz-Uribe, Fiorenza and Neugebauer proved that the Hardy-Littlewood maximal operator M is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself [2, Theorem 1.5]. Pick and Růžička showed that if (1.3) does not hold then the Hardy-Littlewood maximal operator is not bounded. For more details, one can see [2], [3], [13], [16].

Let $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \mathrm{BMO}(\mathbb{R}^n)$ for $i = 1, \ldots, m$, we define

$$M_{\vec{b}} f(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q \prod_{j=1}^m |b_j(x) - b_j(y)||f(y)| \, \mathrm{d}y,$$

where in what follows Q are balls.

Let $\varphi(x) \geqslant 0$ be a smooth and rapidly decreasing function and satisfying the condition:

$$|\varphi(x-y) - \varphi(x)| \leqslant C\frac{|y|}{|x|^{n+1}}, \quad \text{if } |x| > 2|y|.$$

Denote $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. Define the operators

$$\Phi(f)(x) = \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)|f(y)| \, \mathrm{d}y,$$

$$\Phi_{\vec{b}}(f)(x) = \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \prod_{j=1}^m |b_j(x) - b_j(y)|\varphi_\varepsilon(x-y)|f(y)| \, \mathrm{d}y.$$

If $b_j = b$, $j = 1, \ldots, m$, these operators $M_{\vec{b}}$, $\Phi_{\vec{b}}$ were considered on standard Lebesgue spaces in [5].

Now we are ready to state our results.

Theorem 1.1. *Suppose p belongs to $\mathcal{M}(\mathbb{R}^n)$ and $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, \ldots, m$. Then $T_{\vec{b}}$ as in (1.1) can be extended a bounded operator from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.*

Theorem 1.2. *Suppose p belongs to $\mathcal{M}(\mathbb{R}^n)$ and $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, \ldots, m$, then $\Phi_{\vec{b}}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.*

Theorem 1.3. *Suppose p belongs to $\mathcal{M}(\mathbb{R}^n)$ and $b_i \in \mathrm{BMO}(\mathbb{R}^n)$, $i = 1, \ldots, m$, then $M_{\vec{b}}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.*

The remainder of the paper is organized as follows. The proof of Theorem 1.1 will be given in Section 2, and the proof of Theorems 1.2 and 1.3 will be given in Section 3. In this paper, C denotes a positive constant, which may differ in different place.

2. Proof of theorem 1.1

To prove Theorem 1.1, we need some preliminary results. They are the duality and density in spaces $L^{p(\cdot)}(\mathbb{R}^n)$, and the pointwise estimates for sharp maximal functions. The details will follow.

For p satisfying (1.2) the function p' is well defined and the spaces $L^{p(\cdot)}(\mathbb{R}^n)$ can be equipped with the Orlicz type norm

$$\|f\|'_{L^{p(\cdot)}(\mathbb{R}^n)} = \sup\left\{ \int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x\colon\ g \in L^{p'(\cdot)}(\mathbb{R}^n), \quad \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant 1 \right\}.$$

This norm is equivalent to the Luxemburg-Nakano norm (see [9, Theorem 2.3]). This means that

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant \|f\|'_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad f \in L^{p(\cdot)}(\mathbb{R}^n),$$

where $r_p = 1 + 1/p_- - 1/p_+$.

Firstly, the duality in spaces $L^{p(\cdot)}(\mathbb{R}^n)$ can been stated in the following lemma.

Lemma 2.1 (see [9, Theorem 2.1]). *Let* $p\colon \mathbb{R}^n \to [1,\infty)$ *be a measurable function satisfying* (1.2). *If* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *and* $g \in L^{p'(\cdot)}(\mathbb{R}^n)$, *then* fg *is integrable on* $\mathbb{R}^n$ *and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,\mathrm{d}x \leqslant r_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}.$$

Secondly, the density in spaces $L^{p(\cdot)}(\mathbb{R}^n)$ can been stated in the following Lemma 2.2.

Lemma 2.2 (see [7, Lemma 2.2]). *Let* $p\colon \mathbb{R}^n \to [1,\infty)$ *be a measurable function satisfying* (1.2). *Then* $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ *is dense in* $L^{p(\cdot)}(\mathbb{R}^n)$ *and in* $L^{p'(\cdot)}(\mathbb{R}^n)$.

Thirdly, there are some pointwise estimates for sharp maximal functions. Given $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function is defined by

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y.$$

Denote $M_r(f)(x) = M(|f|^r)(x)^{1/r}$, for $r > 0$.

For $\delta > 0$ and $f \in L^\delta_{\mathrm{loc}}(\mathbb{R}^n)$, set also

$$f^\#_\delta(x) = \sup_{x\in Q} \inf_{c\in\mathbb{R}} \left(\frac{1}{Q} \int_Q |f(y) - c|^\delta \,\mathrm{d}y \right)^{1/\delta}.$$

If $\delta = 1$, we denote $f^\#_\delta$ by $f^\#$. A function f is called belong to $\mathrm{BMO}(\mathbb{R}^n)$ if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $f^\#(x) \in L^\infty(\mathbb{R}^n)$. If $f \in \mathrm{BMO}(\mathbb{R}^n)$, the BMO semi-norm of f is given by

$$\|f\|_{\mathrm{BMO}} = \sup_{x\in\mathbb{R}^n} f^\#(x).$$

For a fixed $\lambda \in (0,1)$ and a given measurable function f on $\mathbb{R}^n$, the local sharp maximal function $M^\#_\lambda(f)$ is defined by

$$M^\#_\lambda(f)(x) = \sup_{x\in Q} \inf_{c\in\mathbb{R}} ((f-c)\chi_Q)^*(\lambda|Q|),$$

where $((f-c)\chi_Q)^*$ denotes the non-increasing rearrangement of the function $(f-c)\chi_Q$.

Lemma 2.3 (see [7, Proposition 2.3]). *If $\delta > 0$, $\lambda \in (0,1)$, and $f \in L^{\delta}_{\mathrm{loc}}(\mathbb{R}^n)$, then*

$$M^{\#}_{\lambda}(f)(x) \leqslant (1/\lambda)^{1/\delta} f^{\#}_{\delta}(x), \quad x \in \mathbb{R}^n.$$

To state further results, we need the definition of Orlicz maximal function. Let Φ be a Young function, namely, Φ is a continuous, nonnegative, strictly increasing and convex function on $[0,\infty)$ with

$$\lim_{t\to 0^+} \frac{\Phi(t)}{t} = \lim_{t\to 0^+} \frac{t}{\Phi(t)} = 0.$$

We define the Φ-averages of a function f over a cube Q by

$$\|f\|_{\Phi,Q} = \|f\|_{\Phi(L),Q} = \inf\left\{\lambda > 0\colon \frac{1}{|Q|}\int_Q \left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}x \leqslant 1\right\},$$

where as usual $|Q|$ denotes measure of Q.

Associated to this average, we define the maximal operator M_Φ by

$$M_\Phi f(x) = M_{\Phi(L)} f(x) = \sup_{Q\ni x} \|f\|_{\Phi,Q},$$

where the supremum is taken over all the balls containing x.

If $\Phi(x) = x\log^r(e+x)$, we denote M_Φ as $M_{L(\log L)^r}$. It is well known that $Mf \leqslant CM_{L(\log L)^r} f$ for any $r > 0$, and if $m \in \mathbb{N}$, then

$$M_{L(\log L)^m} \sim M^{m+1}$$

the $m+1$ iterations of the Hardy-Littlewood maximal operator.

Given any positive integer m, for all $1 \leqslant j \leqslant m$, we denote by C^m_j the family of all finite subsets $\sigma = \{\sigma(1),\ldots,\sigma(j)\}$ of j different elements of $\{1,2,\ldots,m\}$. To any $\sigma \in C^m_j$, we associate the complementary sequence σ' given by $\sigma' = \{1,2,\ldots,m\}\setminus\sigma$. For any $\sigma \in C^m_j$, we denote

$$T_{\vec{b}_\sigma} f(x) = \int_{\mathbb{R}^n} (b_{\sigma(1)}(x) - b_{\sigma(1)}(y))\ldots(b_{\sigma(j)}(x) - b_{\sigma(j)}(y)))K(x,y)f(y)\,\mathrm{d}y,$$

and $\|b_\sigma\| = \prod_{j\in\sigma} \|b_j\|_{\mathrm{BMO}}$. In the case of $\sigma = \{1,2,\ldots,m\}$, we denote $T_{\vec{b}_\sigma}$ by $T_{\vec{b}}$ and $\|b_\sigma\|$ by $\|\vec{b}\|$.

Lemma 2.4 (see [15, Lemma 3.1]). *Let $T_{\vec{b}}$ be as in* (1.1) *and let* $0 < \delta < \tau < 1$. *Then there exists a constant* $C > 0$, *depending only on* δ *and the kernel* K, *such that*

$$(T_{\vec{b}}f)^{\#}_{\delta}(x) \leqslant C\Bigg[\|\vec{b}\|M_{L(\log L)^m}f(x) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\|b_\sigma\|M_\tau(T_{\vec{b}_{\sigma'}}f)(x)\Bigg]$$

for any $f \in L_C^{\infty}$.

As the argument in proving [15, Lemma 3.1], we have the following result.

Lemma 2.5. *Let* $T = \int_{\mathbb{R}^n} K(x,y)f(y)\,\mathrm{d}y$ *be a Calderón-Zygmund operator and let* $0 < \delta < 1$. *Then there exists a constant* $C > 0$, *depending only on* δ, *such that*

$$(T(f))^{\#}_{\delta}(x) \leqslant CM(f)(x)$$

for any $f \in L_c^{\infty}$.

Lemma 2.6 (see [10, Theorem 1]). *Suppose* $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and let* φ *be a measurable function satisfying*

$$|\{x\colon\, |\varphi(x)| > \alpha\}| < \infty \qquad \textit{for all } \alpha > 0,$$

then

$$\int_{\mathbb{R}^n}|\varphi(x)g(x)|\,\mathrm{d}x \leqslant C_n\int_{\mathbb{R}^n}M^{\#}_{\lambda_n}\varphi(x)gMg(x)\,\mathrm{d}x.$$

Before giving the proof of Theorem 1.1, we first prove the following result which has its independent role.

Theorem 2.1. *If* $p \in \mathcal{M}(\mathbb{R}^n)$, *then there exists a constant* C_p *such that for any* $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|Tf\|_{L^{p(\cdot)}} \leqslant C_p\|f\|_{L^{p(\cdot)}}.$$

P r o o f. Let $f \in L_C^{\infty}$. For any $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$, since T is of weak type (1.1), according to Lemmas 2.6, 2.3 and 2.5, we have

$$\int_{\mathbb{R}^n}|(Tf)(x)g(x)|\,\mathrm{d}x \leqslant C\int_{\mathbb{R}^n}Mf(x)Mg(x)\,\mathrm{d}x.$$

By Lemma 2.1 and $p, p' \in \mathcal{M}(\mathbb{R}^n)$, we obtain that

$$\int_{\mathbb{R}^n}|(Tf)(x)g(x)|\,\mathrm{d}x \leqslant C_n r_p\|Mf\|_{L^{p(\cdot)}}\|Mg\|_{L^{p'(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}}.$$

This yields

$$\|Tf\|_{L^{p(\cdot)}} \leqslant \|Tf\|'_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

By Lemma 2.2, this completes the proof. □

Proof of Theorem 1.1. Let $f \in L_C^\infty$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\mathrm{loc}}(\mathbb{R}^n)$. We show Theorem 1.1 by induction on m. For $m = 1$, by Theorem 1.5 in [15], $T_b f$ satisfies the conditions of Lemma 2.6. Thus according to Lemmas 2.6, 2.3 and 2.4, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |(T_b f)(x) g(x)|\,\mathrm{d}x &\leqslant C \int_{\mathbb{R}^n} M^{\#}_{\lambda_n}(T_b f)(x) Mg(x)\,\mathrm{d}x \\ &\leqslant C \int_{\mathbb{R}^n} (T_b f)^{\#}_{\delta}(x) Mg(x)\,\mathrm{d}x \\ &\leqslant C \int_{\mathbb{R}^n} \big[\|b\|_{\mathrm{BMO}} M_{L(\log L)} f(x) + \|b\|_{\mathrm{BMO}} M_\tau(Tf)(x)\big]\, Mg(x)\,\mathrm{d}x.\end{aligned}$$

By Lemma 2.1, Theorem 2.1 and p, $p' \in \mathcal{M}(\mathbb{R}^n)$, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} |(Tf)(x) g(x)|\,\mathrm{d}x &\leqslant C_n r_p \|b\|_{\mathrm{BMO}} \|Mf\|_{L^{p(\cdot)}} \|Mg\|_{L^{p'(\cdot)}} \\ &\leqslant C\|b\|_{\mathrm{BMO}} \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}.\end{aligned}$$

This yields

$$\|T_b f\|_{L^{p(\cdot)}} \leqslant \|T_b f\|'_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

Suppose now that for $m-1$ Theorem 1.1 holds, and let us to prove it for m. As above with Theorem 1.5 in [15] again, by Lemmas 2.6, 2.3 and Lemma 2.5, we have

$$\begin{aligned}&\int_{\mathbb{R}^n} |(T_{\vec{b}} f)(x) g(x)|\,\mathrm{d}x \leqslant C \int_{\mathbb{R}^n} M^{\#}_{\lambda_n}(T_{\vec{b}} f)(x) Mg(x)\,\mathrm{d}x \\ &\leqslant C \int_{\mathbb{R}^n} (T_{\vec{b}} f)^{\#}_{\delta}(x) Mg(x)\,\mathrm{d}x \\ &\leqslant C \int_{\mathbb{R}^n} \Bigg[\|\vec{b}\| M_{L(\log L)^m} f(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} \|b_\sigma\| M_\tau(T_{\vec{b}_{\sigma'}} f)(x)\Bigg] Mg(x)\,\mathrm{d}x \\ &\leqslant C \int_{\mathbb{R}^n} \Bigg[\|\vec{b}\| M_{L(\log L)^m} f(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} \|b_\sigma\| M_{L(\log L)^{m-j}} f(x)\Bigg] Mg(x)\,\mathrm{d}x \\ &\leqslant C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \int_{\mathbb{R}^n} \sum_{j=1}^{m} M_{L(\log L)^{m-j}} f(x) Mg(x)\,\mathrm{d}x \\ &\leqslant C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}.\end{aligned}$$

This yields

$$\|T_{\vec{b}} f\|_{L^{p(\cdot)}} \leqslant \|T_{\vec{b}} f\|'_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

By Lemma 2.2, this completes the proof. □

3. Proofs of theorem 1.2 and 1.3

Let
$$\Phi^N f(x) = \sup_{0<\varepsilon<N} \int_{\mathbb{R}^n} \varphi_\varepsilon(x-y)|f(y)|\,\mathrm{d}y,$$
and
$$\Phi^N_{\vec{b}} f(x) = \sup_{0<\varepsilon<N} \int_{\mathbb{R}^n} \prod_{j=1}^m |(b_j(x)-b_j(y))|\varphi_\varepsilon(x-y)|f(y)|\,\mathrm{d}y.$$
To prove Theorem 1.2, we need the following lemma.

Lemma 3.1. *Let $\Phi^N_{\vec{b}}$ be as above and $0<\delta<\tau<1$. Then there exists a constant $C>0$, depending only on δ and τ, such that*
$$(\Phi^N_{\vec{b}} f)^\#_\delta(x) \leqslant C\Big[\|\vec{b}\| M_{L(\log L)^m} f(x) + \sum_{j=1}^m \sum_{\sigma\in C_j^m} \|b_\sigma\| M_\tau(\Phi^N_{\vec{b}_{\sigma'}} f)(x)\Big]$$
for any $f\in L_c^\infty$.

Proof of Lemma 3.1. By homogeneity, we can assume that $\|b_i\|_{\mathrm{BMO}}=1$, $i=1,2,\dots,m$. We first consider the lemma for the case $m=1$. In this case, we have
$$\Phi^N_b f(x) = \sup_{0<\varepsilon<N} \int_{\mathbb{R}^n} |(b(x)-b(y))|\varphi_\varepsilon(x-y)|f(y)|\,\mathrm{d}y.$$

Now, for fixed $x\in\mathbb{R}^n$, let Q denote the ball center at x with radius R. For any $\lambda\in\mathbb{R}$, we choose
$$C_Q = \sup_{0<\varepsilon<N} \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n} |b(y)-\lambda|\varphi_\varepsilon(z-y)|f(y)|\chi_{\mathbb{R}^n\setminus 4Q}(y)\,\mathrm{d}y\,\mathrm{d}z.$$
We first estimate $|\Phi^N_b f(y)-C_Q|$, where $y\in Q$.
$$\begin{aligned}
|\Phi^N_b f(y)-C_Q| &\leqslant \sup_{0<\varepsilon<N}\int_{\mathbb{R}^n}|b(y)-\lambda|\varphi_\varepsilon(y-w)|f(w)|\,\mathrm{d}w\\
&\quad+\sup_{\varepsilon>0}\Big|\int_{\mathbb{R}^n}|b(w)-\lambda|\varphi_\varepsilon(y-w)|f(w)|\,\mathrm{d}w\\
&\quad-\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n}|b(w)-\lambda|\varphi_\varepsilon(z-w)|f(w)|\chi_{\mathbb{R}^n\setminus 4Q}(w)\,\mathrm{d}w\,\mathrm{d}z\Big|\\
&\leqslant \sup_{0<\varepsilon<N}\int_{\mathbb{R}^n}|b(y)-\lambda|\varphi_\varepsilon(y-w)|f(w)|\,\mathrm{d}w\\
&\quad+\sup_{\varepsilon>0}\int_{\mathbb{R}^n}|b(w)-\lambda|\varphi_\varepsilon(y-w)|f(w)|\chi_{4Q}(w)\,\mathrm{d}w\\
&\quad+\sup_{\varepsilon>0}\frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n}|b(w)-\lambda||\varphi_\varepsilon(y-w)-\varphi_\varepsilon(z-w)||f(w)|\chi_{\mathbb{R}^n\setminus 4Q}(w)\,\mathrm{d}w\,\mathrm{d}z\\
&= A_1(y)+A_2(y)+A_3(y).
\end{aligned}$$

So we have

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |\Phi_b^N f(y)| - c_Q|^\delta|\,\mathrm{d}y\right)^{1/\delta}\\
&\leqslant C\left[\left(\frac{1}{|Q|}\int_Q A_1(y)^\delta\,\mathrm{d}y\right)^{1/\delta}+\left(\frac{1}{|Q|}\int_Q A_2(y)^\delta\,\mathrm{d}y\right)^{1/\delta}+\left(\frac{1}{|Q|}\int_Q A_3(y)^\delta\,\mathrm{d}y\right)^{1/\delta}\right]\\
&= I_1+I_2+I_3.
\end{aligned}
$$

Set $\lambda=(b)_Q$, the average of b over the ball Q. For any $1<q<\tau/\delta$, by Hölder's inequality and Jensen's inequality, we have

$$
\begin{aligned}
I_1 &\leqslant C\left(\frac{1}{|Q|}\int_Q |b(y)-\lambda|^{\delta q'}\,\mathrm{d}y\right)^{1/\delta q'}\cdot\left(\frac{1}{|Q|}\int_Q |\Phi^N(f)(y)|^{\delta q}\,\mathrm{d}y\right)^{1/\delta q}\\
&\leqslant C\|b\|_{\mathrm{BMO}} M_{\delta q}(\Phi^N f)(x)\\
&\leqslant CM_\tau(\Phi^N(f))(x).
\end{aligned}
$$

For the term I_2, since $(b-\lambda)f\chi_{4Q}$ is integrable and Φ is of weak type (1.1) (see [18, page 71]), by Kolmogorov's inequality and Lemma 2.3 in [15], we have

$$
I_2 \leqslant \frac{C}{|4Q|}\int_{4Q}|b(y)-\lambda||f(y)|\,\mathrm{d}y \leqslant C\|b\|_{\mathrm{BMO}}\cdot M_{L(\log L)}f(x) = CM_{L(\log L)}f(x).
$$

For the last term I_3, since $x,y,z\in Q$, $w\notin 4Q$, we have

$$
\sup_{\varepsilon>0}|\varphi_\varepsilon(y-w)-\varphi_\varepsilon(z-w)| \leqslant \frac{|y-w|}{|x-w|^{n+1}} \leqslant \frac{CR}{|x-w|^{n+1}}.
$$

where C is independent of ε.

Thus by the argument as in [15, page 683], we obtain

$$
\begin{aligned}
|A_3(y)-C_Q| &\leqslant \sum_{k=2}^{\infty} 2^{-k}\frac{1}{|2^kQ|}\int_{2^kQ}|b(w)-\lambda||f(w)|\,\mathrm{d}w\\
&\leqslant C\sum_{k=2}^{\infty}2^{-k}k\|b\|_{\mathrm{BMO}}M_{L(\log L)}f(x)\\
&\leqslant CM_{L(\log L)}f(x).
\end{aligned}
$$

This yields that

$$
I_3 \leqslant CM_{L(\log L)}f(x).
$$

Thus, Combining the estimates of I_1 to I_3, and taking the supremum over all balls centered at x, we have proved Lemma 3.1 for the case $m=1$.

Now we turn to the case $m \geqslant 2$. For fixed $x \in \mathbb{R}^n$, Q denotes a ball center at x with radius R again, choose

$$C_Q = \sup_{0<\varepsilon<N} \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} \prod_{j=1}^{m} |b_j(y) - \lambda_j| \varphi_\varepsilon(z-y)|f(y)| \chi_{\mathbb{R}^n \setminus 4Q}(y) \, \mathrm{d}y \, \mathrm{d}z.$$

We first estimate $|\Phi_{\vec{b}}^N f(y) - C_Q|$, where $y \in Q$.

$$\begin{aligned}
|\Phi_{\vec{b}}^N f(y) - C_Q| \leqslant & \sup_{0<\varepsilon<N} \bigg| \int_{\mathbb{R}^n} \bigg| \prod_{j=1}^{m} (b_j(y) - \lambda_j) + (-1)^m \prod_{j=1}^{m} (b_j(w) - \lambda_j) \\
& + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} C_{m,j} (b(y) - \bar{\lambda})_\sigma (b(w) - \bar{\lambda})_{\sigma'} |\varphi_\varepsilon(y-w)|f(w)| \, \mathrm{d}w \\
& - \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} \prod_{j=1}^{m} |(b_j(w) - \lambda_j)| \varphi_\varepsilon(z-w)|f(w)| \chi_{\mathbb{R}^n \setminus 4Q}(w) \, \mathrm{d}w \, \mathrm{d}z| \\
\leqslant & \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \prod_{j=1}^{m} |b_j(y) - \lambda_j| \varphi_\varepsilon(y-w)|f(w)| \, \mathrm{d}w \\
& + \sup_{0<\varepsilon<N} \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} |C_{m,j}| |(b(x) - \bar{\lambda})_\sigma| \int_{\mathbb{R}^n} |(b(y) - b(w))_{\sigma'} |\varphi_\varepsilon(y-w)|f(w)| \, \mathrm{d}w \\
& + \sup_{\varepsilon>0} \int_{\mathbb{R}^n} \prod_{j=1}^{m} |b_j(w) - \lambda_j| \varphi_\varepsilon(y-w)|f(w)| \chi_{4Q}(w) \, \mathrm{d}w \\
& + \sup_{\varepsilon>0} \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} \prod_{j=1}^{m} |b_j(w) - \lambda_j| \ |\varphi_\varepsilon(y-w) - \varphi_\varepsilon(z-w)||f(w)| \chi_{\mathbb{R}^n \setminus 4Q}(w) \, \mathrm{d}w \, \mathrm{d}z \\
= & A_1(y) + A_2(y) + A_3(y) + A_4(y).
\end{aligned}$$

We first estimate $A_4(y)$. Since $x, y, z \in Q$, $w \notin 4Q$, we have

$$\sup_{\varepsilon>0} |\varphi_\varepsilon(y-w) - \varphi_\varepsilon(z-w)| \leqslant C \frac{|y-z|}{|x-w|^{n+1}} \leqslant \frac{CR}{|x-w|^{n+1}},$$

where C is independent of ε. Thus

$$A_4(y) \leqslant C \sum_{k=0}^{\infty} 2^{-k} \frac{1}{|2^k Q|} \int_{2^k Q} \bigg| \prod_{j=1}^{m} (b_j(w) - \lambda_j) \bigg| |f(w)| \, \mathrm{d}w.$$

Since $0 < \delta < 1$, we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |\Phi_{\vec{b}}^N f(y) - C_Q|^\delta \,\mathrm{d}y\right)^{1/\delta} &\leqslant C\bigg[\left(\frac{1}{|Q|}\int_Q A_1(y)^\delta \,\mathrm{d}y\right)^{1/\delta} + \left(\frac{1}{|Q|}\int_Q A_2(y)^\delta \,\mathrm{d}y\right)^{1/\delta} \\ &\quad + \left(\frac{1}{|Q|}\int_Q A_3(y)^\delta \,\mathrm{d}y\right)^{1/\delta} + \left(\frac{1}{|Q|}\int_Q A_4(y)^\delta \,\mathrm{d}y\right)^{1/\delta}\bigg] \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Set $\lambda_i = (b_i)_Q$, $i = 1, 2, \ldots, m$. Then

$$\begin{aligned} I_4 &= \left(\frac{1}{|Q|}\int_Q A_4(y)^\delta \,\mathrm{d}y\right)^{1/\delta} \\ &\leqslant C\sum_{k=1}^{\infty} 2^{-k}\frac{1}{|2^kQ|}\int_{2^kQ}\prod_{j=1}^{m}|b_j(w) - \lambda_j||f(w)|\,\mathrm{d}w \\ &\leqslant C\sum_{k=1}^{\infty} 2^{-k}k\|b\|_{\mathrm{BMO}}M_{L(\log L)}f(x) \\ &\leqslant CM_{L(\log L)}f(x).\end{aligned}$$

For I_3, by the same argument as above and making use of Kolmogorov's inequality, the weak type (1,1) of Φ and Lemma 2.3 in [15] (in fact [15, (2.5), page 679]),

$$\begin{aligned} I_3 \leqslant \frac{C}{|4Q|}\int_{4Q}\prod_{j=1}^{m}|b_j - \lambda_j||f(y)|\,\mathrm{d}y &\leqslant C\prod_{j=1}^{m}\|b_j\|_{\mathrm{BMO}}\cdot M_{L(\log L)}f(x) \\ &= CM_{L(\log L)}f(x).\end{aligned}$$

For I_1, we estimate $A_1(x)$. Because

$$A_1(x) \leqslant C\prod_{j=1}^{m}|b_j(x) - \lambda_j|\Phi(f)(x),$$

again with $\lambda_i = (b_i)_Q, i = 1, \ldots, m$, using Hölder's inequality for finitely many functions with $1 < q < \tau/\delta$, we have

$$I_1 \leqslant CM_\tau(\Phi f)(x).$$

Similarly, we have

$$I_2 \leqslant C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} M_\tau(\Phi_{\vec{b}_{\sigma'}}^N f)(x).$$

Combining the estimates of I_1 to I_4, and taking the supremum over all balls centered at x, we obtain the lemma. This finishes the proof of Lemma 3.1. □

P r o o f o f T h e o r e m 1.2. Let $f \in L_C^\infty$. First we prove that $\Phi_{\vec{b}}^N(f)$ belongs to $L^{p(\cdot)}(\mathbb{R}^n)$. To do so we use induction on m. For $m = 1$, by the same arguments as in the proof of Theorem 1.5 in [15], $\Phi_{\vec{b}}^N(f)$ satisfies the conditions of Lemma 2.6. Thus by Lemma 2.6, 2.3 and 3.1, for any $g \in L^{p'(\cdot)}(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |(\Phi_b^N f)(x)g(x)|\,\mathrm{d}x &\leqslant C\int_{\mathbb{R}^n} M_{\lambda_n}^{\#}(\Phi_b^N f)(x)Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} (\Phi_b^N f)_\delta^{\#}(x)Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} \big[\|b\|_{\rm BMO}M_{L(\log L)}f(x) + \|b\|_{\rm BMO}M_\tau(\Phi^N f)(x)\big]\,Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} \big[\|b\|_{\rm BMO}M^2f(x) + \|b\|_{\rm BMO}M_\tau(\Phi f)(x)\big]\,Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} \big[\|b\|_{\rm BMO}M^2f(x) + \|b\|_{\rm BMO}M_\tau(Mf)(x)\big]\,Mg(x)\,\mathrm{d}x\\ &\leqslant C\|b\|_{\rm BMO}\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},\end{aligned}$$

where we have used Lemma 2.1, since $p, p' \in \mathcal{M}(\mathbb{R}^n)$. This yields

$$\|\Phi_b^N f\|_{L^{p(\cdot)}} \leqslant \|\Phi_b^N f\|'_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

Suppose now that for $m-1$, $\Phi_{\vec{b}}^N(f)$ belongs to $L^{p(\cdot)}(\mathbb{R}^n)$, and let us to prove it for m. As the above by Lemmas 2.6, 2.3 and Lemma 3.1 again, we have

$$\begin{aligned}\int_{\mathbb{R}^n} |(\Phi_{\vec{b}}^N f)(x)g(x)|\,\mathrm{d}x &\leqslant C\int_{\mathbb{R}^n} M_{\lambda_n}^{\#}(\Phi_{\vec{b}}^N f)(x)Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} (\Phi_{\vec{b}}^N f)_\delta^{\#}(x)Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} \Big[\|\vec{b}\|M_{L(\log L)^m}f(x) + \sum_{j=1}^m\sum_{\sigma\in C_j^m}\|b_\sigma\|M_\varepsilon(\Phi_{\vec{b}_{\sigma'}}^N f)(x)\Big]Mg(x)\,\mathrm{d}x\\ &\leqslant C\int_{\mathbb{R}^n} \Big[\|\vec{b}\|M_{L(\log L)^m}f(x) + \sum_{j=1}^m\sum_{\sigma\in C_j^m}\|b_\sigma\|M_{L(\log L)^{1/\sigma'}}f(x)\Big]Mg(x)\,\mathrm{d}x\\ &\leqslant C\prod_{j=1}^m\|b_j\|_{\rm BMO}\int_{\mathbb{R}^n} M_{L(\log L)^{1/\sigma'}}f(x)Mg(x)\,\mathrm{d}x\\ &\leqslant C\prod_{j=1}^m\|b_j\|_{\rm BMO}\|f\|_{L^{p(\cdot)}}\|g\|_{L^{p'(\cdot)}},\end{aligned}$$

because $p, p' \in \mathcal{M}(\mathbb{R}^n)$, and we used Lemma 2.1 again. This yields

$$\|\Phi_{\vec{b}}^N f\|_{L^{p(\cdot)}} \leqslant \|\Phi_{\vec{b}}^N f\|'_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

Since $\Phi_{\vec{b}}^{N}(f)(x)$ tends to $\Phi_{\vec{b}}(f)(x)$ pointwise as N tends to ∞, by Fatou's Lemma we have

$$\int_{\mathbb{R}^n} |(\Phi_{\vec{b}}^{N} f)(x)g(x)|\,\mathrm{d}x \leqslant C \prod_{j=1}^{m} \|b_j\|_{\mathrm{BMO}} \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}}.$$

Thus,

$$\|\Phi_{\vec{b}} f\|_{L^{p(\cdot)}} \leqslant \|\Phi_{\vec{b}} f\|'_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

By Lemma 2.2, this completes the proof. □

Proof of Theorem 1.3. In Theorem 1.2, we choose φ such that $\chi_{\{x\colon |x|\leqslant 2\}} \leqslant \varphi$, then we have

$$M_{\vec{b}} f(x) \leqslant C\Phi_{\vec{b}} f(x).$$

By Theorem 1.2, we obtain Theorem 1.3. □

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