Czechoslovak Mathematical Journal, 57 (132) (2007), 1035–1041

HYPERCYCLICITY OF SPECIAL OPERATORS ON HILBERT FUNCTION SPACES

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(Received September 20, 2005)

Abstract. In this paper we give some sufficient conditions for the adjoint of a weighted composition operator on a Hilbert space of analytic functions to be hypercyclic.

Keywords: multiplier, orbit, hypercyclic vector, multiplication operator, weighted composition operator

MSC 2000: 47B37, 47B33

Introduction

Let H be a Hilbert space of functions analytic on a plane domain G such that for each λ in G the linear functional of evaluation at λ given by $f \mapsto f(\lambda)$ is a bounded linear functional on H. By the Riesz representation theorem there is a vector K_λ in H such that $f(\lambda) = \langle f, K_\lambda \rangle$. We call K_λ the reproducing kernel at λ.

Let T be a bounded linear operator on H. For $x \in H$, the orbit of x under T is the set of images of x under the successive iterates of T:

$$\operatorname{orb}(T, x) = \{x, Tx, T^2x, \ldots\}.$$

The vector x is called hypercyclic for T if $\operatorname{orb}(T, x)$ is dense in H. Also a hypercyclic operator is one that has a hypercyclic vector.

The first example of a hypercyclic operator on a Hilbert space was constructed by Rolewicz in 1969 [12]. He showed that if B is the backward shift on $\ell^2(\mathbb{N})$, then λB is hypercyclic if and only if $|\lambda| > 1$.

This paper is a part of the second author's Doctoral thesis written at Shiraz University under the direction of the first author.

A complex-valued function ψ on G is called a multiplier of H if $\psi H \subset H$. The operator of multiplication by ψ is denoted by M_ψ and is given by $f \mapsto \psi f$. By the closed graph theorem M_ψ is bounded. The collection of all multipliers is denoted by $M(H)$. Each multiplier is a bounded analytic function on G. In fact, $\|\varphi\|_G \leqslant \|M_\varphi\|$ ([14]).

If w is a multiplier of H and φ is a mapping from G into G such that $f \circ \varphi \in H$ for all $f \in H$, then C_φ (defined on H by $C_\varphi f = f \circ \varphi$) and $M_w C_\varphi$ are called the composition and the weighted composition operator, respectively. We define the iterates $\varphi_n = \varphi \circ \varphi \circ \ldots \circ \varphi$ (n times). Note that $C_{\varphi_n} = C_\varphi^n$ for all n. In this paper we investigate the hypercyclicity of the adjoint of a weighted composition operator acting on a Hilbert space of analytic functions. For some sources on hypercyclic topic see [1]–[13], [15], [16].

Main results

A nice criterion, namely the Hypercyclicity Criterion is used in the proof of our main theorem. It was developed independently by Kitai [10], Gethner and Shapiro [6]. This criterion has been used to show that hypercyclic operators arise within the classes of composition operators [4], weighted shifts [13], adjoints of multiplication operators [5], and adjoints of subnormal and hyponormal operators [3].

The formulation of the Hypercyclicity Criterion in the following theorem was given by J. Bes in his PhD. thesis [1] (see also [2]).

The Hypercyclicity Criterion Theorem. Suppose X is a separable Banach space and T is a continuous linear mapping on X. If there exist two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that

1. $T^{n_k} y \to 0$ for every $y \in Y$, and
2. there exist functions $S_{n_k}\colon Z \to X$ such that for every $z \in Z$, $S_{n_k} z \to 0$ and $T^{n_k} S_{n_k} z \to z$,

then T is hypercyclic.

Throughout this section let H be a Hilbert space of analytic functions on the open unit disc $\mathbb{D}$ such that H contains constants and the functional of evaluation at λ is bounded for all λ in $\mathbb{D}$. Further, let φ be an analytic univalent map from $\mathbb{D}$ onto $\mathbb{D}$. By φ_n^{-1} we mean the nth iterate of φ^{-1}.

Theorem. *Suppose that the composition operator C_φ is bounded on H and w is a nonconstant multiplier of H such that the sets $\{\lambda \in \mathbb{D}\colon \sup_n |w \circ \varphi_n(\lambda)| < 1\}$ and $\{\lambda \in \mathbb{D}\colon \inf_n |w \circ \varphi_n^{-1}(\lambda)| > 1\}$ have limit points in $\mathbb{D}$. Then the adjoint of the weighted composition operator $M_w C_\varphi$ is hypercyclic.*

P r o o f. First we note that if $\lambda \in \mathbb{D}$ and $f \in H$, then we get

$$\langle f, M_w^* K_\lambda \rangle = \langle wf, K_\lambda \rangle = \langle f, \overline{w(\lambda)} K_\lambda \rangle,$$

which implies that $M_w^* K_\lambda = \overline{w(\lambda)} K_\lambda$. Also

$$\left\langle f, C_\varphi^* K_\lambda \right\rangle = \langle f \circ \varphi, K_\lambda \rangle = f(\varphi(\lambda)) = \left\langle f, K_{\varphi(\lambda)} \right\rangle,$$

hence $C_\varphi^* K_\lambda = K_{\varphi(\lambda)}$. Thus we have

$$(M_w C_\varphi)^* K_\lambda = C_\varphi^* (M_w^* K_\lambda) = \overline{w(\lambda)} C_\varphi^* K_\lambda = \overline{w(\lambda)} K_{\varphi(\lambda)}.$$

Put $A = M_w C_\varphi$ and $\varphi_0 = I$ where I is the identity mapping on $\mathbb{D}$. Then for all $n \in \mathbb{N}$ and all λ in $\mathbb{D}$ we get

$$(A^*)^n K_\lambda = \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\lambda))} \right) K_{\varphi_n(\lambda)}.$$

Put

$$E = \{\lambda \in \mathbb{D}\colon \sup_n |w(\varphi_n(\lambda))| < 1\}$$

and

$$H_E = \operatorname{span}\{K_\lambda\colon \lambda \in E\}.$$

The set H_E is dense in H, because if $f \in H$ and $\langle f, K_\lambda \rangle = 0$ for all λ in E, then $f(\lambda) = 0$ for all λ in E. So by virtue of the hypothesis of the theorem, the zeros of f have a limit point in $\mathbb{D}$, which implies that $f \equiv 0$ on $\mathbb{D}$. Thus H_E is dense in H.

Note that if $\lambda \in E$, then there exists a number α such that $0 < \alpha < 1$ and $\sup_n |w(\varphi_n(\lambda))| < \alpha < 1$. Thus $|w(\varphi_n(\lambda))| < \alpha$ for all n and so we have

$$\prod_{i=0}^{n-1} |w(\varphi_i(\lambda))| \leqslant \prod_{i=0}^{n-1} \alpha = \alpha^n.$$

Since $\lim_{n\to\infty} \alpha^n = 0$, we get

$$\prod_{i=0}^{\infty} |w(\varphi_i(\lambda))| = 0$$

and so we have $\lim_n (A^*)^n K_\lambda = 0$. Thus $(A^*)^n \to 0$ pointwise on H_E which is dense in H.

Now we want to find a right inverse of A^* on a dense subset of H. To see this put

$$F = \{\lambda \in \mathbb{D}\colon \inf_n |w(\varphi_n^{-1}(\lambda))| > 1\}.$$

By a method similar to that we used to prove that H_E is dense in H, we can see that the set

$$H_F = \operatorname{span}\{K_\lambda\colon\ \lambda \in F\}$$

is dense in H, since F has a limit point in $\mathbb{D}$. To find the desired right inverse of A^*, first consider the special case when the collection of linear functionals of point evaluations $\{K_\lambda\colon\ \lambda \in F\}$ is linearly independent. Note that in the next definition there is no possibility of dividing by zero.

Define $B\colon\ H_F \to H$ by extending the definition

$$BK_\lambda = (\overline{w(\varphi^{-1}(\lambda))})^{-1} K_{\varphi^{-1}(\lambda)} \quad (\lambda \in F)$$

linearly to H_F (it is good to note that if $\lambda \in F$, then $\varphi^{-1}(\lambda) \in F$ and indeed B maps H_F to H_F). Now we clearly get

$$B^2 K_\lambda = (\overline{w(\varphi^{-1}(\lambda))})^{-1} (\overline{w(\varphi^{-1}(\varphi^{-1}(\lambda)))})^{-1} K_{\varphi^{-1}(\varphi^{-1}(\lambda))}$$

for all λ in F. Continuing in this manner we can see that

$$B^n K_\lambda = \left(\prod_{i=1}^{n} (\overline{w(\varphi_i^{-1}(\lambda))})^{-1}\right) K_{\varphi_n^{-1}(\lambda)},$$

where φ_i^{-1} is the ith iterate of φ^{-1} and $n \in \mathbb{N}$. By the definition of B we have

$$A^* B K_\lambda = A^*((\overline{w(\varphi^{-1}(\lambda))})^{-1} K_{\varphi^{-1}(\lambda)}) = K_{\varphi(\varphi^{-1}(\lambda))} = K_\lambda$$

for all λ in F. Thus A^*B is identity on the dense subset H_F of H.

Now we want to show that $B^n \to 0$ pointwise on H_F. Note that if $\lambda \in F$, then there exists a number $\beta > 1$ such that

$$\inf_n |w(\varphi_n^{-1}(\lambda))| > \beta > 1.$$

Thus $|w(\varphi_n^{-1}(\lambda))| > \beta > 1$ for all n and so we have

$$\prod_{i=1}^{n} |w(\varphi_i^{-1}(\lambda))|^{-1} \leqslant \left(\frac{1}{\beta}\right)^n.$$

Since $0 < 1/\beta < 1$, we obtain that $\lim\limits_{n\to\infty} (1/\beta)^n = 0$ and so

$$\lim_n \left(\prod_{i=1}^{n} |w(\varphi_i^{-1}(\lambda))|^{-1}\right) K_{\varphi_n^{-1}(\lambda)} = 0.$$

This implies that $B^n \to 0$ pointwise on H_F which is dense in H. Thus by the Hypercyclicity Criterion Theorem or by Corollary 1.5 in [7, p. 235], $A^* = (M_w C_\varphi)^*$ has a hypercyclic vector.

In the case when the linear functionals of point evaluations are not linearly independent, we use a standard method: consider a countable dense subset $F_1 = \{\lambda_n\colon\ n \geqslant 1\}$ of F and inductively choose a subsequence $\{z_n\}$ as follows. Let $z_1 = \lambda_1$. Define

$$F_2 = F_1 \setminus \{\lambda \in F_1\colon\ K_\lambda \in \operatorname{span}\{K_{z_1}\}\}.$$

Denote the first element of F_2 by z_2 and define

$$F_3 = F_2 \setminus \{\lambda \in F_2\colon\ K_\lambda \in \operatorname{span}\{K_{z_1}, K_{z_2}\}\}.$$

Continuing in this manner, we obtain a subset $G = \{z_n\}_n$ of F for which the set

$$H_G = \operatorname{span}\{K_\lambda\colon\ \lambda \in G\}$$

is dense in H with linearly independent linear functionals of point evaluations $\{K_\lambda\colon\ \lambda \in G\}$. Now for each $n \in \mathbb{N}$, define mappings $S_n\colon\ H_G \to H$ by extending the definition

$$S_n K_\lambda = \left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}}\right) K_{\varphi_n^{-1}(\lambda)} \quad (\lambda \in G)$$

linearly to H_G.

Note that if we substitute $\varphi_n^{-1}(\lambda)$ instead of λ in the formula obtained earlier for $(A^*)^n K_\lambda$, we get

$$\begin{aligned}
(A^*)^n K_{\varphi_n^{-1}(\lambda)} &= \left(\prod_{i=0}^{n-1} \overline{w(\varphi_i(\varphi_n^{-1}(\lambda)))}\right) K_{\varphi_n(\varphi_n^{-1}(\lambda))} \\
&= \left(\prod_{i=0}^{n-1} \overline{w(\varphi_{n-i}^{-1}(\lambda))}\right) K_{\varphi_n \circ \varphi_n^{-1}(\lambda)} \\
&= \prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))} K_\lambda
\end{aligned}$$

for all λ in G.

By the definition of S_n we have

$$\begin{aligned}
(A^*)^n S_n K_\lambda &= (A^*)^n \left(\left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}}\right) K_{\varphi_n^{-1}(\lambda)} \right) \\
&= \left(\prod_{i=1}^{n} \overline{(w(\varphi_i^{-1}(\lambda)))^{-1}}\right) (A^*)^n K_{\varphi_n^{-1}(\lambda)} = K_\lambda
\end{aligned}$$

for all λ in G. Thus for all $n \in \mathbb{N}$, $(A^*)^n S_n$ is identity on the dense subset H_G of H. Now, exactly as proved before we have that $B^n \to 0$ pointwise on H_F, we can see that $S_n \to 0$ pointwise on H_G which is dense in H. Thus the conditions of the Hypercyclicity Criterion Theorem are satisfied and so the proof is complete. $\square$

Corollary. *Suppose that w is a nonconstant multiplier of H such that* $\operatorname{ran} w$ *intersects the unit circle. Then the adjoint of the multiplication operator M_w is hypercyclic.*

P r o o f. In the above theorem let φ be identity. Then $\varphi_n(\lambda) = \lambda$ and $\varphi_n^{-1}(\lambda) = \lambda$ for all λ in $\mathbb{D}$. Further, we note that the condition $1 \in H$ implies that $w \in H$ and so w is analytic on the open unit disc $\mathbb{D}$. Now by the Open Mapping Theorem $w(\mathbb{D})$ is open. But $\operatorname{ran} w = w(\mathbb{D})$ intersects the unit circle, thus the sets

$$\{\lambda \in \mathbb{D}\colon\ |w(\lambda)| < 1\}$$

and

$$\{\lambda \in \mathbb{D}\colon\ |w(\lambda)| > 1\}$$

are nonempty open sets in $\mathbb{D}$ and so clearly have limit points in $\mathbb{D}$. Now we can apply the result of the Theorem and so the proof of the Corollary is complete. $\square$

Remark 1. Note that in the above theorem we restrict ourselves for simplicity to the open unit disc but it remains true if we substitute the open unit disc $\mathbb{D}$ by a connected open subset Ω of $\mathbb{C}^n$ where $n \in \mathbb{N}$. In this case H is a Hilbert space of complex valued analytic functions on Ω such that H contains constants and the functional of evaluation at λ is bounded for all λ in Ω. Also, φ is an analytic univalent map from Ω onto Ω and w is an analytic complex valued function on Ω.

Remark 2. Let B, F, G, H_F, H_G and S_n be defined as in the proof of the main theorem. Note that if we define the mapping $B\colon H_G \to H$ exactly as it is defined (in the proof of the theorem) on H_F, then for defining $B^2\colon H_G \to H$ we should have $B(H_G) \subset H_G$, which is not since we do not know that whether $\varphi^{-1}(\lambda) \in G$ whenever $\lambda \in G$. For this reason we have to define the mappings S_n on H_G instead of the operators B^n.

Acknowledgment.
The authors thank the referee for interesting comments and helpful suggestions concerning the paper.

References

[1] *J. Bes*: Three problems on hypercyclic operators. PhD. Thesis. Kent State University, 1998.

[2] *J. Bes, A. Peris*: Hereditarily hypercyclic operators. J. Funct. Anal. *167* (1999), 94–112. zbl

[3] *P. S. Bourdon*: Orbits of hyponormal operators. Mich. Math. J. *44* (1997), 345–353. zbl

[4] *P. S. Bourdon, J. H. Shapiro*: Cyclic Phenomena for Composition Operators. Memoirs of the Am. Math. Soc. 125. Am. Math. Soc., Providence, 1997. zbl

[5] *P. S. Bourdon, J. H. Shapiro*: Hypercyclic operators that commute with the Bergman backward shift. Trans. Am. Math. Soc *352* (2000), 5293–5316. zbl

[6] *R. M. Gethner, J. H. Shapiro*: Universal vectors for operators on spaces of holomorphic functions. Proc. Am. Math. Soc. *100* (1987), 281–288. zbl

[7] *G. Godefroy, J. H. Shapiro*: Operators with dense, invariant cyclic vector manifolds. J. Funct. Anal. *98* (1991), 229–269. zbl

[8] *K. Goswin, G. Erdmann*: Universal families and hypercyclic operators. Bull. Am. Math. Soc. *35* (1999), 345–381. zbl

[9] *D. A. Herrero*: Limits of hypercyclic and supercyclic operators. J. Funct. Anal. *99* (1991), 179–190. zbl

[10] *C. Kitai*: Invariant closed sets for linear operators. Thesis. University of Toronto, Toronto, 1982.

[11] *C. Read*: The invariant subspace problem for a class of Banach spaces. 2. Hypercyclic operators. Isr. J. Math. *63* (1988), 1–40. zbl

[12] *S. Rolewicz*: On orbits of elements. Stud. Math. *32* (1969), 17–22. zbl

[13] *H. N. Salas*: Hypercyclic weighted shifts. Trans. Am. Math. Soc. *347* (1995), 993–1004. zbl

[14] *A. L. Shields, L. J. Wallen*: The commutants of certain Hilbert space operators. Indiana Univ. Math. J. *20* (1971), 777–788. zbl

[15] *B. Yousefi, H. Rezaei*: Hypercyclicity on the algebra of Hilbert-Schmidt operators. Result. Math. *46* (2004), 174–180. zbl

[16] *B. Yousefi, H. Rezaei*: Some necessary and sufficient conditions for Hypercyclicity Criterion. Proc. Indian Acad. Sci. (Math. Sci.) *115* (2005), 209–216. zbl

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