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# ON LOCALLY SOLID TOPOLOGICAL LATTICE GROUPS

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*Abstract.* Let $(G,\tau)$ be a commutative Hausdorff locally solid lattice group. In this paper we prove the following:

(1) If $(G,\tau)$ has the $A$(iii)-property, then its completion $(\widehat{G},\hat{\tau})$ is an order-complete locally solid lattice group.
(2) If $G$ is order-complete and $\tau$ has the Fatou property, then the order intervals of $G$ are $\tau$-complete.
(3) If $(G,\tau)$ has the Fatou property, then $G$ is order-dense in $\widehat{G}$ and $(\widehat{G},\hat{\tau})$ has the Fatou property.
(4) The order-bound topology on any commutative lattice group is the finest locally solid topology on it.

As an application, a version of the Nikodym boundedness theorem for set functions with values in a class of locally solid topological groups is established.

*Keywords*: topological completion, locally solid $\ell$-group, topological continuity, Fatou property, order-bound topology

*MSC 2000*: 46A40, 54H11, 28B15

## 1. Introduction

The theory of topological Riesz spaces is very rich, and vector measures with values in these spaces and order-complete Riesz spaces have been extensively studied (for example, see Aliprantis [1], Fremlin [12], Schmidt [18] and Swartz [19]). In recent years, contributions to the theory of topological groups have been made by Comfort et. al [8–9], Bonales [6] and Raczkowski [17]; in particular, they have studied totally bounded group topologies, Bohr topology and the relevance to locally convex spaces of the celebrated theorem of Pontryagin-Van Kampen which states that every locally compact Abelian group satisfies group duality (see [6], p. 76 for the details). Thereby, the school of mathematicians led by W. W. Comfort has generated tremendous activities in this area of investigations. Topological Riesz groups and their special case, namely, topological lattice groups and measures with values in such groups have been considered by Kalton [15], Khan and Rowlands [16], Avallone and Valente [4] and Jakubík [13]. However, only some partial results have been obtained for functions taking values in this class of ordered groups mainly due to the lack of topology on them (see [5], p. 171). In this paper we establish results specifically related to the topological completion of a Hausdorff locally solid topological lattice group.

This paper is organized as follows: Let $G$ be a commutative lattice group (henceforth called an $\ell$-group), with a locally solid Hausdorff group topology $\tau$. In this paper we investigate some of the properties of $(G,\tau)$ which are inherited by its completion $(\widehat{G},\hat{\tau})$. In particular, in Section 3, we show that, if every bounded monotonic sequence in $G$ is $\tau$-Cauchy, then $(\widehat{G},\hat{\tau})$ is an order-complete locally solid $\ell$-group; this extends Theorem 1 by Kalton [15]. Fremlin [11] has proved (in Theorem 1) that, if $(E,\xi)$ is a topological vector lattice with a Hausdorff locally solid topology which has the Fatou property, then its completion $(\widehat{E},\hat{\xi})$ has the Fatou property and $E$ is order-dense in $\widehat{E}$. A new proof of Fremlin's theorem was given by Aliprantis and Burkinshaw in [2]; using their ideas we prove an analogue of Fremlin's result for a locally solid $\ell$-group with the Fatou property. We also obtain the Nikodym boundedness theorem for lattice group-valued submeasures which extends Theorem 1 of Drewnowski [10]. Finally, in Section 4, we introduce the analogue of the order-bound topology on $G$ and show that it is the finest locally solid group topology on an $\ell$-group.

## 2. Notation and Preliminaries

Throughout this paper all groups are commutative and are written additively. By an *$\ell$-group*, we mean a partially ordered group $G$ in which each pair of elements $x,y$ has a supremum (denoted by $x\vee y$) and an infimum (denoted by $x\wedge y$). We write $x^+=x\vee 0, x^-=(-x)\vee 0, x=x^+-x^-$ and $|x|=x^++x^-$ for any $x\in G$. For the elementary properties of $\ell$-groups we refer the reader to [14].

An $\ell$-group $G$ is said to be *$\sigma$-complete* (resp. *order-complete*) if every bounded increasing sequence (net) in $G$ has a supremum. An $\ell$-subgroup $H$ of $G$ is said to be *order-dense* in $G$, if, for each $x\geqslant 0$ in $G$, $x=\sup\{y\in H\colon\ 0\leqslant y\leqslant x\}$. Let $G^+=\{x\in G\colon\ x\geqslant 0\}$. An $\ell$-group is said to be Archimedean if, for $x,y$ in $G^+$, $ny\leqslant x$ $(n=1,2,\ldots)$ implies $y=0$. Clearly, a $\sigma$-complete group is Archimedean and, if $G$ is an Archimedean $\ell$-group, then an $\ell$-subgroup $H$ is order-dense in $G$ if and only if, for each $u>0$ in $G$, there exists a $v>0$ in $H$ such that $0<v\leqslant u$.

If $G$ is an $\ell$-group, a subset $V$ of $G$ is said to be *solid* if $a\in V$ and $|x|\leqslant|a|$ implies that $x\in V$. We note that a solid set is symmetric; that is, $V=-V$. A group topology $\tau$ on $G$ is said to be *locally solid* if it has a base of $\tau$-neighborhoods of 0 consisting of solid sets. If a subgroup of $G$ is solid, then it is said to be an *ideal* in $G$.

A function $q$ on $G$ is said to be a *quasi-norm* if it has the following properties for all $a, b$ in $G$.

(i) $q(a) \geqslant 0$,

(ii) $q(0) = 0$,

(iii) $q(a+b) \leqslant q(a) + q(b)$,

(iv) $q(-a) = q(a)$.

If, in addition, (v) $q(x) \leqslant q(y)$ for all $x, y$ in $G$ with $|x| \leqslant |y|$, then it is said to be an *$\ell$-quasi-norm*.

A family of quasi-norms (resp. $\ell$-quasi-norms) determines a (locally solid) group topology on $G$; on the other hand, if $\tau$ is a (locally solid) group topology on $G$, then $\tau$ may be determined by the family of all $\tau$-continuous quasi-norms ($\ell$-quasi-norms) on $G$ (cf. [12], 22C)). A subset $B$ of $(G,\tau)$ is said to be *bounded* if $\sup_{x\in B} \eta(x) < +\infty$ for all $\tau$-continuous quasi-norms $\eta$ on $G$. A locally solid topology $\tau$ on $G$ is said to have the *Fatou property* if there exists a base $\mathscr{U}$ of $\tau$-neighborhoods of 0 with the following properties:

(1) each $U \in \mathscr{U}$ is solid, and

(2) if $A \subseteq U$ and $A \uparrow x$ ($A$ upwards directed with supremum $x$), then $x \in U$.

An $\ell$-quasi-norm $\varrho$ is said to have the Fatou property if $A \uparrow x$ in $G^+$ implies $\varrho(x) = \sup\{\varrho(y)\colon\ y \in A\}$. A family of Fatou $\ell$-quasi-norms determines a Fatou topology on $G$; conversely, a locally solid topology $\tau$ with the Fatou property may be determined by the family of all $\tau$-continuous Fataou $\ell$-quasi-norms (cf. [12], 23B)).

A subset $A$ of $G$ is said to be *order-bounded* if there exists an element $x$ in $G$ such that $A \subseteq [-|x|, |x|]$. A sequence $\{u_n\}$ in $G$ is said to *order-converge* to an element $u$ in $G$, written as $u_n \overset{(0)}{\to} u$, if there exists a sequence $\{v_n\}$ in $G$ such that $|u_n - u| \leqslant v_n \downarrow 0$ ($n = 1, 2, \ldots$). Following the notation of [1], a locally solid $\ell$-group $(G,\tau)$ is said to have the $A$(iii)-property (resp. $A$(iv)-property) if and only if every order-bounded increasing sequence (net) is $\tau$-Cauchy.

By modifying the proof of ([15], Lemma 2) we have the following:

**Lemma 1.** *Let $(G,\tau)$ be a locally solid $\ell$-group. Then $(G,\tau)$ has the* $A$(iii)-*property if and only if it has the* $A$(iv)-*property.*

## 3. The completion group

If $(G,\tau)$ is a (Hausdorff) topological group, then there exists a (Hausdorff) complete topological group, $(\widehat{G},\hat{\tau})$ say, such that $G$ may be isomorphically embedded in $\widehat{G}$ as a $\hat{\tau}$-dense subgroup and $\hat{\tau}$ induces the original topology $\tau$ on $G$ (for details of the construction, we refer the reader to [7], pp. 242–250). In the sequel, we shall refer to $(\widehat{G},\hat{\tau})$ as the topological completion of $(G,\tau)$. In this section we investigate some properties of $(G,\tau)$ which are inherited by its completion $(\widehat{G},\hat{\tau})$.

We begin with the following generalization of Theorem 1 due to Kalton [15].

**Theorem 1.** *Let $(G,\tau)$ be a Hausdorff locally solid $\ell$-group with the $A$(iii)-property. Then $(\widehat{G},\hat{\tau})$ is an order-complete locally solid $\ell$-group.*

Proof. Since $\tau$ is locally solid, the mapping $x\to x^+$ is uniformly continuous and so the positive cone $P=\{x\in G\colon\ x\geqslant 0\}$ is $\tau$-closed. Let $\widehat{P}$ denote the $\hat{\tau}$-closure of $P$. Then, as in the proof of ([1], Theorem 2.1), $\widehat{P}$ is a cone in $\widehat{G}$; that is, $\widehat{P}+\widehat{P}\subseteq\widehat{P}$ and $\widehat{P}\cap-\widehat{P}=\{0\}$. The partial ordering induced by $\widehat{P}$ on $\widehat{G}$ extends the partial ordering on $G$ and makes $\widehat{G}$ into an $\ell$-group.

Let $\mathscr{U}$ be a base of solid $\tau$-neighborhoods of 0 in $G$. A base $\widehat{\mathscr{U}}$ of $\hat{\tau}$-neighborhoods of 0 in $\widehat{G}$ consists of the sets $\widehat{U}$, where $U\in\mathscr{U}$ and $\widehat{U}$ denotes the $\hat{\tau}$-closure of $U$. Let $\widehat{V}$ by any $\hat{\tau}$-neighborhood of 0. Then, as in the proof of ([1], Theorem 2.1), there exists a $\hat{\tau}$-neighborhood $\widehat{U}$ of 0 such that the solid hull of $\widehat{U}, S(\widehat{U})$ say, is contained in $\widehat{V}$; that is, $\widehat{U}\subseteq S(\widehat{U})\subseteq\widehat{V}$. It follows that $(\widehat{G},\hat{\tau})$ is a locally solid $\ell$-group.

To show that $(\widehat{G},\hat{\tau})$ is order-complete we modify the proof of ([15], Theorem 1) and use neighborhoods instead of quasi-norms, as follows.

Suppose that $a_n\in\widehat{G}$, $a_n\uparrow$ and $a_n\leqslant 0$ $(n=1,2,\ldots)$. Let $\widehat{V}$ be any $\hat{\tau}$-neighborhood of 0 and let $\widehat{U}$ be a $\hat{\tau}$-neighborhood of 0 in $\widehat{G}$ such that $\widehat{U}+\widehat{U}+\widehat{U}\subseteq\widehat{V}$. For each $n$, there exists a solid $\hat{\tau}$-neighborhood $\widehat{U}_n$ of 0 such that $\widehat{U}_1+\widehat{U}_2+\ldots+\widehat{U}_n\subseteq\widehat{U}$ and an element $x_n$ in $G$, with $x_n\leqslant 0$, such that $|x_n-a_n|\in\widehat{U}_n$.

Let $y_1=x_1$, $y_{n+1}=y_n\vee x_{n+1}$ $(n\geqslant 1)$, so that $y_n\in G$, $y_n\uparrow$ and $y_n\leqslant 0$. Now

$$|y_x\vee x_{n+1}-a_{n+1}|\leqslant|x_{n+1}-a_n|+|y_n-a_n|,$$

which implies that, for $n=1,2,\ldots$,

$$|y_n-a_n|\leqslant|x_1-a_1|+|x_2-a_2|+\ldots+|x_n-a_n|,$$

and so

$$|y_n-a_n|\in\widehat{U}_1+\ldots+\widehat{U}_n\subseteq\widehat{U}.$$

Thus, for any integer $p$ and $n = 1, 2, \ldots$,

$$|a_{n+p} - a_n| \leqslant |a_{n+p} - y_{n+p}| + |y_{n+p} - y_n| + |y_n - a_n|,$$

and so, since $(G, \tau)$ has the A(iii)-property, it follows that, for $n$ sufficiently large,

$$|a_{n+p} - a_n| \in \widehat{U} + \widehat{U} + \widehat{U} \subseteq \widehat{V}.$$

Thus $(\widehat{G}, \hat{\tau})$ has the A(iii)-property and so, by Lemma 1, has the A(iv)-property. Suppose that $\{x_\alpha\colon \alpha \in I\}$ is a bounded increasing net in $\widehat{G}$. Then, since $\widehat{G}$ is complete, $x_\alpha \xrightarrow{\hat{\tau}} x$ in $\widehat{G}$ and it is not difficult to see that $x \geqslant x_\alpha$ $(\alpha \in I)$. Also, if $y \geqslant x_\alpha$ for all $\alpha \in I$, then $y - x_\alpha \in \widehat{P}$ and $y - x_\alpha \xrightarrow{\hat{\tau}} y - x \in \widehat{P}$. Thus $x = \sup x_a$, and so $\widehat{G}$ is order-complete, as required. □

In [2], Aliprantis and Burkinshaw gave a new proof of the following theorem (due to Nakano) and on a subsequent 'revisit' to Nakano's theorem in [3] were able to simplify their proof even further.

**Theorem N** (Nakano). *If $(E, \tau)$ is an order-complete locally solid vector lattice with the Fatou property, then the order intervals are $\tau$-complete.*

We now prove a version of the above theorem for an order-complete locally solid $\ell$-group with the Fatou property; our proof is based on the proof of Theorem N but we make use of $\ell$-quasi-norms instead of neighborhoods. For this, we require the following pair of lemmas.

**Lemma 2.** *Let $(G, \tau)$ be a Hausdorff locally solid $\ell$-group and $(\widehat{G}, \hat{\tau})$ its completion. Then the following are equivalent:*
(i) *$G$ is an ideal in $\widehat{G}$,*
(ii) *every order interval in $\widehat{G}$ is $\tau$-complete.*

Proof. This is a trivial modification of the proof of ([1], Theorem 2.2). □

**Lemma 3.** *Let $G$ be an $\ell$-group and suppose that the sequence $\{u_n\}$ order-converges to $u \in G$. If $\varrho$ is a Fatou $\ell$-quasi-norm, then $\varrho(u) \leqslant \sup_n \varrho(u_n)$.*

Proof. Since $u_n \xrightarrow{(0)} u$, there exists a sequence $\{v_n\}$ in $G$ such that $|u_n - u| \leqslant v_n \downarrow 0$ $(n = 1, 2, \ldots)$. Now $|\,|u_n| - |u|\,| \leqslant |u_n - u| \leqslant v_n \downarrow 0$ implies that $(|u_n| - v_n)^+ \uparrow |u|$ and $(|u| - v_n)^+ \leqslant |u_n|$ $(n = 1, 2, \ldots)$. Since $\varrho$ is a Fatou quasi-norm,

$$\varrho(u) = \varrho(|u|) = \sup_n \varrho((|u| - v_n)^+) \leqslant \sup_n \varrho(u_n),$$

as required. □

**Theorem 2.** *Let $(G,\tau)$ be an order-complete Hausdorff locally solid $\ell$-group with the Fatou property. Then the order intervals of $G$ are $\tau$-complete.*

P r o o f. By Lemma 2 it is sufficient to show that $G$ is an ideal in $\widehat{G}$. We first show that $G$ is order-dense in $A(G)$, the ideal generated by $G$ in $\widehat{G}$; $A(G)=\{\hat{u}\in\widehat{G}\colon \exists x$ in $G$ such that $|\hat{u}|\leqslant|x|\}$.

Let $\hat{u}$ be an element of $A(G)$, so there is an element $u$ in $G$ such that $0<\hat{u}\leqslant u$. Since $\hat{\tau}$ is Hausdorff, there exists a Fatou $\tau$-neighborhood $V$ of 0 such that $\hat{u}\notin\widehat{V}$ and corresponding to $V$ there is a $\tau$-continuous Fatou $\ell$-quasi-norm $\eta$ on $G$ such that $\{x\in G\colon \eta(x)<1\}\subseteq V$. Now $\eta$ has a unique extension to a $\hat{\tau}$-continuous $\ell$-quasi-norm, $\hat{\eta}$ say, on $\widehat{G}$, and we can choose a sequence $\{u_n\}$ in $G$ such that $0\leqslant u_n\leqslant u$ and $\hat{\eta}(\hat{u}-u_n)<2^{-(n+4)}$. Thus, for any positive integer $p$,

$$\eta(u_{n+1}-u_n)\leqslant\hat{\eta}(u_{n+1}-\hat{u})+\hat{\eta}(\hat{u}-u_n)<2^{-(n+3)}\qquad(n=1,2,\ldots)$$

implies that $\eta(u_{n+p}-u_n)<2^{-(n+2)}$.

Let $n$ be any positive integer and let $w_{n,p}=\sup\{u_m\colon\ n\leqslant m\leqslant n+p\}$. Then

$$\begin{aligned}0\leqslant w_{n,p}-u_n&=\sup\{u_m-u_n\colon\ n\leqslant m\leqslant n+p\}\\&\leqslant\sup\{|u_m-u_n|\colon\ n\leqslant m\leqslant n+p\}\\&\leqslant\sum_{m=n}^{n+p-1}|u_{m+1}-u_m|,\end{aligned}$$

and so $\eta(w_{n,p}-u_n)<2^{-(n+2)}$. Since $G$ is order-complete $w_n=\bigvee_{m\geqslant n}u_m$ exists in $G$ and $w_{n,p}\uparrow w_n$. This implies that $0\leqslant w_{n,p}-u_n\uparrow w_n-u_n$ and so, since $\eta$ is a Fatou $\ell$-quasi-norm, $\eta(w_n-u_n)\leqslant2^{-(n+2)}$. Thus, for any positive integer $q$,

$$\begin{aligned}\eta(w_{n+q}-u_n)&\leqslant\eta(w_{n+q}-u_{n+q})+\eta(u_{n+q}-u_n)\\&\leqslant2^{-(n+q+2)}+2^{-(n+2)}\\&<2^{-(n+1)}.\end{aligned}$$

Since $G$ is order-complete, there exists a $w\geqslant0$ in $G$ such that $w_n\downarrow w$. Now $|w_{n+q}-u_n|\overset{(0)}{\to}|w-u_n|$ as $q\to\infty$, and so by Lemma 3, $\eta(|w-u_n|)\leqslant2^{-(n+1)}$. From $\hat{u}-w=\hat{u}-u_n+u_n-w$, we have that $\hat{\eta}(\hat{u}-w)<2^{-n}$ and, since $n$ is any positive integer, it follows that $\hat{\eta}(\hat{u}-w)=0$. Let $S=\{v\in G^{+}\colon\ \eta(w-v)=0\}$. Clearly, $S$ is non-empty. Assume that $s=\inf S\in G$. The set $\mathcal{L}=\{z\in G\colon\ \eta(z)=0\}$ is a solid subgroup of $G$ with the Fatou property and $w-S\subseteq\mathcal{L}$. Now $\sup(w-S)=w-s$ and so, since $\mathcal{L}$ has the Fatou property, $w-s\in\mathcal{L}$. It follows that $s>0$; for, if $s=0$, then $\eta(w)=0$, and so $\hat{\eta}(\hat{u})=0$, which contradicts the fact that $\hat{u}\notin\widehat{V}$.

Next, let $W$ be any Fatou $\tau$-neighborhood of 0 in $G$ such that $W \subseteq V$. Then there exists a $\tau$-continuous Fatou $\ell$-quasi-norm, $\varrho$ say, such that $\{x \in G\colon\ \varrho(x) < 1\} \subseteq W$. Let $\xi(x) = \max(\varrho(x), \eta(x))$. Then $\xi$ is a $\tau$-continuous Fatou $\ell$-quasi-norm and $\{x \in G\colon\ \xi(x) < 1\} \subseteq W$. By repeating the argument used above there exists an element $a$ in $G^+$ such that $\hat{\xi}(\hat{u} - a) = 0$. This implies that $\hat{\eta}(\hat{u} - a) = 0$ and from the identity $a - w = a - \hat{u} + \hat{u} - w$, it follows that $\eta(a - w) = 0$. Thus $a \in S$ and so $s \leqslant a$. This implies that $0 \leqslant (s - \hat{u})^+ \leqslant (a - u)^+$ and, since $\hat{\xi}((a - \hat{u})^+) = 0$ and $\hat{\xi}$ is an $\ell$-quasi-norm, it follows that $\hat{\xi}((s - \hat{u})^+) = 0$. Hence $(s - \hat{u})^+ \in \widehat{W}$ for all Fatou $\tau$-neighborhoods $W$ of 0 in $G$ with $W \subseteq V$. This implies that $(s - \hat{u})^+ = 0$ and so $0 < s \leqslant \hat{u}$; that is, $G$ is order-dense in $A(G)$.

By Theorem 1, $\widehat{G}$ is order-complete and so, in particular, $A(G)$ is Archimedean. Thus $\hat{u} = \sup\{v \in G\colon\ 0 \leqslant v \leqslant \hat{u}\}$. We recall that $\hat{u} \leqslant u$ and so, since $G$ is order-complete, $z = \sup\{v \in G\colon\ 0 \leqslant v \leqslant \hat{u}\}$ exists in $G$. Since $G$ is order-dense in $A(G)$, $z = \hat{u}$ which implies that $G$ is an ideal in $\widehat{G}$. This completes the proof of the theorem. □

In [2], Aliprantis and Burkinshaw gave a new proof of the following theorem due to Fremlin ([11], Theorem 1).

**Theorem F** (Fremlin). *Let $(E, \tau)$ be a Hausdorff locally solid vector lattice with the Fatou property. Then*
(i) *$E$ is order-dense in $\widehat{E}$, and*
(ii) *$(\widehat{E}, \hat{\tau})$ satisfies the Fatou property.*

For a Hausdorff locally solid $\ell$-group with the Fatou property, we have the following.

**Theorem 3.** *Let $(G, \tau)$ be a Hausdorff locally solid $\ell$-group with the Fatou property. Then*
(i) *$G$ is order-dense in $\widehat{G}$, and*
(ii) *$(\widehat{G}, \hat{\tau})$ has the Fatou property.*

Proof. Suppose first that $G$ is order-complete. Then, by Theorem 2, the order intervals of $G$ are $\tau$-complete and so $G$ is an ideal in $\widehat{G}$ by Lemma 2. In particular, this implies that $G$ is order-dense in $\widehat{G}$ (cf. [1], p. 110).

The proof of (ii) and the proof of the theorem when $G$ is not order-complete, follow from the arguments used by Aliprantis and Burkinshaw to prove Theorem F and so will be omitted. □

The topological approach to measure-theoretic studies and the applications of topology to measure theory are very well-studied (see, for example, Drewnowski [10] and the references therein, Kalton [15] and Fremlin [11]). We continue this theme and close this section with a version of the Nikodym boundedness theorem for functions assuming values in a class of locally solid topological groups; our proof is order-theoretic in nature (not related to the completion procedure of the group).

The Nikodym boundedness theorem, from measure theory, has received a great deal of attention and has been generalized in several directions; its versions, for example, for lattice-valued and vector-valued measures may be found in [19] and [20], respectively.

Let us first briefly recall some definitions.

Let $G$ be a Hausdorff topological group and $\mathcal{R}$ a ring of subsets of a set $X$. A function $\mu\colon \mathcal{R} \to G$ is said to be (i) a measure if $\mu(\emptyset) = 0$ and $\mu(E \cup F) = \mu(E) + \mu(F)$ where $E$ and $F$ are in $\mathcal{R}$ with $E \cap F = \emptyset$; (ii) exhaustive if for every sequence $\{E_n\}$ of pairwise disjoint sets in $\mathcal{R}$, $\lim_{n\to\infty} \mu(E_n) = 0$.

The notion of a submeasure, with enormous applications, has been extensively studied by Drewnowski (see [10] and the references therein, [4], [16]). Group-valued submeasures have been introduced by Khan and Rowlands [16] and their work has been recently further investigated by Avallone and Valente [4].

Following Khan and Rowlands [16], a function $\mu$ on $\mathcal{R}$ with values in an $\ell$-quasi-normed group is a submeasure if $\mu(\emptyset) = 0$, $\mu(E \cup F) \leqslant \mu(E) + \mu(F)$ for all $E$, $F$ in $\mathcal{R}$ with $E \cap F = \emptyset$ and $\mu(E) \leqslant \mu(F)$ for all $E, F$ in $\mathcal{R}$ with $E \subseteq F$. Clearly, in this case $\mu(E) \geqslant 0$ for all $E$ in $\mathcal{R}$.

Although Theorem 1 due to Drewnowski [10] has been proved in the context of a quasi-normed group, his proof can be readily modified to the case of any Hausdorff topological group $G$; thereby, we achieve the following version of the Nikodym boundedness theorem.

**Theorem 4.** *Let $M$ be a family of exhaustive $G$-valued measures on a $\sigma$-ring $\mathcal{R}$ such that for each $E \in \mathcal{R}$, $\{\mu(E)\colon \mu \in M\}$ is a bounded subset of $G$. Then $\{\mu(E)\colon E \in \mathcal{R}, \mu \in M\}$ is a bounded subset of $G$.*

The assumption that $\mathcal{R}$ is a $\sigma$-ring is essential in the above theorem (see [10], Example, p. 117).

The next result generalizes Theorem 4 to the case of group-valued submeasures.

**Theorem 5.** *Let $(G, q)$ be an $\ell$-quasi-normed group and $M$ be a family of $G$-valued submeasures on a $\sigma$-ring $\mathcal{R}$ such that*

$$\sup_{\mu\in M} q(\mu(E)) < +\infty$$

*for each $E$ in $\mathcal{R}$. Then* $\sup_{\mu\in M, E\in\mathcal{R}} q(\mu(E)) < +\infty$.

P r o o f. Let $H$ be the group of all $G$-valued mappings on $M$. Clearly, $H$ is a commutative partially ordered group, the ordering being $f \leqslant g$ if and only if $f(\mu) \leqslant g(\mu)$ for all $\mu \in M$. Define the functional $\varphi$ on $H$ by

$$\varphi(f) = \sup_{\mu \in M} q(f(\mu)).$$

Note that $\varphi$ is an $\mathbb{R}_+^*$-valued quasi-norm on $H$ and $\varphi(f) \leqslant \varphi(g)$ if $0 \leqslant f \leqslant g$. Define a mapping $\nu\colon \mathcal{R} \to H$ by

$$\nu(E)(\mu) = \mu(E).$$

Clearly, $\nu$ is an $H$-valued submeasure on $\mathcal{R}$.

Suppose not; with the above notation, $\sup\limits_{E \in \mathcal{R}} \varphi(\nu(E)) = +\infty$. Thus, for each positive integer $n$, there exists a set $E_n$ in $\mathcal{R}$ such that $\varphi(\nu(E_n)) > n$. Let $E = \bigcup\limits_{n=1}^{\infty} E_n$, $E \in \mathcal{R}$ and $\varphi(\nu(E)) = +\infty$. This implies that $\sup\limits_{\mu} q(\mu(E)) = +\infty$, which contradicts the hypothesis. Thus $\sup\limits_{\mu \in M, E \in \mathcal{R}} q(\mu(E))$ is finite. □

## 4. The order-bound topology

Let $G$ be an $\ell$-group and let $\mathcal{S}$ be the family of all quasi-norms such that each $p \in \mathcal{S}$ is bounded on order-bounded subsets of $G$. The topology $\tau_b$ induced by $\mathcal{S}$ on $G$ is the analogue of the order-bound topology in the theory of locally convex partially ordered vector spaces, and so, in the sequel, we refer to $\tau_b$ as the order-bound topology on $G$. It is easy to see that $\tau_b$ is the finest group topology on $G$ such that every order-bounded subset of $G$ is topologically bounded.

**Definition 1.** A homomorphism $\varphi$ of a topological $\ell$-group $(G, \tau)$ into a topological group $(H, \xi)$ is said to be *order-bounded* if it maps order-bounded subsets of $G$ into $\xi$-bounded sets.

Now, we obtain the following useful result.

**Proposition 1.** *Let $(G, \tau)$ be a topological $\ell$-group. Then the following statements are equivalent:*

(i) $\tau_b \subseteq \tau$,

(ii) *every order-bounded homomorphism of $(G, \tau)$ into a topological $\ell$-group $(H, \xi)$ is continuous.*

Proof. (ii) $\Rightarrow$ (i). The identity mapping of $(G,\tau)$ into $(G,\tau_b)$ is an order-bounded homomorphism and so is continuous. It follows that $\tau_b \subseteq \tau$.

(i) $\Rightarrow$ (ii). Let $p$ be any $\xi$-continuous quasi-norm. Then $p \circ \varphi$ is a quasi-norm on $G$ bounded on order-bounded sets. Thus $p \circ \varphi$ is $\tau_b$-continuous and so is $\tau$-continuous. It follows that, for any $\xi$-neighborhood $U$ of 0 in $H$, $\varphi^{-1}(U)$ is a $\tau$-neighborhood of 0 in $G$; that is, $\varphi$ is continuous, as required. □

**Theorem 6.** *Let $G$ be an $\ell$-group. Then the order-bound topology on $G$ is the finest locally solid topology on $G$.*

Proof. First we show that every locally solid topology on $G$ is weaker than $\tau_b$. Let $\xi$ be any locally solid topology on $G$. Then $\xi$ is determined by the family of all $\xi$-continuous $\ell$-quasi-norms. Let $q$ be any $\xi$-continuous $\ell$-quasi-norm on $G$. If $z \in [-|x|,|x|]$, then $q(z) \leqslant q(x)$. This implies that every order-bounded interval is $\xi$-bounded and so $\xi \subseteq \tau_b$.

We now prove that $\tau_b$ is locally solid.

Let $\eta$ be any member of $\mathcal{S}$. Since $\eta$ is bounded on the order-bounded intervals, we can use the same construction as that given by Kalton in [15] to define a new quasi-norm $|\eta|$, as follows. For each $a \in G$ with $a \geqslant 0$, let

$$\eta^*(a) = \sup_{0 \leqslant c \leqslant a} \eta(c),$$

and define

$$|\eta|(a) = \inf\{\eta^*(b)\colon\ -b \leqslant a \leqslant b\}.$$

Clearly, $|\eta|$ is bounded on order-bounded sets; also $|\eta|(a) = \eta^*(a)$ for all $a \geqslant 0$. This implies that $|\eta|(|a|) = |\eta|(a) \quad (a \in G)$, and so the topology $\tau_b'$ defined by the quasi-norms $\{|\eta|\}$ $(\eta \in \mathcal{S})$ is locally solid. Thus by the first part of the proof $\tau_b' \subseteq \tau_b$. On the other hand, if $U$ is any $\tau_b$-neighborhood of 0 in $G$, then there exists a $q \in \mathcal{S}$ and a positive number $\varepsilon$ such that $\{x\colon\ q(x) < \varepsilon\} \subseteq U$. Now $q(x) \leqslant 2|q|(x)$ ([15], Lemma 3), and so we have $\{x\colon\ |q|(x) < \varepsilon/2\} \subseteq U$. This implies that $\tau_b \subseteq \tau_b'$ and so $\tau_b = \tau_b'$. Thus $\tau_b$ is locally solid, as required. □

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*References*

[1] *C. D. Aliprantis*: On the completion of Hausdorff locally solid Riesz spaces. Trans. Amer. Math. Soc. *196* (1974), 105–125. zbl

[2] *C. D. Aliprantis and O. Burkinshaw*: A new proof of Nakano's theorem in locally solid Riesz spaces. Math. Zeit. *144* (1975), 25–33. zbl

[3] *C. D. Aliprantis and O. Burkinshaw*: Nakano's theorem revisited. Michigan Math. J. *23* (1976), 173–176. zbl

[4] *A. Avallone and A. Valente*: A decomposition theorem for submeasures. Atti. Sem. Mat. Fis. Univ. Modena *XLIII* (1995), 81–90. zbl

[5] *A. Boccuto and D. Candeloro*: Uniform $s$-boundedness and convergence results for measures with values in complete $\ell$-groups. J. Math. Anal. Appl. *265* (2002), 170–194. zbl

[6] *F. G. Bonales, F. J. Trigos-Arrieta and R. V. Mendoza*: A characterization of Pontryagin-Van Kampen duality for locally convex spaces. Topology Appl. *121* (2002), 75–89. zbl

[7] *N. Bourbaki*: Elements of Mathematics, General Topology, Part 1. Addison-Wesley, 1966. zbl

[8] *W. W. Comfort, S. Hernandez and F. J. Trigos-Arrieta*: Cross sections and homeomorphism classes of Abelian groups equipped with the Bohr topology. Topology Appl. *115* (2001), 215–233. zbl

[9] *W. W. Comfort, S. U. Raczkowski and F. J. Trigos-Arrieta*: The dual group of a dense subgroup. Czech. Math. J. *54 (129)* (2004), 509–533. zbl

[10] *L. Drewnowski*: Uniform boundedness principle for finitely additive vector measures. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. et. Phys. *21* (1973), 115–118. zbl

[11] *D. H. Fremlin*: On the completion of locally solid vector lattices. Pacific J. Math. *43* (1972), 341–347. zbl

[12] *D. H. Fremlin*: Topological Riesz Spaces and Measure Theory. Cambridge University Press, England, 1974. zbl

[13] *J. Jakubík*: On the affine completeness of lattice ordered groups. Czech. Math. J. *54 (129)* (2004), 423–429. zbl

[14] *G. Jameson*: Ordered Linear Spaces, Lecture Notes in Mathematics No. 141, Springer-Verlag, Berlin, Germany. 1970. zbl

[15] *J. K. Kalton*: Topologies on Riesz groups and applications to measure theory. Proc. London Math. Soc. *28* (1974), 253–273. zbl

[16] *A. R. Khan and K. Rowlands*: A decomposition theorem for submeasures. Glasgow Math. J. *26* (1985), 67–74. zbl

[17] *S. U. Raczkowski*: Totally bounded topological group topologies on the integers. Topology Appl. *121* (2002), 63–74. zbl

[18] *K. D. Schmidt*: Decompositions of vector measures in Riesz spaces and Banach lattices. Proc. Edinburgh Math. Soc. *29* (1986), 23–29. zbl

[19] *C. Swartz*: The Nikodym boundedness theorem for lattice-valued measures. Arch. Math. *53* (1989), 390–393. zbl

[20] *C. Swartz*: An Introduction to Functional Analysis. Marcel Dekker, New York, U.S.A., 1992. zbl



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