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SUBDIRECT PRODUCTS OF CERTAIN VARIETIES OF UNARY ALGEBRAS

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Abstract. J. Płonka in [12] noted that one could expect that the regularization $\mathscr{R}(\boldsymbol{K})$ of a variety $\boldsymbol{K}$ of unary algebras is a subdirect product of $\boldsymbol{K}$ and the variety $\boldsymbol{D}$ of all discrete algebras (unary semilattices), but is not the case. The purpose of this note is to show that his expectation is fulfilled for those and only those irregular varieties $\boldsymbol{K}$ which are contained in the generalized variety $\boldsymbol{TDir}$ of the so-called trap-directable algebras.

Keywords: unary algebra, subdirect product, variety, directable algebra

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The basic algebraic notions are defined here as for algebras in general (cf. [6], [9], for example), but we reformulate some of them, to fit them into specific notation which comes from the theory of automata. In what follows, X is always a nonempty alphabet, X^* denotes the free monoid over X, and e denotes its identity. An *algebra of type* X, or an X*-algebra*, is a system $A = (A, X)$ where A is a nonempty set and every symbol $x \in X$ is realized as a unary operation $x^A\colon A \to A$. For any $a \in A$ and $x \in X$, we write ax^A for $x^A(a)$. For any word $w = x_1x_2\dots x_n \in X^*$, $w^A\colon A \to A$ is defined as the composition of the mappings $x_1^A, x_2^A, \dots, x_n^A$, that is to say, $aw^A = ax_1^Ax_2^A\dots x_n^A$ for any $a \in A$. In particular, e^A is the identity mapping of A. If A is known from the context, we write simply aw instead of aw^A.

We define *terms* of type X over a set V of *variables* as expressions of the form gu, where $g \in V$ and $u \in X^*$, and we denote by $T_X(V)$ the set of all such terms. The *term* X*-algebra* $T_X(V) = (T_X(V), X)$ is defined so that $(gu)x = g(ux)$ for all $gu \in T_X(V)$ and $x \in X$ (see § 1.6 of [8]). An *identity* of type X over V is an expression $gu \approx hv$,

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where $gu, hv \in T_X(V)$. An X-algebra A is said to *satisfy* an identity $gu \approx hv$ if $(g\alpha)u^A = (h\alpha)v^A$ for all valuations $\alpha\colon V \to A$ of the variables. Identities of the form $gu \approx gv$ are called *regular*, whereas identities of the form $gu \approx hv$, with $g \neq h$, are *irregular*. A variety of X-algebras is *regular* if it is determined by a set of regular identities, otherwise it is *irregular*. A class of X-algebras is said to be a *generalized variety* if it is closed under subalgebras, homomorphic images, finite direct products and arbitrary direct powers, or equivalently, if it is a directed union of varieties (see [1], [2], [3]).

In the paper we consider only algebras of a fixed type X, and for brevity we simply say '*algebra*' instead of '*X-algebra*'. The least subalgebra of an algebra A, if it exists, is called the *kernel* of A, and if A has a least nontrivial subalgebra, it is called the *core* of A. The *monogenic subalgebra* of A generated by $a \in A$ is denoted by $\langle a\rangle$. It is obvious that $\langle a\rangle = \{aw\colon w \in X^*\}$. An element $a \in A$ is called a *trap* if $ax = a$, for every $x \in X$. An algebra is called *discrete* if all of its elements are traps. For a set H, Δ_H and ∇_H denote respectively the *diagonal* and the *universal* relation on H. The *Rees congruence* ϱ_B on an algebra A modulo a subalgebra B of A is defined by $\varrho_B = \nabla_B \cup \Delta_A$. The corresponding *Rees quotient* A/ϱ_B is denoted by A/B, and A is said to be an *extension* of B by an algebra C if $A/B \cong C$. If this is the case, C evidently has a trap which corresponds to the image of B under the canonical epimorphism of A onto A/B. In other words, we may regard C as the result of contracting the subalgebra B of A into one element, a trap of‘ C. A *trap-extension* of an algebra is obtained by adjoining to it a trap, that is to say, A is a trap-extension of B if it is an extension of B by a two-element discrete algebra. If B is a subalgebra of A, a congruence θ on A is called a *B-congruence* if $\theta \cap \nabla_B = \Delta_B$, and if Δ_A is the only B-congruence on A, we say that A is a *dense extension* of B. In particular, every algebra is a dense extension of itself.

An algebra A is *connected* if for all $a, b \in A$ there exist $u, v \in X^*$ such that $au = bv$, and it is *strongly connected* if for all $a, b \in A$ there exists $u \in X^*$ such that $au = b$. Obviously, A is strongly connected if and only if $\langle a\rangle = A$, for every $a \in A$. A connected algebra can have at most one trap, and if it has a trap, it is called *trap-connected*. Furthermore, a nontrivial algebra A is *strongly trap-connected* if it has a trap a_0 and $\langle a\rangle = A$, for every $a \in A \setminus \{a_0\}$. Every strongly trap-connected algebra is trap-connected, but the converse does not hold. An algebra A is *directable* if there exists a word $u \in X^*$ such that $au = bu$, for every pair of elements $a, b \in A$. Any directable algebra is connected, so it can have at most one trap, and if it has a trap, it is called *trap-directable*. Let $\boldsymbol{Dir}$, $\boldsymbol{TDir}$ and $\boldsymbol{D}$ denote respectively the classes of all directable, trap-directable and discrete algebras. It is known that $\boldsymbol{D}$ is a variety and $\boldsymbol{Dir}$ and $\boldsymbol{TDir}$ are generalized varieties (see [2], [5], [10]). Moreover, a variety $\boldsymbol{K}$ of unary algebras is regular if and only if it contains $\boldsymbol{D}$, and it is irregular if and only if it is contained in $\boldsymbol{Dir}$ (cf. [2], [5], [10]).

An algebra A is the *direct sum* of algebras A_α, $\alpha \in Y$, if $A = \bigcup_{\alpha \in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \emptyset$, for all $\alpha, \beta \in Y$ such that $\alpha \neq \beta$. If A can not be decomposed into a direct sum of two or more algebras, then it is *direct sum indecomposable.* For more information on direct sum decompositions we refer to [7]. Let $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ be two classes of algebras. The *subdirect product* $\boldsymbol{K}_1 \otimes \boldsymbol{K}_2$ of $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ is the class of all subdirect products of an algebra from $\boldsymbol{K}_1$ and an algebra from $\boldsymbol{K}_2$, and the *Mal'cev product* $\boldsymbol{K}_1 \circ \boldsymbol{K}_2$ is the class of all algebras A which have a congruence ϱ such that $A/\varrho \in \boldsymbol{K}_2$ and every ϱ-class which is a subalgebra of A belongs to $\boldsymbol{K}_1$. In particular, $\boldsymbol{K} \circ \boldsymbol{D}$ is the class of all direct sums of algebras from $\boldsymbol{K}$.

J. Płonka in [11], [12] studied the regularization operator $\mathscr{R}\colon\ \boldsymbol{K} \mapsto \mathscr{R}(\boldsymbol{K})$ on the lattice of varieties of unary algebras and proved, among other things, that

$$\mathscr{R}(\boldsymbol{K}) = \boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \circ \boldsymbol{D}$$

(for some related results we refer to [2]). He also noted in [12] that one could expect that $\mathscr{R}(\boldsymbol{K}) = \boldsymbol{K} \otimes \boldsymbol{D}$, but is not the case. In terminology from the theory of automata, in the example which confirms this note he assumed $\boldsymbol{K}$ to be the variety of *reset* or 1-*definite* algebras, and A to be a trap-extension of a two-element reset algebra, and showed that A belongs to $\mathscr{R}(\boldsymbol{K})$, but does not belong to $\boldsymbol{K} \otimes \boldsymbol{D}$.

In this paper we show that a considerably large class of varieties of unary algebras fulfills the Płonka's expectation. Namely, for an irregular variety $\boldsymbol{K}$ of unary algebras[1] we prove that $\mathscr{R}(\boldsymbol{K}) = \boldsymbol{K} \otimes \boldsymbol{D}$ if and only if $\boldsymbol{K} \subseteq \boldsymbol{TDir}$. For that purpose we use a lot of specific notions which come from the theory of automata (cf. [2], [4], [5], [7], [10]), and a general characterization of subdirectly irreducible unary algebras from [4]. This is the following result:

Theorem 1. *A nontrivial algebra A is subdirectly irreducible if and only if it is a dense extension of a nontrivial subdirectly irreducible subalgebra B by a trap-connected algebra and this B satisfies one of the following conditions:*

(C0) *B is the core of A and strongly connected;*

(C1) *B is the core of A and strongly trap-connected, or B is a trap-extension of the core of A and the core is strongly connected;*

(C2) *B is the core of A and a two-element discrete algebra.*

Moreover, for each $k = 0, 1, 2$, B satisfies the condition (Ck) *if and only if A has exactly k traps.*

[1] Contrary to Płonka, who studied algebras having both unary and nullary fundamental operations, here we consider only algebras all of whose operations are unary.

We also need the following lemma.

Lemma 1. *Let A' be a trap-extension of an algebra A. Then A' is a dense extension of A if and only if A does not have a trap.*

P r o o f. Let $a \in A' \setminus A$ be the trap adjoined to A. We shall prove that A' is not a dense extension of A if and only if A has a trap.

Suppose that A' is not a dense extension of A, i.e., there exists an A-congruence θ on A' different than $\Delta_{A'}$. Then there exists $(b, c) \in \theta$ such that $b \neq c$, and since θ is an A-congruence, then one of b and c, say c, must be equal to a. For every $x \in X$ we have that $(ax, bx) \in \theta$, i.e., $(a, bx) \in \theta$, which together with $(b, a) \in \theta$ yields

$$(b, bx) \in \theta \cap \nabla_A = \Delta_A.$$

Therefore, $b = bx$, and we have obtained that A has a trap b.

Conversely, suppose that A has a trap b. Then $C = \{a, b\}$ is a subalgebra of A' and the Rees congruence on A' modulo C is an A-congruence on A' different than $\Delta_{A'}$, so A' can not be a dense extension of A. $\square$

Recall that every irregular variety of unary algebras is contained in the generalized variety $\boldsymbol{Dir}$ of all directable automata (Corollary 5.1 of [2]).

Theorem 2. *Let $\boldsymbol{K}$ be an irregular variety of algebras. Then the following conditions are equivalent:*

(i) *$\boldsymbol{K} \subseteq \boldsymbol{TDir}$;*

(ii) *$\boldsymbol{K}$ does not contain a nontrivial strongly connected algebra;*

(iii) *$\boldsymbol{K}$ does not contain a nontrivial subdirectly irreducible strongly connected algebra.*

P r o o f. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious, and it remains to prove the implication (iii) $\Rightarrow$ (i).

Suppose that (iii) holds. Let $A \in \boldsymbol{K}$ be any nontrivial subdirectly irreducible algebra. Then A has a nontrivial subdirectly irreducible subalgebra B which satisfies one of the conditions (C0), (C1) and (C2) of Theorem 1. We can immediately exclude the case (C2), since A is directable and can not have two different traps, whereas the case (C0) is excluded by our hypothesis (iii), because $B \in \boldsymbol{K}$. Therefore, B must satisfy (C1), and we conclude that A has a trap. Having in mind that A is directable, the existence of a trap in A implies $A \in \boldsymbol{TDir}$. Hence, we have proved that every subdirectly irreducible algebra from $\boldsymbol{K}$ belongs to $\boldsymbol{TDir}$.

Further, consider an arbitrary algebra $A \in \boldsymbol{K}$. Then A is a subdirect product of subdirectly irreducible algebras A_i, $i \in I$, and evidently, $A_i \in \boldsymbol{K}$, and hence

$A_i \in \boldsymbol{TDir}$, for every $i \in I$. Let P‘ be the direct product of the algebras A_i, $i \in I$. Since every A_i has exactly one trap, P also has exactly one trap. On the other hand, $P \in \boldsymbol{K} \subseteq \boldsymbol{Dir}$, so we conclude that $P \in \boldsymbol{TDir}$, and now $A \in \boldsymbol{TDir}$, as a subalgebra of P. Therefore, we have proved (i). This completes the proof of the theorem. □

Note that a variety $\boldsymbol{K}$ is contained in $\boldsymbol{TDir}$ if and only if it satisfies a set of identities $\{gux \approx hu\colon\ x \in X\}$, for some $u \in X^*$ (see [10] or [5]).

Now we are ready to state and prove the main theorem of the paper.

Theorem 3. *Let $\boldsymbol{K}$ be an irregular variety of algebras. Then*

$$\boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \otimes \boldsymbol{D} \Longleftrightarrow \boldsymbol{K} \subseteq \boldsymbol{TDir}.$$

Proof. Let $\boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \otimes \boldsymbol{D}$. Suppose that $\boldsymbol{K} \nsubseteq \boldsymbol{TDir}$. Then by Theorem 2, there exists a nontrivial subdirectly irreducible strongly connected algebra $A \in \boldsymbol{K}$. Since A is a nontrivial strongly connected algebra, it has no trap, thus $A \in \boldsymbol{K} \setminus \boldsymbol{TDir}$. According to Lemma 1, A' is a dense extension of A. Now, by Theorem 5.1 of [4] it follows that A' is subdirectly irreducible. On the other hand, A' is a direct sum of A and a one-element algebra, which both belong to $\boldsymbol{K}$, so our starting hypothesis yields

$$A' \in \boldsymbol{K} \circ \boldsymbol{D} = \boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \otimes \boldsymbol{D}.$$

But, $A' \in \boldsymbol{K} \otimes \boldsymbol{D}$ and subdirect irreducibility of A' imply that

$$A' \in \boldsymbol{D} \quad \text{or} \quad A' \in \boldsymbol{K},$$

which is not true, because A' is neither discrete nor directable algebra. Therefore, we conclude that $\boldsymbol{K} \subseteq \boldsymbol{TDir}$.

Conversely, let $\boldsymbol{K} \subseteq \boldsymbol{TDir}$. Since every algebra from $\boldsymbol{K} \vee \boldsymbol{D}$ is a subdirect product of subdirectly irreducible algebras from $\boldsymbol{K} \vee \boldsymbol{D}$, it is enough to prove that every subdirectly irreducible algebra from $\boldsymbol{K} \vee \boldsymbol{D}$ belongs either to $\boldsymbol{K}$ or to $\boldsymbol{D}$.

Let $A \in \boldsymbol{K} \vee \boldsymbol{D} = \boldsymbol{K} \circ \boldsymbol{D}$ be an arbitrary subdirectly irreducible algebra. Then A is a direct sum of algebras A_α, $\alpha \in Y$, where $A_\alpha \in \boldsymbol{K} \subseteq \boldsymbol{TDir}$, for each $\alpha \in Y$. This means that every A_α has exactly one trap, and by Theorem 1, $|Y| \leqslant 2$. If $|Y| = 2$, then Theorem 1 says that A has exactly two traps a_1 and a_2, and $B = \{a_1, a_2\}$ is the core of A. If $B \neq A$, then A is connected and direct sum indecomposable, which contradicts the hypothesis $|Y| = 2$. Thus, we conclude that A must be a two-element discrete algebra, and hence $A \in \boldsymbol{D}$. Finally, if $|Y| = 1$, then clearly $A \in \boldsymbol{K}$. This completes the proof of the theorem. □

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