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BOUNDARY VALUE PROBLEMS AND LAYER POTENTIALS ON MANIFOLDS WITH CYLINDRICAL ENDS

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Abstract. We study the method of layer potentials for manifolds with boundary and cylindrical ends. The fact that the boundary is non-compact prevents us from using the standard characterization of Fredholm or compact pseudo-differential operators between Sobolev spaces, as, for example, in the works of Fabes-Jodeit-Lewis [10] and Kral-Wedland [18]. We first study the layer potentials depending on a parameter on compact manifolds. This then yields the invertibility of the relevant boundary integral operators in the global, non-compact setting. As an application, we prove a well-posedness result for the non-homogeneous Dirichlet problem on manifolds with boundary and cylindrical ends. We also prove the existence of the Dirichlet-to-Neumann map, which we show to be a pseudodifferential operator in the calculus of pseudodifferential operators that are "almost translation invariant at infinity."

Keywords: layer potentials, manifolds with cylindrical ends, Dirichlet problem

MSC 2000: 31C12, 58J05, 58J40, 35J05

Introduction

Boundary value problems, mostly on compact manifolds, have long been studied because of their numerous applications to other areas of Mathematics, Physics, and Engineering. Arguably, some of the most important examples arise in connection with the Laplacian and related operators.

A first, simple approach to boundary value problems for the Laplace operator is via the Lax-Milgram theorem which amounts to proving an energy estimate (coercivity)

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for the de Rham differential of certain classes of scalar functions [36], [47]. Another approach commonly used in the literature is via boundary layer potential integrals. While less elementary, this has the advantage that it provides more information about the spaces of Cauchy data, and it allows one to express the solutions via explicit formulas. The same approach may be used to study boundary problems on spaces with weights, on which the Laplace operator may fail to be symmetric.

The second approach, based on the method of layer potentials, became widely used after the pioneering work of Hodge, Kodaira, Kral, Spencer, Duff, and Kohn, among others. See, for instance, [15], [20], [17], [21], or the discussion in the introduction of [35] for further information and references. This method, combining ideas both from the approach based on the Lax-Milgram theorem and the approach based on the Boutet de Monvel calculus, has been successfully employed to solve boundary value problems on compact manifolds with smooth boundary.

More recently, the method of layer potentials has also lead to a solution of the Dirichlet problem for the Laplace operator on compact manifolds with Lipschitz boundaries in [33]. This, in turn, builds on the earlier work from [8], [12], [19], and [50], in the constant coefficient, Euclidean context.

In view of possible applications to boundary value problems on polyhedral domains, we would like to extend the method of layer potentials to various classes of *non-compact* manifolds. There are, however, several technical problems that we need to first overcome for such an extension to be possible—at least along the classical lines. The main contribution of this paper is to systematically study these difficulties in the particular case of manifolds with cylindrical ends, when a number of required results from analysis take a simpler form. See also [14], [22], [9], [41] for earlier results on boundary value problems on non-compact manifolds.

A crucial step in the method of layer potentials is to prove the invertibility of $-1/2I + K$, where K is a suitable pseudodifferential operator. For a domain with smooth boundary, K is compact, and one can use the fact that $-1/2I+K$ is Fredholm of index zero. For a non-compact or non-smooth domain, K in general will not be compact [10], [11], [12], [18]. Nevertheless, $-1/2I + K$ still turns out to be compact (see also [26]). We hope that the approach that we outline in this paper will generalize to more general elliptic Partial Differential Equations on non-compact manifolds.

In fact, in [41], Elmar Schrohe has studied boundary value problems for "asymptotically Euclidean manifolds" (this is a class of non-compact manifolds generalizing the class of manifolds that are Euclidean at infinity) by generalizing to this class of manifolds the Boutet de Monvel's algebra. He has also pointed out the importance and relevance of the spectral invariance of various algebras of pseudodifferential operators. A main analytic difference between his class of manifolds and ours is that while the "Fredholm relevant symbol" is commutative for asymptotically Euclidean manifolds, this is no longer true in the case we intend to study, i.e. that of manifolds with cylindrical ends. Manifolds with cylindrical ends have also appeared in the study of boundary value problems on manifolds with conical points [16], [18], [22], [23]. Manifolds with multicylindrical ends were studied in [27].

The results of this paper were used to prove the well-posedness of the Dirichlet problem in suitable Sobolev spaces with weights in [6]. This well-posedness result was then used in the same paper to obtain fast algorithms for solving the Dirichlet problem on polygonal domains in the plane. A summary of the results in this paper and a sketch of the main ideas of the proof were published in [32].

In order to explain some of the technical difficulties encountered in the setting of manifolds with cylindrical ends, we need to introduce some notation. Let N be a non-compact Riemannian manifold with boundary ∂N and $\Delta_N = d^*d$ be the Laplace operator on N action on scalar functions. A first set of problems consists of defining an elementary solution $E(\,\cdot\,,\,\cdot\,)$ for Δ_N on N and proving that the associated single and double layer potential integrals converge—issues well-understood when ∂N is compact. A second set of problems has to do with the existence of the non-tangential limits of the aforementioned layer potential integral operators. Even if the non-tangential limits exist and are given by pseudodifferential operators on ∂N, these pseudodifferential operators are not expected to be properly supported. Moreover, since ∂N is non-compact, the standard results on the boundedness and compactness of order zero (respectively, negative order) pseudodifferential operators do not (directly) apply. Finally, on non-compact manifolds one is lead to consider various algebras of pseudodifferential operators with a controlled behavior at infinity. These algebras may fail to be "spectrally invariant," in the sense that the inverse of an elliptic, L^2-invertible operator in this algebra may fail to be again in this algebra. (See Definition 2.7 for the definition of a spectrally invariant algebra.)

In order to make the above technical problems more tractable, it is natural to make certain additional assumptions on the non-compact manifolds N and its boundary ∂N. In this paper, we restrict ourselves to the class of manifolds with boundary and cylindrical ends. For the sake of this introduction, let us briefly discuss about pseudodifferential operators in this setting and then describe our main results.

Let M be a *boundaryless* manifold with cylindrical ends. Such manifolds have a product structure at infinity in a strong sense (that is, including also the metric—see Definition 5.1). In this setting, we define two classes of pseudodifferential operators: $\Psi^m_{\mathrm{inv}}(M)$ and $\Psi^m_{\mathrm{ai}}(M)$, whose distribution kernels form a class large enough to contain the distribution kernels appearing in our paper as boundary layer integrals. See also [26], where some of these issues were studied in the case of a polygon.

The first class of operators is the class of order m classical pseudodifferential operators that are "translation invariant in a neighborhood of infinity" (Definition 1.1).

The space $\Psi_{\mathrm{ai}}^{-\infty}(M)$ consists of the closure of $\Psi_{\mathrm{inv}}^{-\infty}(M)$ with respect to a suitable family of semi-norms, including for example the norms of linear maps between the Sobolev spaces $H^m(M) \to H^{m'}(M)$, $m, m' \in 2\mathbb{Z}$ (see Equations (11) and (20); Sobolev spaces on non-integral orders can also be defined, but they are not needed to construct our algebras). Then

(1)
$$\Psi_{\mathrm{ai}}^{m}(M) := \Psi_{\mathrm{inv}}^{m}(M) + \Psi_{\mathrm{ai}}^{-\infty}(M).$$

An operator $P \in \Psi_{\mathrm{ai}}^{m}(M)$ is called *almost invariant in a neighborhood of infinity*. For $P \in \Psi_{\mathrm{ai}}^{m}(M)$, we can characterize when it is Fredholm or compact (between suitable Sobolev spaces), along the classical lines. See [22], [24], [30], [31], [40], [42], [44] and others.

We could have also allowed a power law behavior at infinity for our operators. However, this is technical and would have greatly increased the size of the paper, without really making our results more general. It would have also shifted the focus of our paper, which is on boundary value problems and not on constructing and studying algebras of pseudodifferential operators.

The reason for introducing the algebras $\Psi_{\mathrm{ai}}^{\infty}(M)$ is that $T^{-1} \in \Psi_{\mathrm{ai}}^{-m}(M)$, for any elliptic operator $T \in \Psi_{\mathrm{ai}}^{m}(\partial N)$, provided that $m \geqslant 0$ and T is elliptic and invertible on $L^2(M)$. (Recall that T is invertible as an unbounded operator if T is injective and T^{-1} extends to a bounded operator.) This allows us to define our integral kernels—and implicitly also the boundary layer integrals—much as in [33], namely as follows. First, we embed our manifold with boundary and cylindrical ends N into a boundaryless manifold with cylindrical ends M. We then prove that for suitable $V \geqslant 0$, $V \neq 0$, the operator $\Delta_M + V$ is invertible by checking that it is Fredholm of index zero and injective.

The single layer potential integral is defined then as

(2)
$$\mathscr{S}(f) = (\Delta_M + V)^{-1}(f \otimes \delta_{\partial N}),$$

where $f \in L^2(\partial N)$ and $\delta_{\partial N}$ the conditional measure on ∂N (so that $f \otimes \delta_{\partial N}$ defines the distribution $\langle f \otimes \delta_{\partial N}, \varphi \rangle = \int_{\partial N} f\varphi$, where φ is a test function on N). We shall fix in what follows a vector field ν on M that is normal to ∂N at every point of ∂N, has unit length at ∂N and points outside N (recall that N is a submanifold with boundary of M, so ν is a smooth extension of the outer unit normal to the boundary of N). Similarly, the double layer potential integral is defined as

(3)
$$\mathscr{D}(f) = (\Delta_M + V)^{-1}(f \otimes \delta'_{\partial N}),$$

where $f \in L^2(\partial N)$, again, and $\delta'_{\partial N}$ the normal derivative of the measure $\delta_{\partial N}$ in the sense of distributions (so that $\langle f \otimes \delta'_{\partial N}, \varphi \rangle = \int_{\partial N} f\partial_\nu \varphi$, where ∂_ν is the directional derivative in the direction of ν). Since we are dealing with non-compact manifolds (M and ∂N), the above integrals are defined by relying on mapping properties of the operators in $\Psi^m_{\mathrm{ai}}(M)$.

Next, we show that we can make sense of the restriction to ∂N of the kernel E of $(\Delta_M + V)^{-1}$ and that the restricted kernel gives rise to an operator

$$S := [(\Delta_M + V)^{-1}]_{\partial N} \in \Psi^{-1}_{\mathrm{ai}}(\partial N). \tag{4}$$

We can then relate the non-tangential limits of the single and double layer potentials of some function f using the operator S. This is proved by writing $(\Delta_M + V)^{-1}$ as a sum of an operator $P \in \Psi^{-1}_{\mathrm{inv}}(M)$ and an operator $R \in \Psi^{-\infty}_{\mathrm{ai}}(M)$. The existence and properties of the integrals defined by P follow as in the classical case, because P is properly supported (and hence all our relations can be reduced to the analogous relations on a compact manifold). The existence and properties of the integrals defined by R follow from the fact that R is given by a uniformly smooth kernel, albeit not properly supported.

Similarly, we define

$$K := [(\Delta_M + V)^{-1}\partial^*_\nu]_{\partial N}, \tag{5}$$

by restricting the kernel of $(\Delta_M + V)^{-1}\partial^*_\nu$ to ∂N. Let $f_\pm$ be the non-tangential pointwise limits of some function f defined on $M \setminus \partial N$, provided that they exist. More precisely, f_+ is the *interior* non-tangential limit and f_- is the *exterior* non-tangential limit.

Some of the properties of the single and double layer potentials alluded to above are summarized in the following theorem.

Theorem 0.1. *Given $f \in L^2(\partial N)$, we have*

$$\begin{aligned} &\mathscr{S}(f)_+ = \mathscr{S}(f)_- = Sf\,, \quad \partial_\nu \mathscr{S}(f)_\pm = (\pm\tfrac{1}{2}I + K^*)f\,, \quad \text{and} \\ &\mathscr{D}(f)_\pm = (\mp\tfrac{1}{2}I + K)f, \end{aligned} \tag{6}$$

where K^* is the formal transpose of K.

These theorems are proved by reduction to the compact case [33] (using the decomposition $(\Delta_M + V)^{-1} = P + R$ explained above). As in the classical case of a compact manifold with smooth boundary, we obtain the following result.

Theorem 0.2. *Let N be a manifold with boundary and cylindrical ends. Then*

$$H^s(N) \ni u \mapsto (\Delta_N u, u|_{\partial N}) \in H^{s-2}(N) \oplus H^{s-1/2}(\partial N)$$

is a continuous bijection, for any $s > 1/2$.

A possible application of our results on boundary value problems on manifolds with cylindrical ends is to Gauge theory, where manifolds with cylindrical ends are often used. Also, our techniques and results may also be quite relevant for problem arising in computational mathematics, more precisely for obtaining fast algorithms on three dimensional polyhedral domains using the so called "Boundary Element Method" (see [5], [4], [18]).

The reader is referred to [45], [47], or [48] for definitions and background material on pseudodifferential operators. Note that in our paper we work exclusively with manifolds of bounded geometry. Throughout the paper, a classical pseudodifferential operator P will be called *elliptic* if its principal symbols is invertible outside the zero section.

Let us now briefly review the contents of each section (recall that M is a manifold with cylindrical ends). In Section 1, we introduce the algebra of operators $\Psi_{\text{inv}}^{\infty}(M)$ mentioned above and recall the classical characterizations of Fredholm and compact operators in these algebras. Section 2 deals with the same issues for the algebra $\Psi_{\text{ai}}^{\infty}(M)$, which is a slight enlargement of $\Psi_{\text{inv}}^{\infty}(M)$, but has the advantage that it contains the inverses of its elliptic, L^2-invertible elements. We establish several structure theorems for these algebras. In Section 3, we introduce the double and single layer potentials for manifolds with cylindrical ends and prove that some of their basic properties continue to hold in this setting. In Section 4 we study boundary layer potentials depending on a parameter on compact manifolds using a method initially developed by G. Verchota in [50], and we obtain estimates which are uniform in the parameter. These results then allow us to establish the Fredholmness of the operators S and $\pm\frac{1}{2}I + K$ discussed above. Finally, the last section contains a proof of the Theorem 0.2, which is a statement about the well-posedness of the inhomogeneous Dirichlet problem. This allows us to define and study the Dirichlet-to-Neumann map in the same section.

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1. Operators on manifolds with cylindrical ends

We begin by introducing the class of manifolds with cylindrical ends (without boundary) and by reviewing some of the results on the analysis on these manifolds that are needed in this paper. Here we follow the standard approach [22], [25], [28], [42] to manifolds with cylindrical ends. For simplicity, we shall usually drop the subscript M in the notation for the Laplacian $\Delta_M := d^*d$ on M. Note that in this paper we use the *positive* Laplace operator.

1.1. Manifolds with cylindrical ends and the Laplace operator. Let M_1 be a compact manifold with boundary $\partial M_1 \neq \emptyset$. We assume that a metric g is given on M_1 and that g_1 is a product metric in a tubular neighborhood $V \cong \partial M_1 \times [0,1)$ of the boundary, namely

$$g_1 = g_\partial + (\mathrm{d}x)^2, \tag{7}$$

where $x \in [0,1)$ is the second coordinate in $\partial M_1 \times [0,1)$ and g_∂ is a metric on ∂M_1. Let

$$M := M_1 \cup (\partial M_1 \times (-\infty, 0]), \quad \partial M_1 \equiv \partial M_1 \times \{0\}, \tag{8}$$

be the union of M_1 and $\partial M_1 \times (-\infty, 0]$ along their boundaries. The above decomposition will be called a *standard decomposition* of M. The resulting manifold M is called a *manifold with cylindrical ends.* Note that a manifold with cylindrical ends is a complete, non-compact, Riemannian manifold without boundary.

Let $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ be a manifold with cylindrical ends. Let g be the metric on M and assume, as above, that $g = g_\partial + (\mathrm{d}x)^2$ on the cylindrical end $\partial M_1 \times (-\infty, 0]$, where $x \in (-\infty, 0]$ and g_∂ is a metric on ∂M_1, the boundary of M_1. Let d be the exterior derivative operator on M so that $\Delta = \Delta_M = d^*d$ becomes the (scalar) Laplace operator on M. Also, let $\Delta_{\partial M_1}$ be the Laplace operator on ∂M_1, defined using the metric g_∂. Then

$$\Delta = \Delta_M = -\partial_x^2 + \Delta_{\partial M_1} \tag{9}$$

on the cylindrical end $\partial M_1 \times (-\infty, 0]$.

1.2. Operators that are translation invariant in a neighborhood of infinity. Let $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ be a manifold with cylindrical ends, as above, and let, for any $s \geqslant 0$,

$$\varphi_s\colon\ \partial M_1 \times (-\infty, 0] \to \partial M_1 \times (-\infty, -s] \tag{10}$$

be the isometry given by translation with $-s$ in the x-direction. If $s<0$, then φ_s is defined as the inverse of φ_{-s}. The special form of the operator Δ obtained at the end of the previous subsection suggests the following definition.

Definition 1.1. A continuous linear map $P\colon \mathscr{C}_c^\infty(M)\to\mathscr{C}^\infty(M)$ will be called *translation invariant in a neighborhood of infinity* if its Schwartz kernel has support in

$$V_\varepsilon := \{(x,y)\in M^2,\, \mathrm{dist}(x,y)<\varepsilon\},$$

for some $\varepsilon>0$, and there exists $R>0$ such that $P\varphi_s(f)=\varphi_s P(f)$, for any $f\in\mathscr{C}_c^\infty(\partial M_1\times(-\infty,-R))$ and any $s>0$.

We shall denote by $\Psi^m_{\mathrm{inv}}(M)$ the space of *order m, classical pseudodifferential operators on M that are translation invariant in a neighborhood of infinity.*

As usual, we shall denote by $\Psi^{-\infty}_{\mathrm{inv}}(M)=\bigcap\limits_m \Psi^m_{\mathrm{inv}}(M)$, $\Psi^{\infty}_{\mathrm{inv}}(M)=\bigcup\limits_m \Psi^m_{\mathrm{inv}}(M)$, $m\in\mathbb{Z}$.

We have the following simple lemma.

Lemma 1.2. *Every $R\in\Psi^{-n-1}_{\mathrm{inv}}(M)$, where n is the dimension of M, induces a bounded operator on $L^2(M)$.*

Proof. The classical argument applies. Namely, R is defined by a continuous kernel K. The support condition on K and the translation invariance at infinity then give

$$\int_M |K(x,y)|\,\mathrm{d}x,\ \int_M |K(x,y)|\,\mathrm{d}y\leqslant C$$

for some $C>0$ that is independent of x or y. This proves that R is bounded on $L^2(M)$, via Schur's lemma. □

We shall denote by $\mathscr{D}(T)$ the domain of a possibly unbounded operator T. Recall that an unbounded operator $T\colon\mathscr{D}(T)\to X$ defined on a subset of a Banach space Y and with values in another Banach space X is *Fredholm* if T is *Fredholm* as a bounded operator from its domain $\mathscr{D}(T)$ endowed with the graph norm. Equivalently, T is Fredholm if it is closed and has finite dimensional kernel and cokernel. Also, T is called *invertible* if T is invertible as an operator $\mathscr{D}(T)\to X$. For all differential operators considered below, we shall consider the minimal closed extension, that is, the closure of the operators with domain compactly supported smooth functions.

For each nonnegative, even integer $m\in 2\mathbb{N}$ we shall denote by $H^m(M)$ the domain of the operator $(I+\Delta)^{m/2}$ ($\Delta=\Delta_M$), regarded as an unbounded operator on $L^2(M)$:

$$H^m(M):=\mathscr{D}((I+\Delta)^{m/2}). \qquad (11)$$

We endow $H^m(M)$ with the norm

$$\|u\|_m = \|(I+\Delta)^{m/2}u\|_{L^2}$$

(Below, we shall occasionally write $\|\cdot\|$ instead of $\|\cdot\|_{L^2}$.) Note that $I+\Delta \geqslant I$, and hence

$$\|u\|_m \geqslant \|u\|.$$

(Recall that $m \geqslant 0$.)

As usual, we shall denote by $H^{-m}(M)$ the dual of $H^m(M)$, via a duality pairing that extends the pairing between functions and distributions. We thus identify $H^{-m}(M)$ with a space of distributions on M.

Throughout this paper, we shall denote by T^*M the *cotangent bundle* of M. Also, for any vector bundle $E \to M$, we shall denote by $S^m(E)$ the space of symbols of type $(1,0)$ introduced by Hörmander. Recall that, in local coordinates above an open subset $U \subset M$, we have $a \in S^m(U \times \mathbb{R}^n)$ if, and only if, $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{K,l}(1+|\xi|)^{m-|\beta|}$ for all $x \in K \Subset U$, and all multi-indices α, β with $|\alpha|,|\beta| \leqslant l$. Let $\sigma_m(P) \in S^m(T^*M)/S^{m-1}(T^*M)$ be the principal symbol of an operator $P \in \Psi^m_{\mathrm{inv}}(M)$. See [45], [47], or [48]. The following lemma is a standard result on pseudodifferential operators. More general results can be found in [2], [28].

Lemma 1.3. *Let M be a manifold with cylindrical ends and $P \in \Psi^m_{\mathrm{inv}}(M)$ (so P is an order m pseudodifferential operator that is translation invariant in a neighborhood of infinity).*

(i) *For any s, s', we have $\Psi^s_{\mathrm{inv}}(M)\Psi^{s'}_{\mathrm{inv}}(M) \subset \Psi^{s+s'}_{\mathrm{inv}}(M)$ and the principal symbol*

$$\sigma_s\colon \Psi^s_{\mathrm{inv}}(M)/\Psi^{s-1}_{\mathrm{inv}}(M) \to S^s(T^*M)/S^{s-1}(T^*M)$$

induces an isomorphism onto the subspace of symbols that are translation invariant in a neighborhood of infinity.

(ii) *Any $P \in \Psi^m_{\mathrm{inv}}(M)$ extends to a continuous map $P\colon H^{m'}(M) \to H^{m'-m}(M)$, if $m, m' \in 2\mathbb{Z}$.*

Proof. We include a sketch of the proof for the benefit of the reader. See [2], [3] or [1] for details, where more general results were proved.

(i) follows from the analogous statement for pseudodifferential operators on noncompact manifolds.

To prove (ii) when $m = 0$, we use the symbolic calculus, Lemma 1.2 and Hörmander's trick. For $m' \geqslant m \geqslant 0$, use the fact that $(I+\Delta)^k\colon H^{2k}(M) \to L^2(M)$ is an isomorphism and write

$$P = (I+\Delta)^i Q (I+\Delta)^j + R$$

for suitable $Q, R \in \Psi^0_{\mathrm{inv}}(M)$ and $i + j = m/2$. The other cases are similar. □

Let us now recall a classical and well known construction, see for example [28]. Any operator $P\colon \mathscr{C}_c^\infty(M) \to \mathscr{C}^\infty(M)$ that is translation invariant in a neighborhood of infinity will be properly supported (that is, $P(\mathscr{C}_c^\infty(M)) \subset \mathscr{C}_c^\infty(M)$). Let s be a real number and let φ_s be the translation by s on the cylinder $\partial M_1 \times \mathbb{R}$. Then, for any $f \in \mathscr{C}_c^\infty(\partial M_1 \times (-\infty, 0)) \subset \mathscr{C}_c^\infty(M)$, the function

$$\tilde{P}(f) := \varphi_{-s} P \varphi_s(f) \in \mathscr{C}_c^\infty(\partial M_1 \times (-\infty, 0)) \tag{12}$$

is independent of s, provided that s is large enough. This allows us to define an operator $\tilde{P}\colon \mathscr{C}_c^\infty(\partial M_1 \times \mathbb{R}) \to \mathscr{C}_c^\infty(\partial M_1 \times \mathbb{R})$ by choosing for any $f \in \mathscr{C}_c^\infty(\partial M_1 \times \mathbb{R})$ a large enough s so that

$$\operatorname{supp}(P\varphi_s(f)),\ \operatorname{supp}(\varphi_s(f)) \subset \partial M_1 \times (-\infty, 0) \subset M$$

(this is needed to make sure that $\varphi_{-s} P \varphi_s(f)$ is defined) and so that $\varphi_{-s} P \varphi_s(f)$ is independent of s.

Definition 1.4. The operator $\tilde{P}$ will be called *the indicial operator associated with P*. The resulting map

$$\Phi\colon\ \Psi_{\text{inv}}^\infty(M) \ni P \mapsto \tilde{P} \in \Psi^\infty(\partial M_1 \times \mathbb{R})$$

will be called the *indicial morphism*.

Let us notice now that $\partial M_1 \times \mathbb{R}$ is also a manifold with cylindrical ends. The partially defined action of $\mathbb{R}$ on the ends of M extends to a global action of $\mathbb{R}$ on $\partial M_1 \times \mathbb{R}$. We shall denote by $\Psi_{\text{inv}}^{-\infty}(M_1 \times \mathbb{R})^{\mathbb{R}}$ the operators in $\Psi_{\text{inv}}^{-\infty}(M_1 \times \mathbb{R})$ that are invariant with respect to the action of $\mathbb{R}$ by translations. Let $T \in \Psi_{\text{inv}}^{-\infty}(M_1 \times \mathbb{R})^{\mathbb{R}}$ and η be a smooth function on $\mathbb{R} \times \partial M_1$ with support in $(-\infty, -1) \times \partial M_1$, equal to 1 in a neighborhood of infinity. Then

$$s_0(T) := \eta T \eta \tag{13}$$

defines an operator in $\Psi_{\text{inv}}^{-\infty}(M)$.

Lemma 1.5. *Let s_0 be as in Equation* (13)*. Then $\Phi(s_0(T)) = T$ for all $T \in \Psi_{\text{inv}}^\infty(\partial M_1 \times \mathbb{R})^{\mathbb{R}}$. In particular, the range of the indicial morphism Φ of Definition* 1.4 *is $\Psi_{\text{inv}}^\infty(\partial M_1 \times \mathbb{R})^{\mathbb{R}}$.*

Proof. This is a direct consequence of the definition. □

In order to deal with operators acting on weighted Sobolev spaces, we shall need the following lemma. (See also [28].) Let us denote by $[A, B] := AB - BA$ the commutator of two linear maps A and B.

Lemma 1.6. *Let $P, P_1 \in \Psi^\infty_{\text{inv}}(M)$ be arbitrary and $\varrho\colon\ M \to (-\infty, 0)$ be a smooth function such that $\varrho(y, x) = x$ on a neighborhood of infinity in $\partial M_1 \times (-\infty, 0]$. Then*

(i) $\tilde{Q} = \tilde{P}\widetilde{P_1}$, *if* $Q = PP_1$.

(ii) $\operatorname{ad}_\varrho(P) := [\varrho, P] \in \Psi^\infty_{\text{inv}}(M)$.

P r o o f. Let f be a function with compact support in $\partial M_1 \times (-\infty, 0)$. We have that $\varphi_s PP_1\varphi_{-s} = \varphi_s P\varphi_{-s}\varphi_s P_1\varphi_{-s}$, so the relation (i) follows from the definition of the indicial operator (Definition 1.4).

To prove (ii), we only need to check that $\operatorname{ad}_\varrho(P)$ is translation invariant in a neighborhood of ∞. Since this is checked on a set of the form $\partial M_1 \times (-\infty, 0)$, we can assume that $M = X \times \mathbb{R}$. Let then φ_s, $s \in \mathbb{R}$, be translation by s along $\mathbb{R}$, namely $\varphi_s(x, y) = (x + s, y)$, as before. Let $\varphi_s(f) = f \circ \varphi_s$. We can assume that P is translation invariant, in the sense that $\varphi^*_s(P) := \varphi_s \circ P \circ \varphi_{-s}$, for any $s > 0$. Then

$$\varphi^*_s([x, P]) = [\varphi^*_s(x), \varphi^*_s(P)] = [x + s, P] = [x, P]. \tag{14}$$

Thus $[x, P]$ is also $\mathbb{R}$-invariant.

In general, $\varrho = x$ in a neighborhood of $-\infty$, so the result follows. □

The properties of the indicial operators $\tilde{P}$ are conveniently studied in terms of *indicial families*. Indeed, by considering the Fourier transform in the $\mathbb{R}$ variable, we obtain by Plancherel's theorem an isometric bijection (that is, a unitary operator) defined, using local coordinates y on ∂M_1, by

$$\mathscr{F}\colon\ L^2(\partial M_1 \times \mathbb{R}) \to L^2(\partial M_1 \times \mathbb{R}), \quad \mathscr{F}(f)(y, \tau) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\tau x} f(y, x)\,\mathrm{d}x. \tag{15}$$

Hereafter, $\mathrm{i} := \sqrt{-1}$.

Because $\tilde{P}$ is translation invariant with respect to the action of $\mathbb{R}$, the resulting operator $P_1 := \mathscr{F}\tilde{P}\mathscr{F}^{-1}$ will commute with the multiplication operators in τ, and hence it is a decomposable operator, in the sense that there exist (possibly unbounded) operators $\hat{P}(\tau)$ acting on $\mathscr{C}^\infty(\partial M_1) \subset L^2(\partial M_1)$ such that

$$(P_1 f)(\tau) = \hat{P}(\tau)f(\tau), \quad f(\tau) = f(\cdot, \tau) \in \mathscr{C}^\infty(\partial M_1).$$

In other words,

$$\big[(\mathscr{F}\tilde{P}\mathscr{F}^{-1}f)\big](\tau) = \hat{P}(\tau)f(\tau). \tag{16}$$

Using local coordinates, it is not hard to see that the operators $\hat{P}(\tau)$ are classical pseudodifferential operators and that the map $\tau \mapsto \hat{P}(\tau)f$ is $\mathscr{C}^\infty$ for any $f \in \mathscr{C}^\infty(\partial M_1)$.

One also has $\tilde{P}(\mathrm{e}^{\mathrm{i}\tau x} g) = \mathrm{e}^{\mathrm{i}\tau x} \hat{P}(\tau) g$, for any $g \in L^2(\partial M_1)$. Let $K_{\tilde{P}}$ be the distribution kernel of $\tilde{P}$. Then

$$K_{\tilde{P}}(x_1, x_2, y_1, y_2) = k_{\tilde{P}}(x_1 - x_2, y_1, y_2), \tag{17}$$

for some distribution $k_{\tilde{P}}$ on $\mathbb{R} \times (\partial M_1)^2$. This allows us to write the distribution kernel of $\hat{P}(\tau)$ as

$$K_{\hat{P}(\tau)}(y_1, y_2) = \int_{\mathbb{R}} k(x, y_1, y_2)\, \mathrm{e}^{-\mathrm{i}t\tau}\, \mathrm{d}x. \tag{18}$$

Let $Q = [\varrho, P]$. Then

$$k_{\tilde{Q}} = \mathrm{i} \frac{\partial}{\partial \tau} \hat{P}(\tau). \tag{19}$$

See [25], [28] and the references therein.

2. A spectrally invariant algebra

A serious drawback of the algebra $\Psi^{\infty}_{\mathrm{inv}}(M)$ is that it is not "spectrally invariant," in the sense that the inverse of an elliptic operator $P \in \Psi^{\infty}_{\mathrm{inv}}(M)$ that is invertible on L^2 is not necessarily in this algebra (Definition 2.7 below). In this section we slightly enlarge the algebra $\Psi^{\infty}_{\mathrm{inv}}(M)$ so that it becomes spectrally invariant. This will lead us to an algebra of operators that are "almost translation invariant in a neighborhood of infinity."

2.1. Operators that are almost translation invariant in a neighborhood of infinity. We begin by introducing another algebra of pseudodifferential operators that will be indispensable also later on. Let $\varrho\colon M \to (-\infty, 0)$ be a smooth function such that $\varrho(y, x) = x$ for $(y, x) \in \partial M_1 \times (-\infty, -1)$, as in Lemma 1.6. Recall that $\mathrm{ad}_\varrho(T) := [\varrho, T] = \varrho T - T\varrho$. Assume $T\colon \mathscr{C}^\infty_c(M) \to \mathscr{C}^\infty(M)$ to be a linear map with the property that

$$\mathrm{ad}^k_\varrho(T) := [\varrho, [\varrho, \ldots, [\varrho, T] \ldots]]$$

extends to a continuous map $\mathrm{ad}^k_\varrho(T)\colon H^{-m}(M) \to H^m(M)$, for any $m \in 2\mathbb{Z}_+$. Let $\|T\|_{k,m}$ denote the norm of the resulting operator $\mathrm{ad}^k_\varrho(T)$. Recall the section s_0 defined in Equation (13).

We define $\Psi^{-\infty}_{\mathrm{ai}}(M)$ to be the closure of $\Psi^{-\infty}_{\mathrm{inv}}(M)$ with respect to the countable family of semi-norms

$$T \to \|T\|_{k,m}\,, \quad \text{and} \quad T \to \|\varrho^l (T - s_0(\Phi(T)))\varrho^l\|_{0,m}. \tag{20}$$

where $k, m/2, l \in \mathbb{Z}_+$. Then $\Psi_{\mathrm{ai}}^{-\infty}(M)$ is a Fréchet algebra (that is, a Fréchet space endowed with an algebra structure such that the multiplication is continuous).

Finally, we define

$$\Psi_{\mathrm{ai}}^{m}(M) := \Psi_{\mathrm{inv}}^{m}(M) + \Psi_{\mathrm{ai}}^{-\infty}(M). \tag{21}$$

An element $P \in \Psi_{\mathrm{ai}}^{m}(M)$ will be called *almost translation invariant in a neighborhood of infinity.*

It is interesting to observe now that we can introduce dependence on ϱ at infinity (thus obtaining variants of Melrose's b-calculus, see [28] and [24]). This is done by noticing that for any $P \in \Psi_{\mathrm{inv}}^{m}(M)$ and any $N \in \mathbb{N}$ there exists a bounded operator $R_N\colon\ H^{-k}(M) \to H^{k}(M)$, where $2k \leqslant m - N$, such that

$$(-\varrho)^{-a} P (-\varrho)^{a} - \sum_{j=0}^{N-1} (-\varrho)^{-j} \binom{a}{j} \mathrm{ad}_{\varrho}^{j}(P) = \varrho^{N/2} R_N \varrho^{N/2}. \tag{22}$$

(Above, $\binom{a}{j} = a(a-1)\dots(a-j+1)/j!$ stand for the usual "binomial" coefficients.)

We now define the fractionary Sobolev spaces. Let $s \geqslant 0$ and choose $P_s \in \Psi_{\mathrm{ai}}^{s}(M)$ to be elliptic and to satisfy $P_s \geqslant 1$. We shall denote by $H^s(M)$ the domain of (the closure of) P_s, regarded as an unbounded operator on $L^2(M)$:

$$H^s(M) := \mathscr{D}(\overline{P}_s). \tag{23}$$

This definition is independent of our particular choice of P_s because, if P_s' is another such selection, we can choose $Q \in \Psi_{\mathrm{ai}}^{0}(M)$ and $R \in \Psi_{\mathrm{ai}}^{-\infty}(M)$ such that

$$P_s' = Q P_s + R. \tag{24}$$

Thus, if $\xi \in \mathscr{D}(P_s)$, then there exists a sequence $\xi_n \in \mathscr{C}_c^\infty(M)$, $\xi_n \to \xi$ in $L^2(M)$, such that $P_s \xi_n$ converges in $L^2(M)$. But then $P_s'(\xi_n) = Q(P_s \xi_n) + R\xi_n$ also converges, because Q and R are continuous. See also [24].

We endow $H^s(M)$ with the norm $\|f\|_s := \|P_s f\|_{L^2(M)}$. (Using a quantization map from symbols to pseudodifferential operators, we can assume that $\|f\|_s$ depends analytically on s.) For $s < 0$, $H^s(M)$ is the dual of $H^{-s}(M)$, regarded as a space of distributions on M. The subspace $\mathscr{C}_c^\infty(M) \subset H^s(M)$ is dense. See [1] for more results on Sobolev spaces on manifolds with a Lie structure at infinity, a class of manifolds that includes the class of manifolds with cylindrical ends. For example, $H^s(M)$ can be identified with the domain of $(I + \Delta)^{s/2}$.

We shall also consider *weighted Sobolev spaces* as follows. Let $\varrho\colon\ M \to (-\infty, 0)$, $\varepsilon > 0$, be a smooth functions such that $\varrho(y, x) = x$, for $(y, x) \in \partial M_1 \times (-\infty, -R]$

with R large enough, as before. Then we shall denote by $\varrho^a H^s(M)$ the space of distributions of the form $\varrho^a u$, with $u \in H^s(M)$. We endow $\varrho^a H^s(M)$ with the norm

$$\|f\|_{s,a} := \|\varrho^{-a} f\|_s.$$

We have then the following classical results about almost translation invariant pseudodifferential operators on the manifold with cylindrical ends M [30]. (See [22], [24], [31], [40], [42], [44].) These results generalize the corresponding even more classical results on pseudodifferential operators on compact manifolds.

Theorem 2.1. *Let M be a manifold with cylindrical ends and $P \in \Psi^m_{\mathrm{ai}}(M)$ (so P is an order m pseudodifferential operator that is almost translation invariant in a neighborhood of infinity). Also, let $\varrho > 0$, $\varrho(y,x) = -x$ on a neighborhood of infinity in $\partial M_1 \times (-\infty, 0]$. Let $s, a \in \mathbb{R}$ be arbitrary, but fixed. Then:*

(i) *P extends to a continuous operator $P\colon\ \varrho^a H^s(M) \to \varrho^a H^{s-m}(M)$.*

(ii) *$P\colon\ \varrho^a H^s(M) \to \varrho^{a'} H^{s-m'}(M)$ is compact for any $a' < a$ and $m' > m$.*

(iii) *$P\colon\ \varrho^a H^s(M) \to \varrho^a H^{s-m}(M)$ is compact $\Leftrightarrow$ $\sigma_m(P) = 0$ and $\tilde{P} = 0$.*

(iv) *$P\colon\ \varrho^a H^s(M) \to \varrho^a H^{s-m}(M)$ is Fredholm $\Leftrightarrow$ $\sigma_m(P)$ is invertible and the operator $\tilde{P}\colon\ H^s(\partial M_1 \times \mathbb{R}) \to H^{s-m}(\partial M_1 \times \mathbb{R})$ is an isomorphism.*

Proof. This theorem follows for example from the results in [28], or the older preprint [30]. □

A far reaching program for generalizing the above result to other classes of non-compact manifolds is contained in Melrose's "small red book" [29]. See also [37]. Also, see [43] for an extension of the above results to L^p-spaces, and [7] for some applications to non-linear evolution equations.

As a consequence, we obtain the following result. Recall [39], [38], [47] that if $T\colon\ \mathscr{D}(T) \subset H \to H$ is an unbounded operator on a Hilbert space H, then its adjoint $T^*\colon\ \mathscr{D}(T^*) \subset H \to H$ is defined by its domain $\mathscr{D}(T^*) = \{y \in H,\ \exists z \in H,\ (Tx, y) = (x, z),\ \forall x \in \mathscr{D}(T)\}$ and $T^*(y) = z$. An unbounded operator on a Hilbert space H is called *symmetric* if $(Tx, y) = (x, Ty)$ for all $x, y \in \mathscr{D}(T)$. It is called *self-adjoint* if $T^* = T$. Note that $T = \mathrm{i}\partial_x\colon\ \mathscr{C}^\infty_c(I) \to L^2(I)$, $I = (0, \infty)$, is an example of a symmetric operator that has no self-adjoint extensions. The following lemma shows that this cannot happen if $T \in \Psi^m_{\mathrm{ai}}(M)$. Recall that the closure of an operator T (if there is one) is the operator whose graph is the closure of the graph of T

Corollary 2.2. *Let $P \in \Psi_{\mathrm{ai}}^m(M)$, $m > 0$, be elliptic. If $P\colon \mathscr{C}_c^\infty(M) \to L^2(M)$ satisfies $(Pf, g) = (f, Pg)$, for $f, g \in \mathscr{C}_c^\infty(M)$, then the closure of P is an unbounded self-adjoint operator on $L^2(M)$.*

Proof. We replace P by its closure first. It is enough [39], [38], [47] to prove that $P \pm \mathrm{i}I$ is invertible. Denote the inner product on $L^2(M)$ by $\langle\,\cdot\,,\,\cdot\,\rangle$. Then $\langle (P \pm \mathrm{i}I)\xi, \xi\rangle = \|P\xi\|^2 + \|\xi\|^2$, for any ξ in the domain of P, and hence $P \pm \mathrm{i}I$ is injective and has closed range. We shall work in the space of continuous (i.e., bounded) maps $L^2(M) \to L^2(M)$, endowed with the topology defined by the norm.

Let us prove that the range of $P \pm \mathrm{i}I$ is dense. We deal only with $P + \mathrm{i}I$, because the other case is completely similar. Assume the range of $P \pm \mathrm{i}I$ is not dense, then there exists $\eta \in L^2(M)$ such that

$$\langle (P + \mathrm{i}I)\xi, \eta\rangle = 0$$

for all $\xi \in \mathscr{C}_c^\infty(M)$. Then $(P - \mathrm{i}I)\eta = 0$ in the sense of distributions. Select $Q \in \Psi_{\mathrm{ai}}^{-m}(M)$ such that $Q(P - \mathrm{i}I) = I - R$, where $R \in \Psi_{\mathrm{ai}}^{-\infty}(M)$. Then $\eta = R\eta$. Choose $\eta_n \in \mathscr{C}_c^\infty(M)$, $\eta_n \to \eta$ in $L^2(M)$. By the definition of $\Psi_{\mathrm{ai}}^{-\infty}(M)$, we can find operators $R_n \in \Psi_{\mathrm{inv}}^{-\infty}(M)$ such that

$$\|R - R_n\|_{0,m'} := \|(I + \Delta)^{m'/2}(R - R_n)(I + \Delta)^{m'/2}\| \to 0,$$

for $m' \geqslant m$. Then $\xi_n := R_n\eta_n \to \eta$, as well, and $\xi_n \in \mathscr{C}_c^\infty(M)$. Moreover,

$$(P - \mathrm{i}I)\xi_n = (P - \mathrm{i}I)R_n\eta_n \to (P - \mathrm{i}I)R\eta,$$

because the operators $(P - \mathrm{i}I)R_n$ are continuous on $L^2(M)$ and converge in norm to $(P - \mathrm{i}I)R \in \Psi_{\mathrm{ai}}^{-\infty}(M)$. This proves that η is in the domain of the closure of P, which is a contradiction, since we have already seen that $P - \mathrm{i}I$ is injective. $\square$

We now investigate the structure of the ideals of the algebras $\Psi_{\mathrm{inv}}^{-\infty}(M)$ and, most importantly, $\Psi_{\mathrm{ai}}^{-\infty}(M)$. For any compact manifold X, we shall denote by $\mathscr{S}(\mathbb{R}^k \times X)$ the space of Schwartz functions on $\mathbb{R}^k \times X$.

Lemma 2.3. *The range of the map*

$$\Phi\colon\ \Psi_{\mathrm{ai}}^{-\infty}(M) \ni P \mapsto \tilde{P} \in \Psi^{-\infty}(\partial M_1 \times \mathbb{R}) \tag{25}$$

identifies with $\mathscr{S}(\mathbb{R} \times (\partial M_1)^2)$, via the map χ that sends the kernel $K(t_1, t_2, y_1, y_2) \in \mathscr{C}^\infty(\mathbb{R}^2 \times (\partial M_1)^2)$ of $\tilde{P}$ to the function $k(t, y_1, y_2) = K(t, 0, y_1, y_2) \in \mathscr{S}(\mathbb{R} \times (\partial M_1)^2)$.

In particular, $\Phi(\Psi_{\mathrm{ai}}^{-\infty}(M)) = \Psi_{\mathrm{ai}}^{-\infty}(\partial M_1 \times \mathbb{R})^{\mathbb{R}}$.

Proof. The indicial map

$$\Phi\colon\ \Psi_{\text{inv}}^{-\infty}(M) \to \Psi_{\text{inv}}^{-\infty}(\partial M_1 \times \mathbb{R})$$

of Definition 1.4 is by definition continuous. It is also surjective by Proposition 1.5. It has a canonical continuous section s_0, which associates to $T \in \Psi_{\text{inv}}^{\infty}(M_1 \times \mathbb{R})^{\mathbb{R}}$ the operator $s_0(T) := \eta T \eta$, where η is a smooth function on $\mathbb{R} \times \partial M_1$ and with support in $(-\infty, -1) \times \partial M_1$, and equal to 1 in a neighborhood of infinity (cf. Equation (13)).

Moreover, s_0 sends properly supported operators to $\Psi_{\text{inv}}^{-\infty}(M)$. This shows that

$$\Psi_{\text{inv}}^{-\infty}(M) \cong \ker(\Phi) \oplus s_0(\Psi_{\text{inv}}^{\infty}(M_1 \times \mathbb{R}))^{\mathbb{R}},$$

as Fréchet spaces. We also see that the quotient seminorms defined by the seminorms of Equation (20) on the range of Φ are the same as the seminorms defining the topology on $\mathscr{S}(\mathbb{R}\times(\partial M_1)^2)$. Since $\mathscr{S}(\mathbb{R}\times(\partial M_1)^2)$ is the closure of $\chi(\Psi_{\text{inv}}^{-\infty}(M_1\times\mathbb{R}))$, the result follows. $\square$

The same proof as above also gives the following result.

Corollary 2.4. *Let Y be a compact, smooth manifold without boundary. Then the algebra $\Psi_{ai}^{-\infty}(Y \times \mathbb{R})^{\mathbb{R}}$ is the space of operators T on $L^2(Y \times \mathbb{R})$ such that $(I + \Delta_{Y\times\mathbb{R}})^m \mathrm{ad}_{\varrho}^k(T)(I + \Delta_{Y\times\mathbb{R}})^m$ is continuous on $L^2(M)$ and $\mathbb{R}$-invariant for any $m, k \in \mathbb{Z}_+$. The resulting family of seminorms is the family of seminorms of Equation* (20) *defining the topology on $\Psi_{ai}^{-\infty}(Y \times \mathbb{R})^{\mathbb{R}}$.*

Let $\mathfrak{I}$ be the kernel of the map $\Phi\colon\ \Psi_{\text{ai}}^{-\infty}(M) \to \Psi^{-\infty}(\partial M_1 \times \mathbb{R})$ of Definition (25). It is an ideal of $\Psi_{\text{ai}}^{\infty}(M)$ (i.e., $\Psi_{\text{ai}}^{\infty}(M)\mathfrak{I} = \mathfrak{I}\Psi_{\text{ai}}^{\infty}(M)$). We also have the following description of $\mathfrak{I}$ that is similar in spirit to Corollary 2.4.

Lemma 2.5. *The space $\mathfrak{I}$ is the space of all continuous operators T on $L^2(M)$ such that $(I + \Delta)^m \varrho^l T \varrho^l (I + \Delta)^m$ is bounded on $L^2(M)$ for any $m, l \in \mathbb{Z}_+$. The resulting family of seminorms is the family of seminorms of Equation* (20) *defining the topology on $\mathfrak{I}$.*

Proof. It is clear from the definition that

$$T \to \|(I + \Delta)^m \varrho^l T \varrho^l (I + \Delta)^m\|$$

is one of the seminorms of Equation (20), namely $\|\cdot\|_{l,m}$.

Conversely, let T be an operator on $L^2(M)$ such that for each $m, l \in \mathbb{Z}_+$ the operator $(I+\Delta)^m \varrho^l T \varrho^l (I+\Delta)^m$ is bounded. The family of seminorms $T \to \|\varrho^l (I+\Delta)^m T (I+\Delta)^m \varrho^l\|$ is equivalent to the family $\|\cdot\|_{l,m}$. We shall use this family instead.

The Schwartz kernel of T is $K_T(x,y) = \langle T\delta_y, \delta_x\rangle$ and it satisfies

$$\varrho^l(x)\varrho^l(y)|K_T(x,y)| \leqslant C^2\|\varrho^l(I+\Delta)^m T(I+\Delta)^m\varrho^l\| \tag{26}$$

where $C \geqslant \|\delta_x\|_{-m}$, uniformly in $x \in M$, for some $m > n/2$. (We have used here the Sobolev embedding theorem for manifolds with cylindrical ends [1].)

We shall prove now that T is in the closure of $\ker \Phi \subset \Psi^{-\infty}_{\rm inv}(M)$ (recall that $\Phi(T) = \tilde{T}$ is the indicial map). Let $\alpha_n = 1-\varphi_n(\eta) \in \mathscr{C}^\infty_c(M)$, where φ_n is translation by $-n$ on the cylindrical end, and $\eta \in \mathscr{C}^\infty(M)$ is equal to 1 in a neighborhood of infinity and is supported on $\partial M_1 \times (-\infty, 0]$, if

$$M = M_1 \cup \partial M_1 \times (-\infty, 0]$$

is a standard decomposition of M.

We have that $T_n := \alpha_n T \alpha_n$ has the compactly supported Schwartz kernel

$$K_{T_n}(x,y) = \alpha_n(x)K_T(x,y)\alpha_n(y).$$

Taking $l > 1$ in the Equation (26), we see using Shur's lemma (as in the proof of Lemma 1.2) that $\|T_n - T\| \to 0$ (the norm here is that of bounded operators on $L^2(M)$). The proof that $\|T_n - T\|_{l,m} \to 0$ for $l > 0$ or $m > 0$ is completely similar.

□

Let $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ be a standard decomposition of M (so M_1 is a smooth, compact manifold with smooth boundary). We consider a tubular neighborhood $\partial M_1 \times [0,1) \subset M_1$ of the boundary ∂M_1 of M_1. Consider, as before, a diffeomorphism ψ from M to the interior of M_1 that coincides with $(y,t) \mapsto (y, -t^{-1}) \in \partial M_1 \times (0,1) \subset M_1$ in a neighborhood of infinity. This diffeomorphism can be assumed to be the identity outside the tubular neighborhood $\partial M_1 \times [0,1)$ and to correspond to a diffeomorphism of $(-\infty, 0]$ onto $(0, 1/2]$.

Corollary 2.6. *The diffeomorphism ψ above identifies $\mathfrak{I}$ with $\mathscr{C}^\infty_0(M_1^2)$, that is, the space of smooth functions on M_1^2 that vanish to infinite order at the boundary $\partial(M_1^2) = (\partial M_1 \times M_1) \cup (M_1 \times \partial M_1)$.*

Proof. This follows right away from the proof of Lemma 2.5. □

To formulate the following results, it is convenient to use the following classical concept (see [41], for example). Let $\mathscr{L}(\mathscr{H})$ denote the algebra of bounded operators on some Hilbert space $\mathscr{H}$.

Definition 2.7. Let $\mathscr{A}$ be an algebra of bounded operators on some Hilbert space $\mathscr{H}$ (i.e., $\mathscr{A} \subset \mathscr{L}(\mathscr{H})$). We say that $\mathscr{A}$ is *spectrally invariant* if, and only if, $(I+T)^{-1} \in I + \mathscr{A}$, for any $T \in \mathscr{A}$ such that $I+T$ is invertible as an operator on $\mathscr{H}$.

Lemma 2.8. *The algebras $\mathfrak{I} \subset \mathscr{L}(L^2(M))$ and the algebra $\Psi_{\text{ai}}^{-\infty}(\partial M_1 \times \mathbb{R})^{\mathbb{R}} \subset \mathscr{L}(L^2(\partial M_1 \times \mathbb{R}))$ are spectrally invariant.*

P r o o f. Both are well known results (see [24] or [42] and the references therein). An easy proof is obtained using Lemma 2.5 or, respectively, Corollary 2.4. □

The property of being spectrally invariant is preserved under extensions of algebras (see [24]). Using this twice, we obtain the following result.

Corollary 2.9. *The algebras $\Psi_{\text{ai}}^{-\infty}(M) \subset \mathscr{L}(L^2(M))$ and $\Psi_{\text{ai}}^{0}(M) \subset \mathscr{L}(L^2(M))$ are spectrally invariant.*

A proof of this corollary is also contained in the following theorem, which is the main result of this section. It states that $\Psi_{\text{ai}}^{\infty}(M)$ is, in a certain sense, also spectrally invariant, its proof does not rely on the above corollary.

Theorem 2.10. *Let $T \in \Psi_{\text{ai}}^{m}(M)$, $m \geqslant 0$, be such that T is invertible as a (possibly unbounded) operator on $L^2(M)$. If $m > 0$, we assume also that T is elliptic. Then $T^{-1} \in \Psi_{\text{ai}}^{-m}(M)$.*

P r o o f. Note that for $m = 0$, it is a consequence of the invertibility that T must again be elliptic, as in the case $m > 0$.

Let Q_1 be a parametrix of T, namely, $Q_1 \in \Psi_{\text{ai}}^{-m}(M)$ and

$$Q_1 T - I,\ TQ_1 - I \in \Psi_{\text{ai}}^{-\infty}(M).$$

Let ξ be a distribution such that $\xi\,,\ T\xi \in L^2(M)$. Then

$$\xi = Q_1(T\xi) - (Q_1 T - I)\xi \in H^m(M).$$

This shows that the maximal domain of T is $H^m(M)$. Since T is invertible, the graph topology on the domain of T coincides with the topology of $H^m(M)$. It follows then that $T\colon\ H^m(M) \to L^2(M)$ is Fredholm (in fact, even invertible) and hence $\hat{T}(\tau)$ is invertible in $\mathscr{L}(L^2(\partial M_1))$ for any $\tau \in \mathbb{R}$.

Let $R_1 \in \Psi_{\text{ai}}^{-\infty}(M)$ be such that R_1 and $\hat{R}_1(\tau)$ are injective, for any $\tau \in \mathbb{R}$. (This is possible because $L^2(M)$ has a countable orthonormal basis. It would not be possible to chose such an $R_1 \in \Psi_{\text{inv}}^{-\infty}(M)$, we owe this comment to the referee.) Then $Q_2 := Q_1^* Q_1 + R_1^* R_1$ is a parametrix of TT^* such that $Q_2\colon\ L^2(M) \to H^{2m}(M)$ is an isomorphism. Let $R_2 := TT^*Q_2 - I \in \Psi_{\text{ai}}^{-\infty}(M)$. By construction, $I + R_2$ is invertible on $L^2(M)$ and $I + \hat{R}_2(\tau)$ are invertible on $L^2(\partial M_1)$ for any $\tau \in \mathbb{R}$.

By Lemma 2.8 applied to $I + \hat{R}_2(\tau)$ and the algebra $\Psi_{\text{ai}}^{-\infty}(\partial M_1 \times \mathbb{R})^{\mathbb{R}}$, we can find $R_3 \in \Psi_{\text{ai}}^{-\infty}(M)$ such that $(I + R_2)(I + R_3) - I \in \mathfrak{I}$. (We can take $R_3 =$

$s_0[(I+\tilde{R}_2)^{-1}-I]$. We can also assume that $I+R_3$ is injective, by replacing $I+R_3$ with

$$(I+R_2)^*[(I+R_3)^*(I+R_3)+R_4^*R_4],$$

where $R_4 \in \mathfrak{I}$ is injective.

We are now ready to complete our proof. The operator $R_5 := TT^*Q_2(I+R_3)-I \in \mathfrak{I}$ is such that $I+R_5$ is injective. It follows that $I+R_5 = TT^*Q_2(I+R_3)$ is Fredholm of index zero and, hence, invertible on $L^2(M)$. Using again Lemma 2.8, we obtain that there exists $R_6 \in \mathfrak{I}$ such that $(I+R_5)(I+R_6) = I$. Thus,

$$TT^*Q_2(I+R_3)(I+R_6) = I.$$

This means that $P := T^*Q_2(I+R_3)(I+R_6)$ is a right inverse to T. We can prove in exactly the same way that T has a left inverse in $\Psi_{\mathrm{ai}}^{-m}(M)$ and, hence, that it is invertible in $\Psi_{\mathrm{ai}}^{\infty}(M)$. $\square$

The above theorem applied to $T = I + \Delta$ gives the following result.

Corollary 2.11. *Let M be a manifold with cylindrical ends and $\Delta = \Delta_M$ be the Laplace operator on M. Then $(I+\Delta)^{-1} \in \Psi_{\mathrm{ai}}^{-2}(M)$.*

2.2. Perturbation by potentials. We shall need also a further extension of the above corollary. To state it, recall that an operator L, mapping L^2_{loc} into distributions, is said to have the *unique continuation property* if

$$Lu = 0 \quad \& \quad u \text{ vanishes in an open set} \Longrightarrow u = 0 \text{ on } M.$$

Proposition 2.12. *Let $L \in \Psi_{\mathrm{ai}}^{m}(M)$ be nonnegative (that is, $(Lf, f) \geqslant 0$ for all $f \in \mathscr{C}_c^\infty(M)$) and satisfy the unique continuation property. Also, let $V \in \mathscr{C}^\infty(M) \cap \Psi_{\mathrm{ai}}^{0}(M)$ (that is, V is translation invariant in a neighborhood of infinity), $V \geqslant 0$, such that V is strictly positive on some open subset of M. Then, if $L + V\colon H^m(M) \to L^2(M)$ is Fredholm, it is also invertible.*

P r o o f. The assumptions $L \geqslant 0$ and $V \geqslant 0$ imply $L+V \geqslant 0$, as well. Assume by contradiction that $L+V\colon H^m(M) \to L^2(M)$ is Fredholm but not invertible. Then $L+V$ is Fredholm as an unbounded operator on $L^2(M)$ and is not invertible. This shows that 0 must be an eigenvalue of $L+V$.

Let $u \neq 0$, $u \in L^2(M)$ be an associated eigenvector:

$$(L+V)u = 0.$$

Then

$$\langle Lu, u\rangle + \langle Vu, u\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(M)$.

Since $\langle Lu, u\rangle \geqslant 0$ and $\langle Vu, u\rangle \geqslant 0$, we must have both $\|L^{1/2}u\|^2 = \langle Lu, u\rangle = 0$ and $\|V^{1/2}u\|^2 = \langle Vu, u\rangle = 0$. Thus $Lu = 0$ and $Vu = 0$. The second relation gives that u vanishes on some open subset of M. Since L has the unique continuation property, u must vanish identically. This contradicts the original assumptions and the proof is now complete. □

Example. If $T \in \Psi^k_{\mathrm{ai}}(M)$ has the unique continuation property then $L := T^*T$ satisfies the hypotheses of the above proposition (with $m = 2k$). In particular, this is the case for $\Delta = \Delta_M = d^*d$, since the kernel of $d = d_M$ consists of only locally constant functions.

The following theorem is crucial for our approach to extending the method of layer potentials to manifolds with cylindrical ends.

Theorem 2.13. *Let M be a manifold with cylindrical ends and $V \geqslant 0$ be a smooth function on M that is translation invariant in a neighborhood of infinity and does not vanish at infinity. Denote by $\Delta = \Delta_M$ the Laplace operator on M. Then $\Delta + V$ is invertible as an unbounded operator on $L^2(M)$ and $(\Delta + V)^{-1} \in \Psi^{-2}_{\mathrm{ai}}(M)$.*

P r o o f. For starters, Δ is non-negative ($\Delta \geqslant 0$) and has the unique continuation property (cf. the previous example). Since the potential V is non-negative, as well as strictly positive on some non-empty open set, our result will follow from Proposition 2.12 as soon as we show that $\Delta + V\colon\ H^2(M) \to L^2(M)$ is Fredholm.

Since Δ is elliptic, $P := \Delta + V\colon\ H^2(M) \to L^2(M)$ will be Fredholm if, and only if, $\tilde{P}$ is invertible. In turn, to show that $\tilde{P}$ is invertible it suffices to prove the norm of the inverse of $\hat{P}(\tau)\colon\ H^2(\partial M_1) \to L^2(\partial M_1)$ is bounded uniformly in $\tau \in \mathbb{R}$.

More specifically, let $V_\infty \in \mathscr{C}^\infty(\partial M_1)$ be the limit at infinity of the function V. (This limit exists because we assumed V to be translation invariant in a neighborhood of infinity.) Denote $\Delta = \Delta_{\partial M_1}$, to simplify notation in what follows. By definition, we have

$$\hat{P}(\tau) = \Delta + \tau^2 + V_\infty.$$

Since $V_\infty + \tau^2 \geqslant 0$ and does not vanish identically for any $\tau \in \mathbb{R}$, by assumption, we obtain as in [33] that $\hat{P}(\tau)$ is indeed invertible for any $\tau \in \mathbb{R}$. (One can also justify this using the methods used to prove Proposition 2.12.)

Let $\mathscr{L}(X, Y)$ denote the normed space of all linear bounded operators between two Banach spaces X, Y.

The invertibility of $\Delta + V_\infty$ implies that $\Delta_{\partial M_1} + V_\infty \geqslant cI$, for some $c > 0$. The functional calculus gives that $(\Delta + \tau^2 + V_\infty)^2 \geqslant c^2 I$ and that

$$(\Delta + \tau^2 + V_\infty)^2 \geqslant (\Delta + V_\infty)^2 \geqslant \frac{1}{4}\Delta^2 + \frac{1}{2}(\Delta + V_\infty)^2 - V_\infty^2 \geqslant \frac{1}{4}\Delta^2 - \|V_\infty\|_\infty^2 .$$

Consequently,

$$(\Delta + \tau^2 + V_\infty)^2 \geqslant \frac{\varepsilon}{4}(\Delta^2 - 4\|V_\infty\|_\infty) + (1-\varepsilon)c^2 \geqslant 2C^2(\Delta^2 + 1) \geqslant C^2(\Delta + 1)^2, \tag{27}$$

if $\varepsilon > 0$ and $C > 0$ are small enough. In particular, we obtain from Equation (27) that

$$\|(\Delta + \tau^2 + V_\infty)(\Delta + 1)^{-1}\| \geqslant C,$$

and, ultimately,

$$\begin{aligned} &\|(\Delta_{\partial M_1} + \tau^2 + V_\infty)^{-1}\|_{\mathscr{L}(L^2(\partial M_1), H^2(\partial M_1))} \\ &\qquad = \|(\Delta_{\partial M_1} + 1)(\Delta_{\partial M_1} + \tau^2 + V_\infty)^{-1}\|_{\mathscr{L}(L^2(\partial M_1), L^2(\partial M_1))} \leqslant C^{-1}, \end{aligned}$$

for any $\tau \in \mathbb{R}$. This completes the proof of our theorem. □

Let us mention that in the proof of the above theorem we used an ad-hoc argument to prove a result that holds in much greater generality. Namely, assume that P is elliptic of order m. Then there exists $R > 0$ such that $\hat{P}(\tau)$ is invertible as a map $H^m(\partial M_1) \to L^2(\partial M_1)$, for any $|\tau| > R$. Moreover, $\hat{P}(\tau)^{-1}$ depends continuously on τ on its domain of definition. In particular, if P is elliptic of order $m > 0$ and $\hat{P}(\tau)$ is invertible for any τ, then $\|\hat{P}(\tau)^{-1}\|$ is uniformly bounded as a map $L^2(\partial M_1) \to H^m(\partial M_1)$. See [46], especially Theorem 9.2, for details.

2.3. Products. We shall need also the following product decomposition result for the ideal of regularizing, almost invariant pseudodifferential operators.

First, let us observe that if M is a manifold with cylindrical ends and X is a smooth, compact, Riemannian manifold without boundary, then $M \times X$ is also a manifold with cylindrical ends.

For any Fréchet algebra A, we shall denote by $\mathscr{C}^\infty(X^2, A)$ the space of smooth functions on $X \times X$ and values in A, with the induced topology and the product:

$$f \star g(x, x'') = \int_X f(x, x')g(x', x'')\, \mathrm{d}x', \tag{28}$$

the integration being with respect to the volume element obtained from the Riemannian metric on X. For example, $\Psi^{-\infty}(X) \cong \mathscr{C}^\infty(X^2, \mathbb{C})$.

Theorem 2.14. *Let M be a manifold with cylindrical ends and X be a smooth, compact, Riemannian manifold without boundary. Then $\Psi_{\mathrm{ai}}^{-\infty}(M \times X)$ is isomorphic to $\mathscr{C}^\infty(X^2, \Psi_{\mathrm{ai}}^{-\infty}(M))$.*

Proof. Let us denote by $\mathscr{S}(\mathbb{R}, V)$ the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$ with values in a Fréchet space V. Also, let $\mathscr{C}_0^\infty$ denote the space of smooth functions on a manifold with boundary that vanish to infinite order at the boundary, as in the statement of Corollary 2.6.

The statement of the theorem follows from Lemma 2.3, Corollary 2.6, and the relations

$$\begin{aligned} &\mathscr{S}(\mathbb{R}, \mathscr{C}^\infty((\partial M_1 \times X)^2)) \simeq \mathscr{C}^\infty(X^2, \mathscr{S}(\mathbb{R}, \mathscr{C}^\infty((\partial M_1)^2))), \quad \text{and} \\ &\mathscr{C}_0^\infty((\partial M_1 \times X)^2) \simeq \mathscr{C}^\infty(X^2, \mathscr{C}_0^\infty((\partial M_1)^2)). \end{aligned} \tag{29}$$

□

3. Boundary layer potential integrals

We want to extend the method of boundary layer potential to manifolds with cylindrical ends. We begin by introducing the class of manifolds with boundary that we plan to study in this paper.

3.1. Submanifolds with cylindrical ends. Let $N \subset M$ be a submanifold with boundary of a manifold with cylindrical ends. We want to generalize the method of layer potentials to this non-compact case. We notice that N plays a role in the method of boundary layer potentials mostly through its boundary ∂N. (We shall make our assumptions on N more precise below in Definition 5.1.) Because of this, we shall formulate some of our results in the slightly more general setting when ∂N is replaced by a suitable submanifold of codimension one.

Definition 3.1. Let $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ be a manifold with cylindrical ends. A submanifold with cylindrical ends of M is a submanifold $Z \subset M$ such that

$$Z \cap (\partial M_1 \times (-\infty, 0]) = Z' \times (-\infty, 0],$$

for some submanifold $Z' \subset \partial M_1$. We shall write then $Z \sim Z' \times (-\infty, 0]$.

We shall fix Z, Z' as above in what follows. Our main interest is of course when $Z = \partial N$, but for certain reasonings, it is useful to allow this slightly greater level of generality.

Let us recall from [48, vol. II, Proposition 2.8], that a distribution L on $\mathbb{R}^n \times \mathbb{R}^n$ is the kernel of a classical pseudodifferential operator of order $-j$, $j = 1, 2, \ldots,$ if, and only if,

$$L \sim \sum_{l=0}^{\infty} (q_l(x, z) + p_l(x, z) \ln |z|) \tag{30}$$

where q_l are smooth functions of x with values distributions in z that are homogeneous of degree $j + l - n$ and smooth for $z \neq 0$, and p_l are polynomials homogeneous of degree $j + l - n$. (The sign "$\sim$" in Equation (30) above means that the difference $L - \sum_{l=0}^{N} (q_l(x, z) + p_l(x, z) \ln |z|)$ is as smooth as we want if N is chosen large enough.)

It is not difficult to check that the converse holds true also for $j = 0$ under some additional conditions, for example when $p_0 = 0$ and $q_0(x, z)$ is odd in z and the associated distribution is defined by a principal value integral.

Theorem 3.2. *Let M be a manifold with cylindrical ends and let $Z \subset M$ be a codimension one submanifold with cylindrical ends, as in Definition* 3.1. *If $P \in \Psi^m_{\mathrm{inv}}(M)$, $m < -1$, is given by the kernel $K \in \mathscr{C}^\infty(M^2 \setminus M)$, then the restriction of K to $Z^2 \setminus Z$ extends uniquely to the kernel of an operator $P_Z \in \Psi^{m+1}_{\mathrm{inv}}(Z)$. The same result holds true with $\Psi^m_{\mathrm{ai}}(M)$ and $\Psi^{m+1}_{\mathrm{ai}}(Z)$ replacing $\Psi^m_{\mathrm{inv}}(M)$ and $\Psi^{m+1}_{\mathrm{inv}}(Z)$.*

Moreover, if $\sigma_m(P)$ is odd, then we can also allow $m = -1$, provided that we define P_Z by using a principal value integral.

P r o o f. Let $P \in \Psi^m_{\mathrm{inv}}(M)$. Then K is supported in a set of the form

$$V_\varepsilon := \{(x, y) \in M^2, \operatorname{dist}(x, y) < \varepsilon\},$$

by Definition 1.1. Clearly the restriction of K to $Z^2 \setminus Z$ will be supported in $V_\varepsilon \cap Z^2$. Moreover, by standard (local) arguments, namely Equation (30) above, $K|_{Z \times Z}$ is the kernel of a unique pseudodifferential operator on Z of order $\leqslant m + 1$. (See [47], [48]). The translation invariance of this operator follows from the definition.

To prove the same result for operators that are almost translation invariant in a neighborhood of infinity, it is enough to do this for order $-\infty$ operators. More precisely, we need to check that if $T \in \Psi^{-\infty}_{\mathrm{ai}}(M)$, then $T_Z \in \Psi^{-\infty}_{\mathrm{ai}}(Z)$. This statement is local in a neighborhood of Z in the following sense. Let φ be a smooth function on M that is translation invariant in a neighborhood of infinity, $\varphi = 1$ in a neighborhood of Z and with support in a small neighborhood of Z. The statement for T is equivalent to the corresponding statement for $\varphi T \varphi$. We can assume then that $M = Z \times S^1$, with Z identified with $Z \times \{1\}$. By the Theorem 2.14, we can write $T = T(\theta, \theta')$, $\theta, \theta' \in S^1$ to be a smooth function with values in $\Psi^{-\infty}_{\mathrm{ai}}(Z)$. The result then follows because $T_Z = T(1, 1)$. $\square$

We need now to investigate the relation between restriction to the submanifold Z of codimension one in M and indicial operators.

Proposition 3.3. *Let $Z \subset M$ be as in Definition 3.1, with Z of codimension one, $Z \sim Z' \times (-\infty, 0]$, in a neighborhood of infinity. Let $P \in \Psi^m_{\mathrm{ai}}(M)$, $m \leqslant -1$. Then $\tilde{P}_{Z' \times \mathbb{R}} = \widetilde{P_Z}$ and $[\hat{P}(\tau)]_{Z'} = \widehat{P_Z}(\tau)$.*

Proof. This follows from definitions, as follows. First we notice that both statements of the Proposition are local in a neighborhood of infinity, so we can assume that $Z = Z' \times \mathbb{R}$. The first relation then is automatic. For the second relation we also use the fact that the restriction to Z' and the Fourier transform in the $\mathbb{R}$-direction commute. □

3.2. Boundary layer potential integrals. We now proceed to define the boundary layer potential integrals. Let M be a manifold with cylindrical ends and $Z \subset M$ be a submanifold with cylindrical ends *of codimension one*. (Later on we shall restrict ourselves to the case when $Z = \partial N$, where $N \subset M$ is a submanifold with boundary and cylindrical ends. For now though, it is more convenient to continue to consider this more general case.)

Let δ_Z be the surface measure on Z, regarded as a distribution on M. If $f \in L^2(Z)$, then

$$f \otimes \delta_Z \in H^{-a}(M)\,, \quad a > 1/2. \tag{31}$$

Similarly, if δ'_Z is the normal derivative of δ_Z, then

$$f \otimes \delta'_Z \in H^{-a-1}(M)\,, \quad a > 1/2. \tag{32}$$

Definition 3.4. Fix a smooth function $V \geqslant 0$, $V \in \Psi^0_{\mathrm{inv}}(M)$, V not identical equal to 0 on M. As before, we shall continue to denote by $\Delta = \Delta_M$ the Laplace operator on M. Let $f \in L^2(Z)$ and $a > 1/2$. The *single layer potential integral associated to $Z \subset M$ and $\Delta + V$* is defined as

$$\mathscr{S}(f) := (\Delta + V)^{-1}(f \otimes \delta_Z) \in H^{2-a}(M),$$

and the *double layer potential integral associated to $Z \subset M$ and V* is defined as

$$\mathscr{D}(f) := (\Delta + V)^{-1}(f \otimes \delta'_Z) \in H^{1-a}(M).$$

Assume that the normal bundle of Z in M is oriented (so there will be a *positive side* and *negative side* of Z in M). As in [33] we shall denote by $f_\pm$ the non-tangential limits of some function defined on $M \setminus Z$, when we approach Z from the positive side $(+)$, respectively from the negative side $(-)$, provided, of course, that these limits exist pointwise almost everywhere. (It is here where we need the normal bundle to Z to be oriented.)

We now begin to follow the strategy of [33]. Let

$$S := [(\Delta + V)^{-1}]_Z \in \Psi_{\mathrm{ai}}^{-1}(Z). \tag{33}$$

We shall fix in what follows a vector field ∂_ν on M that is normal to Z at every point of Z. The principal symbol of the order -1 operator $(\Delta + V)^{-1}\partial_\nu^*$ is odd, so we can also define

$$K := [(\Delta + V)^{-1}\partial_\nu^*]_Z \in \Psi_{\mathrm{ai}}^{0}(Z). \tag{34}$$

Proposition 3.5. *With the above notation, the operator S of Equation* (33) *is elliptic. Moreover, the zero principal symbol of K vanishes, $\sigma_0(K) = 0$, and hence actually $K \in \Psi_{\mathrm{ai}}^{-1}(Z)$.*

P r o o f. First, the fact that S is elliptic follows from a symbol calculation (which is local in nature) analogous to [35, (3.42), p. 33]. In fact, similar considerations show that $\sigma_0(K) = 0$ so, in fact, $K \in \Psi_{\mathrm{ai}}^{-1}(Z)$. See also the discussion in [48, vol. II, Proposition 11.2, p. 36]. $\square$

Let $Z \subset M$ be a codimension one submanifold with cylindrical ends with oriented normal bundle to Z. Then we shall denote by f_+ the non-tangential limits at Z from the positive part of the normal bundle. Similarly, we shall denote by f_- the non-tangential limits at Z from the negative part of the normal bundle.

Theorem 3.6. *Let $Z \subset M$ be a codimension one submanifold with cylindrical ends. Assume the normal bundle to Z is oriented. Given $f \in L^2(Z)$, we have*

$$\mathscr{S}(f)_+ = \mathscr{S}(f)_- = Sf$$

as pointwise a.e. limits. Also, using the notation of Equation (34) *above, we have*

$$\partial_\nu \mathscr{S}(f)_\pm = (\pm\tfrac{1}{2}I + K^*)f,$$

where K^* is the formal transpose of K.

P r o o f. Let us write $T := (\Delta + V)^{-1} = P + R$, where $P \in \Psi_{\mathrm{inv}}^{m}(M)$ (so it is translation invariant in a neighborhood of infinity) and $R \in \Psi_{\mathrm{ai}}^{-\infty}(M)$. The first statement of the proposition, namely

$$[T(f \otimes \delta_Z)]_\pm = T_Z f$$

is clearly linear in $T \in \Psi_{\mathrm{ai}}^{m}(M)$, $m < -1$. It is enough then to prove it for P and R separately.

For $T = (\Delta + V)^{-1}$ replaced by P, this is a local statement (because P is properly supported), which then follows from [33, Proposition 3.8].

For T replaced by R, we argue as in the proof of Theorem 3.2 that we can assume that $M = Z \times S^1$, with Z identified with the submanifold $Z \times \{1\}$. Then we use again Theorem 2.14 to write $R = R(\theta, \theta')$, for some smooth function with values in $\Psi_{\mathrm{ai}}^{-\infty}(Z)$.

This gives

$$R(f \otimes \delta_Z)(z, \theta) = [R(\theta, 1)f](z)$$

and $R_Z = R(1,1)$. Let $g_\theta(z) = R(f \otimes \delta_Z)(z, \theta)$. The assumptions on the function $R(\theta, \theta')$ guarantee that the function

$$S^1 \ni \theta \mapsto g_\theta \in H^m(M)$$

is continuous (in fact, even $\mathscr{C}^\infty$) for any m. Then

$$[R(f \otimes \delta_Z)]_\pm = \lim_{\theta \to 1 \pm 0} g_\theta = g_1 := R(1,1)f = R_Z f.$$

$\square$

The following theorem is proved in a completely similar way, following the results of [33, Proposition 3.8].

Theorem 3.7. *Let Z be a codimension one submanifold with cylindrical ends of M with oriented normal bundle. Given $f \in L^2(Z)$, we have*

$$\mathscr{D}(f)_\pm = (\mp \tfrac{1}{2} I + K) f$$

as pointwise a.e. limits.

We can replace the pointwise almost everywhere limits with L^2-limits both for the tangential limits of the single and double layer potentials; see Theorem 3.12.

For further reference, let us discuss now the "trace theorem" for codimension one submanifolds in our setting. See [1] for more details and results of this kind for manifolds with a Lie structure at infinity.

Proposition 3.8. *Let $Z \subset M$ be a submanifolds with cylindrical ends of the manifold with cylindrical ends M. Then the restriction map $\mathscr{C}_c^\infty(M) \to \mathscr{C}_c^\infty(Z)$ extends to a continuous map $H^s(M) \to H^{s-1/2}(Z)$, for any $s > 1/2$.*

Proof. We can assume, as in the proof of Theorem 3.6, that $M = Z \times S^1$. Since the Sobolev spaces $H^s(M)$ and $H^{s-1/2}(Z)$ do not depend on the metric on M and Z, as long as these metrics are compatible with the structure of manifolds with cylindrical ends, we can assume that the circle S^1 is given the invariant metric making it of length 2π and that M is given the product metric.

Then $\Delta = \Delta_Z + \Delta_{S^1}$ and $\Delta_{S^1} = -\partial_\theta^2$ has spectrum $\{4\pi^2 n^2\}$, $n \in \mathbb{Z}$. Let $L^2(Z \times S^1)_n \subset L^2(Z \times S^1)$ denote the eigenspace corresponding to the eigenvalues $n \in \mathbb{Z}$ of $(2\pi\mathrm{i})^{-1}\partial_\theta$. Then

$$L^2(Z \times S^1) \simeq \bigoplus_{n\in\mathbb{Z}} L^2(Z \times S^1)_n \simeq \bigoplus_{n\in\mathbb{Z}} L^2(Z),$$

where the isomorphism $L^2(Z\times S^1)_n \simeq \bigoplus_{n\in\mathbb{Z}} L^2(Z)$ is obtained by restricting to $1 \in S^1$.

To prove our proposition, it is enough to check that if $\xi_n \in L^2(Z)$ is a sequence such that

$$\sum_n \|(1 + n^2 + \Delta_Z)^{s/2}\xi_n\|^2 < \infty \tag{35}$$

then $\sum(1 + \Delta_Z)^{s/2-1/4}\xi_n$ is convergent.

Let $C = 1 + \int_{\mathbb{R}}(1 + t^2)^{-s}\,\mathrm{d}t$ and assume that each ξ_n is in the spectral subspace of Δ_Z corresponding to $[m, m+1) \subset \mathbb{R}_+$. Then

$$(1 + m^2)^{s-1/2}\Big(\sum_n \|\xi_n\|\Big)^2 \leqslant C\sum_n \|(1 + n^2 + m^2)^{s/2}\xi_n\|^2.$$

Since the constant C is independent of m and the spectral spaces of Δ_Z corresponding to $[m, m+1) \subset \mathbb{R}$ give an orthogonal direct sum decomposition of $L^2(Z)$, this checks Equation (35) and completes the proof. □

3.3. Higher regularity of the layer potentials. We shall not need the following results in what follows. We include them for completeness and because they give a better intuitive picture of the properties of layer potentials. Choose a small open tubular neighborhood U of Z in M, such that $U \simeq Z \times (-\varepsilon, \varepsilon)$ via a diffeomorphism that is compatible with the cylindrical ends structure of Z and M. For example, assume that ∂_ν is a vector field on M that is normal to Z and translation invariant in a neighborhood of infinity. Denote by $\exp(t\partial_\nu)$ the one-parameter group of diffeomorphisms generated by ∂_ν. (This group exists because ∂_ν extends to the canonical compactification of M to a manifold with boundary $\simeq M_1$.) Then the range $U = U_\varepsilon$ of the map

$$Z \times (-\varepsilon, \varepsilon) \ni (z, t) \mapsto \Psi(z, t) := \exp(t\partial_\nu)z \in M$$

is a good choice, for $\varepsilon > 0$ small enough. In particular, for ε small enough, the complement U_ε^c of U_ε is a smooth submanifold with boundary, such that its boundary ∂U_ε^c is a submanifold with cylindrical ends. Moreover, $\partial U_\varepsilon^c = Z_{-\varepsilon} \cup Z_{+\varepsilon}$ is the disjoint union of two manifolds diffeomorphic to Z via $Z \simeq Z \times \{\pm\varepsilon\} \simeq Z_{\pm\varepsilon}$, where the second map is given by Ψ.

Denote by $H^m(U_\varepsilon^c)$ the space of restrictions of distributions in $H^m(M)$ to (the interior of) the complement of U_ε.

The following two theorems describe the mapping properties of the single and double layer potentials. Since the statements and proofs work actually in greater generality, we begin with some more general results, which we shall then specialize to the case of single and double layer potentials.

Theorem 3.9. *Let $U \simeq Z \times (-\varepsilon, \varepsilon)$ be a tubular neighborhood of Z in M (as above) and let $T \in \Psi_{\mathrm{ai}}^m(M)$. Restriction to U^c defines for any s continuous maps*

$$H^s(Z) \ni f \mapsto T(f \otimes \delta_Z) \in H^\infty(U^c),$$

which are translation invariant in a neighborhood of infinity, for any tubular neighborhood U of Z.

Proof. Let ψ_0 and ψ_1 be smooth functions on M and $T \in \Psi_{\mathrm{ai}}^m(M)$. Assume the following: ψ_0 and ψ_1 are translation invariant in a neighborhood of infinity; ψ_0 is equal to 1 in a neighborhood of Z; ψ_1 vanishes in a neighborhood of the support of ψ_0; and ψ_0 is equal to 1 in a neighborhood of U^c. Then

$$T(f \otimes \delta_Z)|_{U^c} = (\psi_1 T \psi_0)(f \otimes \delta_Z)$$

and $\psi_1 T \psi_0 \in \Psi_{\mathrm{ai}}^{-\infty}(M)$ because the supports of ψ_0 and ψ_1 are disjoint. □

Consider now $U = U_\varepsilon \simeq Z \times (-\varepsilon, \varepsilon)$, for $\varepsilon > 0$ small enough, where the last diffeomorphism is given by the exponential map. Then decompose $\partial U_\varepsilon^c = Z_{+\varepsilon} \cup Z_{-\varepsilon}$ as a disjoint union, as above. In particular, we fix the diffeomorphisms $Z \simeq Z_{\pm\varepsilon}$ defined by the exponential, as above. Then the traces of the restrictions to U_ε^c

$$H^s(Z) \ni f \to T_{\pm\varepsilon} f := T(f \otimes \delta_Z)|_{Z_{\pm\varepsilon}} \in H^{s'}(Z_{\pm\varepsilon}) \simeq H^{s'}(Z) \tag{36}$$

define continuous operators $T_{\pm\varepsilon}\colon H^s(Z) \to H^{s'}(Z)$, for any $s, s' \in \mathbb{R}$.

We fix in what follows $\varepsilon > 0$ as above. Similarly, we obtain operators $T_{\pm t}\colon H^s(Z) \to H^{s'}(Z)$, for any $t \in (0, \varepsilon]$ and any $s, s' \in \mathbb{R}$.

Theorem 3.10. *Let $T \in \Psi_{\mathrm{ai}}^m(M)$ and T_t be as above, Equation (36). Then $T_{\pm t} \in \Psi_{\mathrm{ai}}^{-\infty}(Z)$ and the two functions*

$$(0, \varepsilon] \ni t \to t^l \partial_t^k T_{\pm t} \in \Psi_{\mathrm{ai}}^{m+1+k-l+\delta}(Z)$$

extend by continuity to $[0, \varepsilon]$ if $\delta > 0$. These extensions are bounded for $\delta = 0$.

Proof. The proof is based on the ideas in [48, vol. II, Ch. 7, Sec. 12], especially Theorem 12.6, and some local calculations. Here are some details.

Since the statement of the theorem is "linear" in T, it is enough to prove it for $T \in \Psi^m(M)$ and for $T \in \Psi_{\mathrm{ai}}^{-\infty}(M)$. The later case is obvious—in fact, it is already contained in the proof of Theorem 3.6. Then, we can further reduce the proof to the case when $T = s_0(T_1)$, with $T_1 \in \Psi_{\mathrm{ai}}(\partial M_1 \times \mathbb{R})^{\mathbb{R}}$, and to the case when T has compactly supported Schwartz kernel. Again, the second case is easier, being an immediate consequence of the corresponding result for the compact case. Because the second case involves a similar argument, we shall nevertheless discuss this here.

Assume, for the next argument, that M is compact. Since the result is true for regularizing operators, we can use a partition of unity to localize to the domain of a coordinate chart. This allows then to further replace M with $\mathbb{R}^n$, Z with $\mathbb{R}^{n-1}$, and T with an operator of the form $T = a(x, D)$, with $a(\ ,\ )$ in Hörmander's symbol class $S^m = S^m(\mathbb{R}^n)$ [48, vol. II] of functions that satisfy uniform estimates in the space variable x (recall that $a \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ is in $S^m(\mathbb{R}^n)$ if, and only if, $|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leqslant C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|}$ for all multi-indices α and β).

Let $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $(\xi', \xi_n) \in \mathbb{R}^{n-1*} \times \mathbb{R}^*$ be the usual decomposition of the variables. Also, let

$$a_t(x', \xi') = (2\pi \mathrm{i})^{-1} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\xi_n} a(x, t, \xi', \xi_n)\, \mathrm{d}\xi_n.$$

Then a_t is such that $T_t = a_t(x, D)$ and the (two) functions $t^l \partial_t^k a_{\pm t}$ extend to continuous functions $[0, \varepsilon] \to S^{m+1+l-k+\delta} = S^{m+1+l-k+\delta}(\mathbb{R}^n)$, for any $\delta > 0$. These extensions are bounded as functions with values in $S^{m+1+l-k}$. This completes the proof of our result for the case M compact.

Let us consider now to the case when $T = s_0(T_1)$. We can assume that $M = \partial M_1 \times \mathbb{R}$ and that T is $\mathbb{R}$-invariant. The proof is then the same as in the case M compact, but using local coordinates on ∂M_1 instead of on M, and making sure that all our symbols and all maps preserve the $\mathbb{R}$-invariance. This completes the proof of our result. □

A consequence of the above theorem is the following continuity result.

Corollary 3.11. *Let $T \in \Psi_{\mathrm{ai}}^{-1}(M)$.*

(i) *If $f \in H^m(Z)$, then the functions $(0,\varepsilon] \ni t \mapsto T_{\pm t} f \in H^m(Z)$ extend by continuity at* 0.

(ii) *If $f \in H^\infty(Z)$, then the mappings $(0,\varepsilon] \times Z \mapsto \big(T_{\pm t} f\big)(z)$ extend to functions in $H^\infty([0,\varepsilon] \times Z)$.*

P r o o f. Denote by $\mathscr{L}(X,Y)$ the normed space of bounded operators between two Banach spaces X and Y. Theorem 3.10 ensures that $(0,\varepsilon] \to T_{\pm t}$ have continuous extensions to functions

$$[0,\varepsilon] \to \mathscr{L}(H^{m+\delta}(Z), L^2(Z)),$$

for $\delta > 0$. For $\delta = 0$ these extensions will be bounded.

This proves the first part of our result as follows. If $f \in H^{m+\delta}(Z)$, then the functions $T_{\pm t} f \in L^2(Z)$ extend by continuity on $[0,\varepsilon]$ because $T_{\pm t}$ extend by continuity on $[0,\varepsilon]$ as maps to $\mathscr{L}(H^{m+\delta}(Z), L^2(Z))$. Since $H^{m+\delta}(Z)$, $\delta > 0$, is dense in $H^m(Z)$ and $T_{\pm t}$ are bounded as maps $[0,\varepsilon] \to \mathscr{L}(H^m(Z), L^2(Z))$, the result follows from an $\varepsilon/3$-type argument.

To prove (ii), it is enough to prove then that $\partial_t^b (I + \Delta_Z)^a T_{\pm t} f$ is in $L^2([0,\varepsilon] \times Z)$, for any $a, b \in \mathbb{N}$. Using again Theorem 3.10, we know that $\partial_t^b (I + \Delta_Z)^a T_{\pm t}$ extend to continuous functions $[0,\varepsilon] \to \Psi_{\mathrm{ai}}^c(M)$, with $c = m + 2 + b + 2a$, (take $\delta = 1$). Since $f \in H^\infty(Z) \subset H^c(Z)$, the functions $(0,\varepsilon] \ni t \to \partial_t^b (I + \Delta_Z)^a T_{\pm t} f \in L^2(Z)$ extend by continuity to a function defined on $[0,\varepsilon]$. This extension is then in $L^2([0,\varepsilon] \times Z)$. $\square$

We can specialize all the above results to $T = (\Delta + V)^{-1}$ or $T = (\Delta + V)^{-1} \partial_\nu^*$. This gives maps $S_{\pm t}(f) := \mathscr{S}(f)|_{Z_{\pm t}}$ and $D_{\pm t}(f) := \mathscr{D}(f)|_{Z_{\pm t}}$, where $t \in (0,\varepsilon]$.

Theorem 3.12. *Using the notation we have just introduced, we have*

(i) *$S_{\pm t}, D_{\pm t} \in \Psi_{\mathrm{ai}}^{-\infty}(Z)$ and the functions $(0,\varepsilon] \ni t \to t^l \partial_t^k S_{\pm t} \in \Psi_{\mathrm{ai}}^{\delta-1+k-l}(Z)$ and $(0,\varepsilon] \ni t \to t^l \partial_t^k D_{\pm t} \in \Psi_{\mathrm{ai}}^{\delta+k-l}(Z)$ extend by continuity to $[0,\varepsilon]$, for $\delta > 0$. For $\delta = 0$ these functions are bounded.*

(ii) *If $f \in L^2(Z)$, then the functions $t \to S_{\pm t} f,\ D_{\pm t} f \in L^2(Z)$ extend by continuity to $[0,\varepsilon]$.*

(iii) *If $f \in H^\infty(Z)$, then the restrictions of $\mathscr{S}(f)$ and $\mathscr{D}(f)$ to $Z \times [-\varepsilon, 0)$ and, respectively, $Z \times (0,\varepsilon]$ extend to functions in $H^\infty(Z \times [-\varepsilon, 0])$, respectively in $H^\infty(Z \times [0,\varepsilon])$.*

The aim of this section is to investigate the invertibility of layer potential operators which depend on a parameter $\tau \in \mathbb{R}$, via a method initially developed by G. Verchota in [50], for the case of the flat-space Laplacian. The novelty here is to derive estimates which are *uniform* with respect to the real parameter τ.

Let $\mathfrak{M}$ be a smooth, *compact, boundaryless* Riemannian manifold, and fix a reasonably regular subdomain $\Omega \subset \mathfrak{M}$ (Lipschitz will do). Here, $\mathfrak{M}$ will play the role of ∂M_1 in our standard notation and, anticipating notation introduced in the next section, Ω will play the role of the *exterior* of X.

Set ν for the outward unit conormal to Ω and $d\sigma$ for the surface measure on $\partial\Omega$ (naturally inherited from the metric on $\mathfrak{M}$). The departure point is the following Rellich type identity:

$$
\begin{aligned}
(37)\qquad & \int_{\partial\Omega} \langle \nu, w\rangle \big\{ |\nabla_{\tan} u|^2 - |\partial_\nu u|^2 \big\}\, \mathrm{d}\sigma \\
&= 2\mathrm{Re} \int_{\partial\Omega} \langle w_{\tan}, \nabla u\rangle \partial_\nu \bar{u}\, \mathrm{d}\sigma - 2\mathrm{Re} \int_{\Omega} \langle \nabla \bar{u}, w\rangle \Delta_{\mathfrak{M}}\, u\, \mathrm{d}x \\
&\quad + \mathrm{Re} \int_{\Omega} \big\{ (\operatorname{div} w)|\nabla u|^2 - 2(\mathscr{L}_w g)(\nabla u, \nabla \bar{u}) \big\}\, \mathrm{d}x,
\end{aligned}
$$

which, so we claim, is valid for a (possibly complex-valued) scalar function u and a real-valued vector field w (both sufficiently smooth, otherwise arbitrary) in Ω. Hereafter, the subscript 'tan' denotes the tangential component relative to $\partial\Omega$. At the level of vector fields, ∇ is used to denote the Levi-Civita connection on $\mathfrak{M}$. Also, $\mathscr{L}_w g$ stands for the Lie derivative of the metric tensor g with respect to the field w; recall that, in general,

$$
\mathscr{L}_w g(X, Y) = \langle \nabla_X w, Y\rangle + \langle \nabla_Y w, X\rangle,
$$

for any two vector fields X, Y.

To prove (37), consider the vector field $F := |\nabla u|^2 w - 2(\partial_w u)\nabla \bar{u}$ and compute

$$
\begin{aligned}
(38)\qquad \langle \nu, F\rangle &= |\nabla u|^2 \langle \nu, w\rangle - 2\,(\partial_w u)(\partial_\nu \bar{u}) \\
&= |\nabla u|^2 \langle \nu, w\rangle - 2\,\langle w_{\tan}, \nabla u\rangle\, \partial_\nu \bar{u} - 2\,|\partial_\nu u|^2 \langle \nu, w\rangle \\
&= \langle \nu, w\rangle (|\nabla_{\tan} u|^2 - |\partial_\nu u|^2) - 2\langle w_{\tan}, \nabla u\rangle\, \partial_\nu \bar{u},
\end{aligned}
$$

by decomposing $w = w_{\tan} + \langle \nu, w\rangle \nu$ and $|\nabla u|^2 = |\nabla_{\tan} u|^2 + |\partial_\nu u|^2$. Furthermore,

$$
(39)\qquad \operatorname{div} F = (\operatorname{div} w)|\nabla u|^2 + w(|\nabla u|^2) - 2\,(\partial_w u)\Delta_{\mathfrak{M}}\, \bar{u} - 2\,\nabla u(\partial_w \bar{u}).
$$

Given the current goal, the first and the third terms suit our purposes; for the rest we write

$$\begin{aligned}
w(|\nabla u|^2) - 2\,\nabla u(\partial_w \bar{u}) &= w(|\nabla u|^2) - 2\,\nabla u(w(\bar{u})) \\
&= w(|\nabla u|^2) - 2\,[\nabla u, w]\bar{u} - 2\,w(\nabla u(\bar{u})) \\
&= w(|\nabla u|^2) + 2\langle \partial_w(\nabla u), \nabla \bar{u}\rangle - 2\langle \nabla_{\nabla u} w, \nabla \bar{u}\rangle - 2\,w|\nabla u|^2 \\
&= -w(|\nabla u|^2) + \mathrm{Re}\,[w(|\nabla u|^2)] - 2(\mathscr{L}_w\, g)(\nabla u, \nabla \bar{u}),
\end{aligned}$$

where the third equality utilizes the fact that ∇ is torsion-free. Since the real parts of the first two terms in the last expression above cancel out, it ultimately follows that

$$\mathrm{Re}\,(\mathrm{div}\, F) = (\mathrm{div}\, w)|\nabla u|^2 - 2\,\mathrm{Re}\,[(\partial_w \bar{u})\Delta_{\mathfrak{M}}\, u] - 2\,\mathrm{Re}\,(\mathscr{L}_w\, g)(\nabla u, \nabla \bar{u}). \tag{40}$$

Thus, the Rellich identity (37) follows from (40), (38), and the Divergence Theorem, after taking the real parts.

Another general identity (in fact, a simple consequence of the Divergence Theorem) that is useful here is

$$\int_{\partial\Omega} |u|^2 \langle w, \nu\rangle\, \mathrm{d}\sigma = \mathrm{Re} \int_{\Omega} \{2u\langle \nabla \bar{u}, w\rangle + (\mathrm{div}\, w)|u|^2\}\, \mathrm{d}x. \tag{41}$$

To proceed, fix a nonnegative scalar potential $W \in C^\infty(\mathfrak{M})$ and for the remainder of this subsection assume that

$$(\Delta_{\mathfrak{M}} + \tau^2 + W)u = 0 \quad \text{in} \quad \Omega, \tag{42}$$

where $\tau \in \mathbb{R}$ is an arbitrary parameter (fixed for the moment). Our immediate objective is to show that

$$\int_{\partial\Omega} |\partial_\nu u|^2\, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} \{|\nabla_{\tan} u|^2 + (1+\tau^2)|u|^2\}\, \mathrm{d}\sigma, \tag{43}$$

uniformly in τ, and that for each $\varepsilon > 0$ there exists a finite constant $C = C(\Omega, \varepsilon) > 0$ so that

$$\int_{\partial\Omega} \{|\nabla_{\tan} u|^2 + \tau^2|u|^2\}\, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} |\partial_\nu u|^2\, \mathrm{d}\sigma + \varepsilon \int_{\partial\Omega} |u|^2\, \mathrm{d}\sigma \tag{44}$$

uniformly in the parameter $\tau \in \mathbb{R}$. We shall also need a strengthened version of (44) to the effect that

$$W > 0 \text{ in } \Omega \Longrightarrow \int_{\partial\Omega} \{|\nabla_{\tan} u|^2 + (1+\tau^2)|u|^2\}\, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} |\partial_\nu u|^2\, \mathrm{d}\sigma \tag{45}$$

uniformly in the parameter $\tau \in \mathbb{R}$.

With an eye on (44), let us recall Green's first identity for the function u that we assumed to satisfy Equation (42)

$$\int_\Omega \{|\nabla u|^2 + \tau^2|u|^2 + W|u|^2\}\,\mathrm{d}x = \mathrm{Re}\int_{\partial\Omega} \bar{u}\,\partial_\nu u\,\mathrm{d}\sigma$$

which readily yields the energy estimate

$$\int_\Omega \{|\nabla u|^2 + \tau^2|u|^2 + W|u|^2\}\,\mathrm{d}x \leqslant \int_{\partial\Omega} |u||\partial_\nu u|\,\mathrm{d}\sigma. \tag{46}$$

In turn, this further entails

$$\int_\Omega \tau^2|\nabla u||u|\,\mathrm{d}x \leqslant C|\tau|\int_\Omega \{\tau^2|u|^2 + |\nabla u|^2\}\,\mathrm{d}x \leqslant C|\tau|\int_{\partial\Omega} |u||\partial_\nu u|\,\mathrm{d}\sigma, \tag{47}$$

uniformly in τ.

Let us now select w to be transversal to $\partial\Omega$, i.e.

$$\operatorname{ess\,inf}\langle w,\nu\rangle > 0 \text{ on } \partial\Omega, \tag{48}$$

something which can always be arranged given that $\partial\Omega$ is assumed to be Lipschitz. This, in concert with (41), then gives

$$\int_{\partial\Omega} |u|^2\,\mathrm{d}\sigma \leqslant C\int_\Omega \{|u|^2 + |\nabla u||u|\}\,\mathrm{d}x. \tag{49}$$

Multiplying (49) with τ^2 and then invoking (46)–(47) eventually justifies the estimate

$$\int_{\partial\Omega} \tau^2|u|^2\,\mathrm{d}\sigma \leqslant C\int_{\partial\Omega} (1+|\tau|)|\partial_\nu u||u|\,\mathrm{d}\sigma. \tag{50}$$

Next, make the (elementary) observation that for every $\varepsilon, \delta > 0$ there exists $C = C(\varepsilon,\delta) > 0$ so that

$$(1+|\tau|)|\partial_\nu u||u| \leqslant \delta\tau^2|u|^2 + C|\partial_\nu u|^2 + \varepsilon|u|^2, \tag{51}$$

uniformly in τ. When considered in the context of (50), the boundary integral produced by the first term in the right side of (51) can be absorbed in the left side of (50), provided δ is sufficiently small. Thus, with this alteration in mind, (50) becomes

$$\int_{\partial\Omega} \tau^2|u|^2\,\mathrm{d}\sigma \leqslant C\int_{\partial\Omega} |\partial_\nu u|^2\,\mathrm{d}\sigma + \varepsilon\int_{\partial\Omega} |u|^2\,\mathrm{d}\sigma$$

which is certainly in the spirit of (44). In fact, in order to fully prove the latter estimate, there remains to control the tangential gradient in a similar fashion. To this end, observe that (48) and Rellich's identity (37) give

$$\int_{\partial\Omega} |\nabla_{\tan} u|^2 \, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} |\partial_\nu u|^2 \, \mathrm{d}\sigma + C \int_\Omega \tau^2 |\nabla u||u| \, \mathrm{d}x + C \int_\Omega |\nabla u|^2 \, \mathrm{d}x,$$

uniformly in τ. With this at hand, the same type of estimates employed before can be used once again to further bound the solid integrals in terms of (suitable) boundary integrals. The bottom line is that

$$\int_{\partial\Omega} |\nabla_{\tan} u|^2 \, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} |\partial_\nu u|^2 \, \mathrm{d}\sigma + \varepsilon \int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma, \tag{52}$$

uniformly in τ, and (44) follows.

It is now easy to prove (45), having disposed off (44). One useful ingredient in this regard is

$$\int_\Omega |u|^2 \, \mathrm{d}x \leqslant C \int_\Omega \{|\nabla u|^2 + W|u|^2\} \, \mathrm{d}x, \tag{53}$$

itself a version of Poincaré's inequality. When used in conjunction with (46) and (49), this readily yields

$$\int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma \leqslant C \int_\Omega \{|\nabla u|^2 + W|u|^2\} \, \mathrm{d}x \leqslant C \int_{\partial\Omega} |u||\partial_\nu u| \, \mathrm{d}\sigma \tag{54}$$

so that, ultimately,

$$\int_{\partial\Omega} |u|^2 \, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} |\partial_\nu u|^2 \, \mathrm{d}\sigma, \tag{55}$$

in the case we are currently considering. In concert with (44), this concludes the proof of (45). Let us now turn our attention to the estimate (43). For starters, Rellich's identity (37) can also be employed, along with the condition (48), to produce

$$\int_{\partial\Omega} |\partial_\nu u|^2 \, \mathrm{d}\sigma \leqslant C \int_{\partial\Omega} |\nabla_{\tan} u|^2 \, \mathrm{d}\sigma + C \int_\Omega \tau^2 |\nabla u||u| \, \mathrm{d}x + C \int_\Omega |\nabla u|^2 \, \mathrm{d}x, \tag{56}$$

uniformly in τ. Then, much as before,

$$\begin{aligned} \int_\Omega \{\tau^2 |\nabla u||u| + |\nabla u|^2\} \, \mathrm{d}x &\leqslant C \int_{\partial\Omega} (1 + |\tau|)|\partial_\nu u||u| \, \mathrm{d}\sigma \\ &\leqslant \delta \int_{\partial\Omega} |\partial_\nu u|^2 \, \mathrm{d}\sigma + C \int_{\partial\Omega} (1 + \tau^2)|u|^2 \, \mathrm{d}\sigma, \end{aligned} \tag{57}$$

where $\delta > 0$ is chosen small and C depends only on Ω and δ. With these two estimates at hand, the endgame in the proof of (43) is clear.

After these preliminaries, we can finally address the main theme of this subsection. More concretely, for each $\tau \in \mathbb{R}$, let S_τ, K_τ be, respectively, the single and the double layer potential operators associated with $\Delta_{\mathfrak{M}}+\tau^2+W$ on $\partial\Omega$ (recall that the potential function W was first introduced in connection with (42)). From the work in [33], it is known that if Ω has a Lipschitz boundary then both

$$S_\tau\colon\ L^2(\partial\Omega) \longrightarrow H^1(\partial\Omega) \quad \text{and} \quad \tfrac{1}{2}I + K_\tau\colon\ L^2(\partial\Omega) \longrightarrow L^2(\partial\Omega)$$

are invertible operators for each $\tau \in \mathbb{R}$. Our objective is to study how the norms of their inverses depend on the parameter τ. To discuss this issue, for each $\tau \in \mathbb{R}$ and $f \in H^1(\partial\Omega)$, set

$$\|f\|_{H^1_\tau(\partial\Omega)} := \|f\|_{H^1(\partial\Omega)} + |\tau|\|f\|_{L^2(\partial\Omega)}. \tag{58}$$

Thus, $\mathbb{R} \ni \tau \mapsto \|\cdot\|_{H^1_\tau(\partial\Omega)}$ is a one-parameter family of equivalent norms on the Sobolev space $H^1(\partial\Omega)$. The main result of this subsection is as follows.

Proposition 4.1. *Assume that Ω is a fixed, Lipschitz subdomain of $\mathfrak{M}$, and retain the notation introduced above. Then there exits a finite constant $C = C(\partial\Omega) > 0$, depending exclusively on the Lipschitz character of Ω, such that for each $\tau \in \mathbb{R}$, we have*

$$\|S_\tau^{-1} f\|_{L^2(\partial\Omega)} \leqslant C\|f\|_{H^1_\tau(\partial\Omega)} \tag{59}$$

uniformly for $f \in H^1(\partial\Omega)$.

Furthermore, if $W > 0$ on a set of positive measure in Ω, then for any $\tau \in \mathbb{R}$ we also have

$$\|(\tfrac{1}{2}I + K_\tau)^{-1} f\|_{L^2(\partial\Omega)} \leqslant C\|f\|_{L^2(\partial\Omega)}, \tag{60}$$

uniformly for $f \in L^2(\partial\Omega)$.

Proof. Consider first (60). Let $\Omega_+ := \Omega$, $\Omega_- := \mathfrak{M} \setminus \bar{\Omega}$, and for $f \in L^2(\partial\Omega)$, set $u := \mathscr{S} f$ in $\Omega_\pm$. Thus,

$$(u)_+ = (u)_-, \quad (\nabla_{\tan} u)_+ = (\nabla_{\tan} u)_-, \quad (\partial_\nu u)_\pm = (\pm\tfrac{1}{2}I + K_\tau^*)f. \tag{61}$$

In turn, (61), (43) and (45) allow us to write

$$\begin{aligned}
\|(-\tfrac{1}{2}I + K_\tau^*)f\|_{L^2(\partial\Omega)} &= \|(\partial_\nu u)_-\|_{L^2(\partial\Omega)} \\
&\leqslant C\|(u)_-\|_{H^1_\tau(\partial\Omega)} = C\|(u)_+\|_{H^1_\tau(\partial\Omega)} \leqslant C\|(\partial_\nu u)_+\|_{L^2(\partial\Omega)} \\
&= C\|(\tfrac{1}{2}I + K_\tau^*)f\|_{L^2(\partial\Omega)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|f\|_{L^2(\partial\Omega)} &\leqslant \|(-\tfrac{1}{2}I + K_\tau^*)f\|_{L^2(\partial\Omega)} + \|(\tfrac{1}{2}I + K_\tau^*)f\|_{L^2(\partial\Omega)} \\
&\leqslant C\|(\tfrac{1}{2}I + K_\tau^*)f\|_{L^2(\partial\Omega)}
\end{aligned} \tag{62}$$

for some constant $C = C(\partial\Omega) > 0$ independent of τ. Going further, if $\mathscr{L}(X) := \mathscr{L}(X, X)$, the normed algebra of all bounded operators on a Banach space X, then (62) entails

$$\|(\tfrac{1}{2}I + K_\tau)^{-1}\|_{\mathscr{L}\left(L^2(\partial\Omega)\right)} = \|(\tfrac{1}{2}I + K_\tau^*)^{-1}\|_{\mathscr{L}\left(L^2(\partial\Omega)\right)} \leqslant C.$$

This takes care of (60).

As for (59), the argument is rather similar, the main step being the derivation of the estimate

$$\|f\|_{L^2(\partial\Omega)} \leqslant C\|\nabla_{\tan}(S_\tau f)\|_{L^2(\partial\Omega)} + C(1 + |\tau|)\|S_\tau f\|_{L^2(\partial\Omega)},$$

out of (61) and (43), when the latter is written both for Ω_+ and Ω_-. Once again, the crux of the matter is that the intervening constant $C = C(\partial\Omega) > 0$ is independent of τ. The proof is finished. □

5. The Dirichlet problem

We now apply the results we have established to solve the inhomogeneous Dirichlet problem on manifolds with boundary and cylindrical ends.

The class of manifolds with boundary and cylindrical ends that we consider have a product structure at infinity (including the boundary and the metric). It is possible to relax somewhat these conditions, but for simplicity we do not address this technical question in this paper.

Definition 5.1. Let N be a Riemannian manifold with boundary ∂N. We shall say that N is a *manifold with boundary and cylindrical ends* if there exists an open subset V of N isometric to $(-\infty, 0) \times X$, where X is a compact manifold with boundary, such that $N \setminus V$ is compact.

The manifolds M, N, and X are all assumed to be smooth (we only work with smooth manifolds in this paper). Even the potential applications to manifolds with conical points are via a Kondratiev type transformation that maps the boundary to a manifold with cylindrical ends and maps the conical point to infinity. (These are the "blow-up" transformations use by Melrose.)

Lemma 5.2. *Let N be a Riemannian manifold with boundary ∂N. Then N is a manifold with boundary and cylindrical ends if, and only if, there exists a manifold with cylindrical ends (without boundary) M with a standard decomposition*

$$M = M_1 \cup \big(\partial M_1 \times (-\infty, 0]\big)$$

and containing N such that

$$N \cap \big(\partial M_1 \times (-\infty, 0]\big) = X \times (-\infty, 0],$$

for some compact manifold with boundary $X \subset \partial M_1$.

Proof. If there is a manifold M with the indicated properties, then it follows from the definition that N is a manifold with boundary and cylindrical ends. If the metric on N is a product metric on a tubular neighborhood of ∂N, then we can take $M := N \cup (-N)$ to be the *double* of N. (That is, the manifold obtained by gluing two copies of N along their common boundary). The general case can be reduced to this one, because any metric on N is equivalent to a product metric in a small tubular neighborhood of ∂N. (This is proved by taking the exponential map of $-\nu$. A more general tubular neighborhood theorem can be found in [1].) $\square$

Let $M = M_1 \cup (\partial M_1 \times (-\infty, 0])$ be a manifold with cylindrical ends. The transformation

$$(-\infty, -1] \ni x \to t := x^{-1} \in [-1, 0)$$

then extends to a diffeomorphism ψ between M and the interior $M_0 := M_1 \setminus \partial M_1$ of M_1:

$$\psi \colon\ M \to M_0 := M_1 \setminus \partial M_1. \tag{63}$$

If $N \subset M$ is a manifold with boundary and cylindrical ends, as in Lemma 5.2, then the above diffeomorphism will map N to a subset $N_0 \subset M_0$, whose closure N_1 is a *compact* manifold with corners of codimension at most two, $N_1 \subset M_1$. We can identify N_1 with the disjoint union $N_0 \cup X$, if X is as in the definition above.

We shall fix $N \subset M$ as above in what follows. We define then $H^s(N)$ to be the space of restrictions to the interior of N of distributions $u \in H^s(M)$. Recall that the main goal of this paper is to prove that the map

$$H^s(N) \ni u \to ((\Delta_N + V)u, u|_{\partial N}) \in H^{s-2}(N) \oplus H^{s-1/2}(\partial N) \tag{64}$$

is an isomorphism for $s > 1/2$, where $V \geqslant 0$ a smooth function that is asymptotically translation invariant in a neighborhood of infinity (that is $V \in \Psi^0_{\mathrm{ai}}(M)$).

We shall use the results of the previous subsections for the particular case when $Z = \partial N$.

Proposition 5.3. *Assume that the function V is chosen so that V is not identically zero on $\partial M_1 \setminus \overline{X}$. Then the map $-\frac{1}{2}I + K^*\colon\ L^2(\partial N) \to L^2(\partial N)$ is injective.*

P r o o f. Just follow word for word [33, Proposition 4.1]. □

Note that our signs are opposite to those in [33] or [49], because we use the definition that makes the Laplace operator is *positive*.

To prove the Fredholm property of the operators $\frac{1}{2}I + K$ and $\frac{1}{2}I + K^*$, we need to slightly change the corresponding argument in [33]. Recall that the index of a Fredholm operator P is the dimension of the kernel of P minus the dimension of the cokernel of P.

Proposition 5.4. *Retain the same assumptions as in Proposition* 5.3*. Then the operator*

$$-\tfrac{1}{2}I + K\colon\ L^2(\partial N) \longrightarrow L^2(\partial N) \tag{65}$$

is Fredholm of index zero.

P r o o f. The above proposition is known when M is compact [33, Corollary 4.5].

To check that it is Fredholm, we shall rely on (iv) in Theorem 2.1 which, in view of Proposition 3.5, (15), and (16), amounts to studying the associated indicial family.

Let $W := V_{\partial M_1}$, where $V_{\partial M_1}(y) = V(y,x)$, for $(y,x) \in \partial M_1 \times (-\infty, R)$, for some large R. The existence of such a $V_{\partial M_1}$ follows from the assumption that V is translation invariant in a neighborhood of infinity. Also, let $T = (\Delta + V)^{-1}\partial_\nu^*$. Recall that $K := T_{\partial N}$ and that $\partial N \sim \partial X \times (-\infty, 0]$ in a neighborhood of infinity. Then Proposition 3.3 gives

$$\hat{K}(\tau) = \widehat{T_{\partial N}}(\tau) = [\hat{T}(\tau)]_{\partial X} = [(\Delta_{\partial M_1} + \tau^2 + W)^{-1}\partial_\nu^*]_{\partial X} = K_\tau, \tag{66}$$

where K_τ is the double layer potential operator associated with the perturbed Laplacian $\Delta_{\partial M_1} + \tau^2 + W$ on ∂X (cf. the discussion in §3.3). Let $f_\tau(x)$ be the Fourier transform in the t-variable of $f(x,t)$ ($t \in \mathbb{R}$). In light of this and (16), there remains to prove that the map

$$L^2(\partial X \times \mathbb{R}) \ni f(x,t) \mapsto \mathscr{F}^{-1}[(-\tfrac{1}{2}I + K_\tau)\hat{f}_\tau(x)](t) \in L^2(\partial X \times \mathbb{R}) \tag{67}$$

is an isomorphism. To see this, let $g \in L^2(\partial X \times \mathbb{R})$ be arbitrary and, for each $\tau \in \mathbb{R}$, introduce $h_\tau := (-\frac{1}{2}I + K_\tau)^{-1}\hat{g}_\tau$. From Proposition 4.1 (utilized for $\Omega := \partial M_1 \setminus \overline{X}$, which accounts for a change in sign as far as the coefficient $1/2$ is concerned), it follows that this is meaningful, $h_\tau \in L^2(\partial X)$ and

$$\|h_\tau\|_{L^2(\partial X)} \leqslant C\|\hat{g}_\tau\|_{L^2(\partial X)}, \qquad \text{uniformly for } \tau \in \mathbb{R}. \tag{68}$$

If we now set $h(x,t) := \mathscr{F}^{-1}(h_\tau(x))(t)$ then, thanks to (68) and Plancherel's formula,

$$\begin{aligned}(69)\quad \int_{\partial X}\int_{\mathbb{R}} |h(x,t)|^2\,\mathrm{d}t\,\mathrm{d}\sigma_x &= \int_{\partial X}\int_{\mathbb{R}} |h_\tau(x)|^2\,\mathrm{d}\tau d\sigma_x = \int_{\mathbb{R}} \|h_\tau\|^2_{L^2(\partial X)}\,\mathrm{d}\tau \\ &\leqslant C\int_{\mathbb{R}} \|\hat{g}_\tau\|^2_{L^2(\partial X)}\,\mathrm{d}\tau = C\int_{\partial X}\int_{\mathbb{R}} |g(x,t)|^2\,\mathrm{d}t\,\mathrm{d}\sigma_x.\end{aligned}$$

That is, $h \in L^2(\partial X \times \mathbb{R})$ and $\|h\|_{L^2(\partial X\times\mathbb{R})} \leqslant C\|g\|_{L^2(\partial X\times\mathbb{R})}$. Furthermore,

$$\begin{aligned}(70)\qquad \mathscr{F}^{-1}[(-\tfrac{1}{2}I + K_\tau)\hat{h}_\tau(x)](t) &= \mathscr{F}^{-1}[(-\tfrac{1}{2}I + K_\tau)h_\tau(x)](t) \\ &= \mathscr{F}^{-1}(\hat{g}_\tau)(x) = g(x,t)\end{aligned}$$

which proves that the map (67) is onto. The fact that (67) is also one-to-one, follows more or less directly from the analogue of (60) in our context.

Thus, at this stage, we may conclude that (65) is indeed a Fredholm operator; there remains to compute its index. To set the stage, let us observe that Proposition 5.3 and duality can now be used to justify that

$$(71)\qquad -\tfrac{1}{2}I + K\colon\ L^2(\partial N) \longrightarrow L^2(\partial N) \quad \text{is onto.}$$

Next, so we claim,

$$(72)\qquad -\tfrac{1}{2}I + K\colon\ H^1(\partial N) \longrightarrow H^1(\partial N) \text{ is Fredholm and onto}$$

as well. Indeed, since $K \in \Psi^{-1}_{\mathrm{ai}}(\partial N)$, it follows that for each s,

$$(73)\qquad f \in H^s(\partial N)\ \&\ (-\tfrac{1}{2}I + K)f \in H^{s+1}(\partial N) \Longrightarrow f \in H^{s+1}(\partial N).$$

In concert with (71), this shows that the operator in (72) is onto. Also, since

$$(74)\qquad \dim\operatorname{Ker}(-\tfrac{1}{2}I + K; H^1(\partial N)) \leqslant \dim\operatorname{Ker}(-\tfrac{1}{2}I + K; L^2(\partial N)) < +\infty,$$

the claim (72) is proved. In particular,

$$(75)\qquad \operatorname{index}(-\tfrac{1}{2}I + K; L^2(\partial N)) \leqslant 0 \quad \text{and} \quad \operatorname{index}(-\tfrac{1}{2}I + K; H^1(\partial N)) \leqslant 0.$$

We now take an important step by proving that

$$(76)\qquad S\colon\ L^2(\partial N) \longrightarrow H^1(\partial N) \text{ is Fredholm.}$$

(Later on we shall prove that this operator is in fact invertible). This task is accomplished much as before, i.e. by relying on Theorem 2.1 and Proposition 4.1, and we only sketch the main steps. First, as pointed out in Proposition 3.5, S is elliptic. Second, the first estimate in Proposition 4.1 eventually allows us to conclude that the assignment

$$L^2(\partial X \times \mathbb{R}) \ni f(x,t) \mapsto \mathscr{F}^{-1}[S_\tau \hat{f}_\tau(x)](t) \in H^1(\partial X \times \mathbb{R})$$

is an isomorphism, concluding the proof of the claim (76).

Having dealt with (76), we next invoke an intertwining identity, to the effect that

$$\big(-\tfrac{1}{2}I + K\big)S = S\big(-\tfrac{1}{2}I + K^*\big).$$

This can be seen by starting with Green's formula $u = \mathscr{D}(u|_{\partial N}) - \mathscr{S}(\partial_\nu u)$ written for the harmonic function $u := \mathscr{S}(f)$, and then using the jump-relations deduced in Theorems 3.6–3.7. The identity (5) allows us to obtain

$$\begin{aligned}\operatorname{index}\big(-\tfrac{1}{2}I + K; H^1(\partial N)\big) &= \operatorname{index}\big(-\tfrac{1}{2}I + K^*; L^2(\partial N)\big)\\ &= -\operatorname{index}\big(-\tfrac{1}{2}I + K; L^2(\partial N)\big).\end{aligned}$$

From this and (75) we may finally conclude that the operator (65) has index zero, as desired. □

Corollary 5.5. *Let V be as before. Then the operator*

$$-\tfrac{1}{2}I + K\colon\ H^s(\partial N) \longrightarrow H^s(\partial N)$$

is invertible for each $s \in \mathbb{R}$.

P r o o f. To begin with, the case $s = 0$ is easily proved by putting together the above two propositions. In particular, the operator $-\frac{1}{2}I + K\colon\ H^s(\partial N) \longrightarrow H^s(\partial N)$, in the statement of this corollary, is injective for each $s \geqslant 0$. Since the fact that this operator is also surjective is a consequence of the corresponding claim in the case $s = 0$ and the smoothing property (73), the desired conclusion follows for $s \geqslant 0$. As for the case $s < 0$, a similar reasoning shows that

$$-\tfrac{1}{2}I + K^*\colon\ H^{-s}(\partial N) \longrightarrow H^{-s}(\partial N) \tag{77}$$

is invertible for each $s < 0$. This and duality then yield the invertibility of $-\frac{1}{2}I + K\colon$ $H^s(\partial N) \longrightarrow H^s(\partial N)$ for $s < 0$, as wanted. □

Another proof of the above result can be obtained from Theorem 2.10, for the case $m = 0$, the "easy one."

Recall that $H^s(N)$ is the space of restrictions of distributions in $H^s(M)$ to the interior of N. (Both M and N are smooth manifolds.) After these preliminaries, we are finally in a position to discuss the following basic result.

Theorem 5.6. *Let $V \in \Psi^0_{\mathrm{ai}}(M)$ be a smooth positive function. For any $s > 0$ and any $f \in H^s(\partial N)$, there exists a unique function $u \in H^{s+1/2}(N)$ such that $u|_{\partial N} = f$ and $(\Delta_N + V)u = 0$.*

P r o o f. Extend first V to a smooth positive function in $\Psi^0_{\mathrm{ai}}(M)$ (that is, asymptotically translation invariant in a neighborhood of infinity) which is not identically zero on the complement of N. The conclusion in Corollary 5.5 will hold for this function. First we claim that

$$\mathscr{D}\colon\ H^s(\partial N) \longrightarrow H^{s+1/2}(N), \qquad s \in \mathbb{R}, \tag{78}$$

is well-defined and bounded. Indeed, if $s < 0$, then this is a consequence of the implication

$$f \in H^s(\partial N),\ s < 0 \implies f \otimes \delta_{\partial N} \in H^{s-1/2}(\partial N) \tag{79}$$

along with the factorization $\mathscr{D}(g) = (\Delta + V)^{-1}\partial_\nu^*(g \otimes \delta_{\partial N})$. For $s = 0$, one can employ the techniques of [35]. The case $s > 0$ then follows inductively from what we have proved so far with the aid of a commutator identity which essentially reads $\nabla\mathscr{D}f = \mathscr{D}(\nabla_{\tan} f) +$ lower order terms; see (8.19) in [34] as well as (6.17) in [35].

Having disposed off (78) the existence part in the theorem is then easily addressed. Specifically, if $s > 0$, consider $g := (-\frac{1}{2}I + K)^{-1}f \in H^s(\partial N)$ and then set $u := \mathscr{D}(g) \in H^{s+1/2}(N)$ by (78).

To prove uniqueness, assume that $u \in H^{s+1/2}(N)$ is a null solution for the Dirichlet problem in N. For an arbitrary function $\varphi \in \mathscr{C}^\infty_c(N)$, let v solve the Dirichlet problem

$$(\Delta_N + V)v = 0, \qquad v|_{\partial N} = -[(\Delta + V)^{-1}\varphi]|_{\partial N},$$

and then set $w := v + (\Delta + V)^{-1}\varphi$. It follows that $(\Delta_N + V)w = \varphi$ in N and $w|_{\partial N} = 0$. Consequently, Green's formula gives

$$(u, \varphi) = (u, (\Delta_N + V)w) = ((\Delta_N + V)u, w) = 0$$

since $u|_{\partial N} = w|_{\partial N} = 0$. Since φ is arbitrary, this forces $u = 0$ in N as desired. $\square$

We are now ready to prove Theorem 0.2 which, for the convenience of the reader, we restate below.

Theorem 5.7. *Let N be a manifold with boundary and cylindrical ends and $V \geqslant 0$ be a smooth functions that is asymptotically translation invariant in a neighborhood of infinity. Then*

$$H^s(N) \ni u \to \tilde{\Delta}_N(u) := ((\Delta_N + V)u, u|_{\partial N}) \in H^{s-2}(N) \oplus H^{s-1/2}(\partial N)$$

is a continuous bijection, for any $s > 1/2$.

P r o o f. First we extend V to M, making sure that it is still $\geqslant 0$, smooth, and asymptotically translation invariant. The continuity of the map $\tilde{\Delta}_N$ follows from the continuity of $\Delta_N + V\colon\ H^s(N) \to H^{s-2}(N)$ and from the continuity of the trace map $H^s(N) \to H^{s-1/2}(\partial N)$.

As before, we fix a smooth function $V \geqslant 0$ which vanishes in a neighborhood of N. We take V to be not identically equal to 0 and to be translation invariant in a neighborhood of infinity, as before. This is seen to be possible using a tubular neighborhood of infinity. Let $g \in H^{s-2}(N)$ be arbitrary. First extend g to a distribution (denoted also g) in $H^{s-2}(M)$, then set $u_1 = (\Delta + V)^{-1} g \in H^s(M)$ and $f_1 = u_1|_{\partial N} \in H^{s-1/2}$. Finally, choose $u_2 \in H^s(N)$ such that $(\Delta_N + V)u_2 = 0$ and $u_2|_{\partial N} = f - f_1$. Then $u := u_1 + u_2$ satisfies $(\Delta_N + V)u = g$ and $u|_{\partial N} = f$. This proves the surjectivity of $\tilde{\Delta}_N$. The injectivity of this map then follows from the uniqueness part in Theorem 5.6. □

It is likely that some versions of the above two theorems extend to weighted Sobolev spaces. This will likely requires techniques similar to those used in [13]. In [42], Schrohe and Schulze have generalized the Boutet de Monvel calculus to manifolds with boundary and cylindrical ends. With some additional work, their results can probably be used to prove our Theorem 0.2 above. Our approach, however, is shorter and also leads to a characterization of the Dirichlet-to-Neumann boundary map, Theorem 5.8. It is worth pointing out that our methods can also handle non-smooth structures (cf. §4) and seem amenable to other basic problems of mathematical physics in non-compact manifolds (such as Maxwell's equations in infinite cylinders). We hope to return to these issues at a later time.

5.1. The Dirichlet-to-Neumann map. Theorem 5.6 allows us to define the Dirichlet-to-Neumann map $\mathscr{N}$

$$\mathscr{N}(f) = (\partial_\nu u)_+$$

for $f \in L^2(\partial N)$ and u solution of $(\Delta_N + V)u = 0$, $u_+ := u|_{\partial N} = f$.

Theorem 5.8. *Let N be a manifold with boundary and cylindrical ends. Then the operator $S\colon H^s(\partial N) \to H^{s+1}(\partial N)$ of Equation (33) is invertible for any s and $(\frac{1}{2}I + K^*)S^{-1} = \mathscr{N}$, the "Dirichlet-to-Neumann map." In particular, $\mathscr{N} \in \Psi^1_{\mathrm{ai}}(\partial N)$.*

Proof. The operator S is elliptic by Proposition 3.5. For further reference, let us note here that

$$f \in H^s(\partial N) \ \&\ Sf \in H^{s+1}(\partial N) \Longrightarrow f \in H^{s+1}(\partial N), \tag{80}$$

by elliptic regularity.

Next, using the notation of Proposition 4.1, we have $\hat{S}(\tau) = S_\tau$. By the results of the same proposition, $\hat{S}(\tau)$ is invertible for any τ, and the norm of the inverse is uniformly bounded (this can be proved also by using the results of [33] or [35] and the estimates in [46]). Consequently, $S\colon H^s(\partial N) \to H^{s+1}(\partial N)$ is Fredholm (cf. Theorem 2.1).

Checking that S is injective when $s = 0$ is done much as in the last part of §6 in [33]. In short, the idea is as follows. Assume that $Sf = 0$ for some $f \in H^s(\partial N)$ and let $N \subset M$, where M is a manifold with cylindrical ends without boundary, as in Lemma 5.2. Then $u := \mathscr{S}(f)$ satisfies $(\Delta + V)u = 0$ on $M \setminus \partial N$, and

$$u_{\partial N} = u_+ = u_- = Sf = 0.$$

Furthermore, thanks to (80), (79) and the factorization $\mathscr{S}(f) = (\Delta + V)^{-1}(f \otimes \delta_{\partial N})$, the function u is sufficiently regular so that (the uniqueness part in) Theorem 5.6 holds both in N and in $M \setminus N$. Hence, by Theorem 3.6,

$$f = (\partial_\nu u)_+ - (\partial_\nu u)_- = 0,$$

as desired. Thus, $S\colon H^s(\partial N) \to H^{s+1}(\partial N)$ is injective, first for $s \geqslant 0$ (via a simple embedding), then for $s \in \mathbb{R}$ via (80).

Since S is formally self-adjoint, we get that S has also dense range. Using now the fact that S is Fredholm, we obtain that S is bijective, as desired. $\square$

Corollary 5.9. *The Cauchy data space*

$$\{(u|_{\partial N}, \partial_\nu u|_{\partial N});\ u \in H^s(N),\ (\Delta_N + V)u = 0\}$$

is a closed subspace of $H^{s-1/2}(\partial N) \oplus H^{s-3/2}(\partial N)$ for any $s > 1/2$.

P r o o f. By Theorem 5.7, the Cauchy data space

$$\mathscr{C} := \{(u|_{\partial N}, \partial_\nu u|_{\partial N});\ u \in H^s(N),\ (\Delta_N + V)u = 0\}$$

is given by the graph of $\mathscr{N}$, namely

$$\mathscr{C} = \Gamma(\mathscr{N}) := \{(f, \mathscr{N} f), f \in H^{s-1/2}\} \subset H^{s-1/2}(\partial N) \oplus H^{s-3/2}(\partial N).$$

Theorem 5.8 shows that this space is closed, since $\mathscr{N} \in \Psi^1_{\mathrm{ai}}(\partial N)$ and hence it defines a continuous (everywhere defined) map $H^{s-1/2}(\partial N) \to H^{s-3/2}(\partial N)$. □

We conclude this section with yet another integral representation formula for the Dirichlet problem.

Corollary 5.10. *Retain the usual set of assumptions. Then, for each $s > 0$, the solution to the boundary problem*

$$u \in H^{s+1/2}(N), \quad (\Delta_N + V)u = 0, \quad u|_{\partial N} = f \in H^s(\partial N),$$

(first treated in Theorem 5.6) can also be expressed in the form

$$u = \mathscr{S}(S^{-1} f) \quad \text{in } N.$$

P r o o f. The starting point is the claim (which can be justified in a manner similar to (78)) that

$$\mathscr{S}\colon\ H^s(\partial N) \longrightarrow H^{s+3/2}(M), \qquad s \in \mathbb{R}, \tag{81}$$

is well-defined and bounded. In concert with the fact that $S\colon\ H^s(\partial N) \to H^{s+1}(\partial N)$ is invertible, this finishes the proof of the corollary. □

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