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BMO-SCALE OF DISTRIBUTION ON $\mathbb{R}^n$

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Abstract. Let S' be the class of tempered distributions. For $f \in S'$ we denote by $J^{-\alpha} f$ the Bessel potential of f of order α. We prove that if $J^{-\alpha} f \in \mathrm{BMO}$, then for any $\lambda \in (0,1)$, $J^{-\alpha}(f)_\lambda \in \mathrm{BMO}$, where $(f)_\lambda = \lambda^{-n} f(\varphi(\lambda^{-1} \cdot))$, $\varphi \in S$. Also, we give necessary and sufficient conditions in order that the Bessel potential of a tempered distribution of order $\alpha > 0$ belongs to the VMO space.

Keywords: BMO, VMO, John and Niereberg, Bessel potential

MSC 2000: 32A37

1. Introduction

The space of functions of bounded mean oscillation first appeared in the work of John and Nirenberg [1] in the context of nonlinear partial differential equations that arise in the study of minimal surfaces. The space of function of bounded mean oscillation, or BMO, naturally arises as the class of functions whose deviation from their means over cubes is bounded; L_∞ functions have this property, but there exist unbounded functions with bounded mean oscillation. Such functions are slowly growing and they typically have at most logarithmic blow up. The space BMO shares similar properties with the space L_∞ and it often serves as a substitute for it. For instance classical singular integrals do not map L_∞ into L_∞ but L_∞ into BMO. And in many instances interpolation between L_p and BMO works just as well as between L_p and L_∞. But the role of the space BMO is deeper and more far reaching than that. This space crucially arises in many situations in analysis, such as in the characterization of the L_2 boundedness of nonconvolution singular integral operators with standard kernels.

The following remarkable result is due C. Fefferman (announced in Characterizations of bounded mean oscillation, Bull. Amer. Soc. 77 (1971), 587–588): the dual of H^1 is BMO.

In this paper we prove that if $J^{-\alpha}f \in \mathrm{BMO}$, where f is a tempered distribution and $J^{-\alpha}f$ is the Bessel potential of f of order α, then for any $\lambda \in (0,1)$, $J^{-\alpha}(f)_\lambda \in \mathrm{BMO}$ (see Theorem 1), where $(f)_\lambda = \lambda^{-n}f(\varphi(\lambda^{-1}\cdot))$, $\varphi \in S$. Also, we prove (see Theorem 2) that $J^{-\alpha}f \in \mathrm{VMO}$ if and only if $J^{-\alpha}f \in \mathrm{BMO}$ and $\lim\limits_{\lambda\to 0^+} \lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}} = 0$.

2. Definition and notations

In this section, we gather definitions and notations that will be used throughout the paper.

For any $r > 0$ and $x \in \mathbb{R}^n$ we will denote by $B(x,r)$ the closed ball of radius r in $\mathbb{R}^n$ centered at x. A function is said to be of bounded mean oscillation if its mean oscillation over all cubes is bounded. Precisely, given a locally integrable function f on $\mathbb{R}^n$ and a measurable set Q in $\mathbb{R}^n$, denote by

$$f_Q = \frac{1}{|Q|}\int_Q f(y)\,\mathrm{d}y,$$

the mean of f over Q, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^n$. Then the oscillation of f over Q is the function $|f - f_Q|$ and the mean oscillation of f over Q is

$$\frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x.$$

Definition 1. For f a real-valued locally integrable function on $\mathbb{R}^n$, set

$$\|f\|_{\mathrm{BMO}} = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\,\mathrm{d}x,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$. The function is called of bounded mean oscillation if $\|f\|_{\mathrm{BMO}} < \infty$.

For instance, if F is an integrable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that $F(\cdot,y) \in \mathrm{BMO}$ for all $y \in \mathbb{R}^n$, then the function f given by

$$f(x) = \int_{\mathbb{R}^n} F(x,y)\,\mathrm{d}y$$

satisfies

$$\left\|\int_{\mathbb{R}^n} F(\cdot,y)\,\mathrm{d}y\right\|_{\mathrm{BMO}} \leqslant \int_{\mathbb{R}^n} \|F(\cdot,y)\|_{\mathrm{BMO}}\,\mathrm{d}y. \tag{1}$$

Indeed, let Q be a cube in $\mathbb{R}^n$, then using Fubini's theorem we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x &= \frac{1}{|Q|}\int_Q \left|\int_{\mathbb{R}^n} F(x,y)\,\mathrm{d}y - \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n} F(z,y)\,\mathrm{d}y\,\mathrm{d}z\right|\mathrm{d}x \\ &\leqslant \frac{1}{|Q|}\int_Q\int_{\mathbb{R}^n}\left|F(x,y)-\frac{1}{|Q|}\int_Q F(z,y)\,\mathrm{d}z\right|\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n}\frac{1}{|Q|}\int_Q\left|F(x,y)-\frac{1}{|Q|}\int_Q F(z,y)\,\mathrm{d}z\right|\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^n}\|F(\cdot,y)\|_{\mathrm{BMO}}\,\mathrm{d}y.\end{aligned}$$

This completes the proof of (1).

The space of functions of vanishing mean oscillation VMO was introduced by Sarason [2] as the set of integrable functions f on $\mathbb{R}^n$ satisfying

$$\lim_{\delta\to 0}\sup_{Q:|Q|\leqslant\,\delta}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x\right)=0$$

or

$$\lim_{N\to\infty}\sup_{Q:l(Q)\geqslant N}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|\,\mathrm{d}x\right)=0,$$

here Q denotes cubes in $\mathbb{R}^n$. Then VMO $(\mathbb{R}^n)$ is the closure in the BMO$(\mathbb{R}^n)$ norm of the space of continuous functions that vanish at infinity. It is known that $f\in$ VMO if and only if

$$\lim_{h\to 0}\|f-f(\cdot-h)\|_{\mathrm{BMO}}=0 \tag{2}$$

(see Theorem 1 in [2]).

Let S' denote the space of tempered distributions on $\mathbb{R}^n$, note that BMO $\subset S'$. For every $f\in S'$ and $s>0$, we define the dilation $(f)_s$ of the distribution f as follows:

$$(f)_s(\varphi)=s^{-n}f(\varphi(s^{-1}\cdot)),\qquad \varphi\in S.$$

If f is a function, them $(f)_s(x)=f(sx)$, for almost all $x\in\mathbb{R}^n$.

Let $-\infty<\alpha<\infty$. The Bessel potential $J^{-\alpha}$ of order α is defined by

$$F(J^{-\alpha}\psi)=(1+|\xi|^2)^{-\alpha/2}F(\psi),$$

$\xi\in\mathbb{R}^n$, $\psi\in S'$. Here $F: S'\to S'$ denotes the Fourier transform on S'. For $f\in L_1$ we have

$$F(f)(\xi)=\int_{\mathbb{R}^n} f(x)\mathrm{e}^{-i\xi\cdot x}\,\mathrm{d}x.$$

If $\alpha > 0$, then the Bessel potential $J^{-\alpha}$ is a convolution type operator,

$$J^{-\alpha} f(x) = \int_{\mathbb{R}^n} f(y) G_\alpha(x-y)\,\mathrm{d}y,$$

where the Bessel potential kernel G_α is defined by

$$G_\alpha(x) = \frac{1}{(2\pi)^{n/2} 2^{\alpha/2} \Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty t^{(\alpha-n)/2} \mathrm{e}^{-|x|^2/2t} \mathrm{e}^{-t/2} \frac{\mathrm{d}t}{t}. \tag{3}$$

It is also known that

$$\int_{\mathbb{R}^n} G_\alpha(x)\,\mathrm{d}x = 1,$$

(see [3]).

3. Main results

Suppose $f \in \mathrm{BMO}$ and $g \in L_1$. In general, the convolution $f * g$ is not defined as an absolutely convergent integral. However, the operator $f \to f * g$ is defined on the space BMO as the adjoint to the corresponding convolution operator on H^1. Moreover we have the following.

Lemma 1. *If $f \in \mathrm{BMO}$ and $g \in L_1$, then $f * g \in \mathrm{BMO}$ and*

$$\|f * g\|_{\mathrm{BMO}} \leqslant \|g\|_1 \|f\|_{\mathrm{BMO}}.$$

Proof. We have,

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |(f*g)(x) - (f*g)_Q|\,\mathrm{d}x \\ &\quad = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q ((f*g)(x) - (f*g)(y))\,\mathrm{d}y \right| \mathrm{d}x \\ &\quad = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q \left[\int_Q (f(x-t) - f(y-t)) g(t)\,\mathrm{d}t \right] \mathrm{d}y \right| \mathrm{d}x; \end{aligned}$$

by Fubini's theorem

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |(f*g)(x) - (f*g)_Q|\,\mathrm{d}x \\ &\quad = \frac{1}{|Q|} \int_Q \left| \frac{1}{|Q|} \int_Q \left[\int_Q (f(x-t) - f(y-t))\,\mathrm{d}y \right] g(t)\,\mathrm{d}t \right| \mathrm{d}x \\ &\quad \leqslant \frac{1}{|Q|} \int_Q \int_Q |f(x-t) - f_Q| |g(t)|\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \left(\int_Q |g(t)|\,\mathrm{d}t \right) \|f\|_{\mathrm{BMO}}. \end{aligned} \tag{4}$$

Thus

$$\|f * g\|_{\mathrm{BMO}} \leqslant \|g\|_1 \|f\|_{\mathrm{BMO}}.$$

This completes the proof of Lemma 1. $\square$

Theorem 1. *Let $f \in S'$, $\alpha > 0$, and $J^{-\alpha} f \in \mathrm{BMO}$. Then for every λ with $0 < \lambda < 1$ we have $J^{-\alpha}(f)_\lambda \in \mathrm{BMO}$. Moreover, there exists a constant $C_\alpha > 0$, depending only on α, such that*

$$\sup_{0<\lambda<1} \lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}} \leqslant C_\alpha \|J^{-\alpha} f\|_{\mathrm{BMO}}.$$

Proof. Let us begin by observing that

$$J^{-\alpha}(f)_\lambda = (f)_\lambda * G_\alpha,$$

so

$$\begin{aligned}\lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}} &= \lambda^\alpha \sup_Q |Q|^{-1} \int_Q |J^{-\alpha}(f)_\lambda(x) - (J^{-\alpha}(f)_\lambda)_Q| \,\mathrm{d}x \\ &= \lambda^\alpha \sup_Q |Q|^{-1} \int_Q \left| \int_{\mathbb{R}^n} (f)_\lambda(y) G_\alpha(x-y) \,\mathrm{d}y - |Q|^{-1} \int_Q J^{-\alpha}(f)_\lambda(z) \,\mathrm{d}z \right| \mathrm{d}x \\ &= \lambda^{\alpha-n} \sup_Q |Q|^{-1} \int_Q |f * (G_\alpha)_{\frac{1}{\lambda}}(x) - (f * (G_\alpha)_{\frac{1}{\lambda}})_Q| \,\mathrm{d}x;\end{aligned}$$

thus

$$\lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}} = \lambda^{\alpha-n} \|f * (G_\alpha)_{\frac{1}{\lambda}}\|_{\mathrm{BMO}}. \tag{5}$$

On the other hand we have

$$(G_\alpha)_{\frac{1}{\lambda}} = G_\alpha * Y_\lambda \tag{6}$$

where

$$F(Y_\lambda)(\xi) = \lambda^n \left(\frac{1+|\xi|^2}{1+\lambda^2|\xi|^2} \right)^{\alpha/2}. \tag{7}$$

Consider the expansion

$$(1-t)^{\alpha/2} = 1 + \sum_{m=1}^{\infty} A_{m,\alpha} t^m,$$

where $|t| < 1$. The explicit formula for the coefficients $A_{m,\alpha}$ is as follows:

$$A_{m,\alpha} = \frac{(-1)^m \frac{\alpha}{2}(\frac{\alpha}{2}-1)\dots(\frac{\alpha}{2}-m+1)}{m!} \tag{8}$$

for all $m \geqslant 1$. We already know that

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)}; \tag{9}$$

thus we can write (8) as follows

$$A_{m,\alpha} = \frac{(-1)^m \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\dots\left(\frac{\alpha}{2}-m+1\right)\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+m+1\right)}{(m!)^2 m^{(1+\alpha/2)}} \times \left[\frac{m!\, m^{(1+\alpha/2)}}{\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+m+1\right)}\right].$$

By (9) we obtain

$$\Gamma\left(1+\frac{\alpha}{2}\right) = \lim_{m\to\infty} \frac{m!\, m^{(1+\alpha/2)}}{\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+m+1\right)}.$$

Thus there exists M_α such that

$$\left|\frac{m!\, m^{(1+\alpha/2)}}{\left(\frac{\alpha}{2}+1\right)\dots\left(\frac{\alpha}{2}+m+1\right)}\right| \leqslant M_\alpha,$$

for all m. Then

$$|A_{m,\alpha}| \leqslant \frac{\left|\left(\frac{\alpha}{2}+m+1\right)\dots\left(\frac{\alpha}{2}+1\right)\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\dots\left(\frac{\alpha}{2}-m+1\right)\right|}{(m!)^2 m^{(1+\alpha/2)}} M_\alpha,$$

thus

$$|A_{m,\alpha}| \leqslant C_\alpha m^{-(1+\alpha/2)} \tag{10}$$

where

$$C_\alpha = \left|\left(\frac{\alpha}{2}+m+1\right)\dots\left(\frac{\alpha}{2}+1\right)\frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)\dots\left(\frac{\alpha}{2}-m+1\right)\right| M_\alpha.$$

Next, since

$$\frac{1+|\xi|^2}{1+\lambda^2|\xi|^2} = \lambda^{-2}\left(1-\frac{(1-\lambda^2)}{1+\lambda^2|\xi|^2}\right),$$

we get from (7)

$$F(Y_\lambda)(\xi) = \lambda^n\Big(\frac{1+|\xi|^2}{1+\lambda^2|\xi|^2}\Big)^{\alpha/2} = \lambda^n\Big[\lambda^{-2}\Big(1-\frac{(1-\lambda^2)}{1+\lambda^2|\xi|^2}\Big)\Big]^{\alpha/2}$$
$$= \lambda^{n-\alpha}\Big(1-\frac{1-\lambda^2}{1+\lambda^2|\xi|^2}\Big)^{\alpha/2} = \lambda^{n-\alpha}\Big[1+\sum_{m=1}^{\infty} A_{m,\alpha}\Big(\frac{1-\lambda^2}{1+\lambda^2|\xi|^2}\Big)^m\Big]$$
$$= \lambda^{n-\alpha} + \lambda^{n-\alpha}\sum_{m=1}^{\infty} A_{m,\alpha}(1-\lambda^2)^m(1+\lambda^2|\xi|^2)^{-m},$$

therefore

$$Y_\lambda = \lambda^{n-\alpha}\delta + \lambda^{n-\alpha}\sum_{m=1}^{\infty} A_{m,\alpha}(1-\lambda^2)^m(G_{2m})_\lambda, \tag{11}$$

where δ denote the delta-measure at 0.

Let us denote

$$X_\lambda = \sum_{m=1}^{\infty} A_{m,\alpha}(1-\lambda^2)^m(G_{2m})_\lambda.$$

Then

$$\int |X_\lambda| \leqslant \int \sum_{m=1}^{\infty} |A_{m,\alpha}||(1-\lambda^2)^m||(G_{2m})_\lambda|$$
$$\leqslant C_\alpha \sum_{m=1}^{\infty} m^{-\left(1+\frac{\alpha}{2}\right)}(1-\lambda^2)^m \int |(G_{2m})_\lambda|$$

for $0<\lambda\leqslant 1$; but $\int (G_{2m})_\lambda = \int |(G_{2m})_\lambda| = F((G_{2m})_\lambda)(0) = 1$, thus $X_\lambda \in L_1$ for all $0<\lambda\leqslant 1$.

Now we can write (11) as follows

$$\lambda^{\alpha-n}Y_\lambda = \delta + X_\lambda. \tag{12}$$

By (5), (6) and (12) we have

$$\lambda^{\alpha-n}\|f*(G_\alpha)_{\frac{1}{\lambda}}\|_{\mathrm{BMO}} = \lambda^{\alpha-n}\|f*G_\alpha*Y_\lambda\|_{\mathrm{BMO}}$$
$$= \|f*G_\alpha*\lambda^{\alpha-n}Y_\lambda\|_{\mathrm{BMO}}$$
$$= \|f*G_\alpha*(\delta+X_\lambda)\|_{\mathrm{BMO}}$$
$$= \|f*G_\alpha*\delta + f*G_\alpha*X_\lambda\|_{\mathrm{BMO}}$$
$$\leqslant \|f*G_\alpha\|_{\mathrm{BMO}} + \|f*G_\alpha*X_\lambda\|_{\mathrm{BMO}}$$
$$\leqslant \|f*G_\alpha\|_{\mathrm{BMO}} + \|X_\lambda\|_1\|f*G_\alpha\|_{\mathrm{BMO}}$$
$$\leqslant (1+\|X_\lambda\|_1)\|f*G_\alpha\|_{\mathrm{BMO}},$$

where we have used Lemma 1. Finally

$$\sup_{0<\lambda<1} \{\lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}}\} \leqslant C_\alpha \|J^{-\alpha} f\|_{\mathrm{BMO}}.$$

This completes the proof of Theorem 1. □

Theorem 2. *Let $f \in S'$ and $\alpha > 0$. Then the following are equivalent:*
(i) $J^{-\alpha} f \in \mathrm{BMO}$ *and*

$$\lim_{\lambda\to 0} \lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}} = 0,$$

(ii) $J^{-\alpha} f \in \mathrm{VMO}$.

Proof. Let $f \in S'$. The following equality can be easily checked:

$$\frac{1}{(1+|\xi|^2)^{\alpha/2}} - \frac{1}{(1+|\xi|^2)^{\alpha/2}} \frac{(1+\lambda^2|\xi|^2)^{\alpha/2} - \lambda^\alpha|\xi|^\alpha}{(1+\lambda^2|\xi|^2)^{\alpha/2}} = \frac{\lambda^\alpha}{(1+\lambda^2|\xi|^2)^{\alpha/2}} + \frac{\lambda^\alpha|\xi|^\alpha - \lambda^\alpha(1+|\xi|^2)^{\alpha/2}}{(1+|\xi|^2)^{\alpha/2}(1+\lambda^2|\xi|^2)^{\alpha/2}}. \tag{13}$$

It is known that there exists $\Phi_\alpha \in L_1$ such that

$$\frac{(1+|\xi|^2)^{\alpha/2} - |\xi|^\alpha}{(1+|\xi|^2)^{\alpha/2}} = F(\Phi_\alpha)(\xi) \tag{14}$$

for all $\xi \in \mathbb{R}^n$ (see [3], p. 134). It follows from (13) and (14) that

$$J^{-\alpha} f - J^{-\alpha} f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot)) = \lambda^\alpha f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot)) - \lambda^\alpha [f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))] * \Phi_\alpha. \tag{15}$$

Thus by Lemma 1 we have

$$\begin{aligned} &\|J^{-\alpha} f - J^{-\alpha} f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \\ &\quad\leqslant \lambda^\alpha \|f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} + \lambda^\alpha \|f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \|\Phi_\alpha\|_1 \\ &\quad= \lambda^\alpha \|J^{-\alpha}(f)_\lambda\|_{\mathrm{BMO}} (1 + \|\Phi_\alpha\|_1) \to 0 \quad \text{as } \lambda \to 0. \end{aligned} \tag{16}$$

On the other hand by (2) and using Lemma 1 together with the Lebesgue dominated convergence theorem one can see that the functions

$$g_\lambda = J^{-\alpha} f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))$$

belong to the space VMO for every $0 < \lambda \leqslant 1$. Thus, since the space VMO is a closed subspace of the space BMO, (16) implies $J^{-\alpha} f \in \mathrm{VMO}$. This proves the implication (i) ⇒ (ii).

Let $f \in S'$ and $J^{-\alpha} f \in \mathrm{VMO}$. It is not hard to check that

$$\begin{aligned}\frac{\lambda^\alpha}{(1+\lambda^2|\xi|^2)^{\alpha/2}} &= \frac{(1+|\xi|^2)^{\alpha/2}}{1+|\xi|^\alpha} \\ &\quad\times\Big[\frac{1}{(1+|\xi|^2)^{\alpha/2}} - \frac{1}{(1+|\xi|^2)^{\alpha/2}}\frac{(1+\lambda^2|\xi|^2)^{\alpha/2}-\lambda^\alpha|\xi|^\alpha}{(1+\lambda^2|\xi|^2)^{\alpha/2}} \\ &\qquad + \frac{\lambda^\alpha}{(1+\lambda^2|\xi|^2)^{\alpha/2}(1+|\xi|^2)^{\alpha/2}}\Big].\end{aligned}$$

There exists a function $\psi_\alpha \in L_1$ such that

$$F(\psi_\alpha)(\xi) + 1 = \frac{(1+|\xi|^2)^{\alpha/2}}{1+|\xi|^\alpha}, \tag{17}$$

for all $\xi \in \mathbb{R}^n$ (see [3], p. 134). Now we get from (17) and (14) that

$$\begin{aligned}\lambda^\alpha f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot)) &= \psi_\alpha * [J^{-\alpha} f - J^{-\alpha} f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))] \\ &\quad + [J^{-\alpha} f - J^{-\alpha} f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))] \\ &\quad + \psi_\alpha * (\lambda^\alpha[J^{-\alpha} f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))]) \\ &\quad + \lambda^\alpha[J^{-\alpha} f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))].\end{aligned} \tag{18}$$

Using Lemma 1, we obtain

$$\begin{aligned}&\|\psi_\alpha * (\lambda^\alpha[J^{-\alpha} f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))])\|_{\mathrm{BMO}} \\ &\quad\leqslant \lambda^\alpha \|\psi_\alpha\|_1 \|J^{-\alpha} f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \\ &\quad\leqslant \lambda^\alpha \|\psi_\alpha\|_1 \|J^{-\alpha} f\|_{\mathrm{BMO}} \|(\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))\|_1\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\|\lambda^\alpha[J^{-\alpha} f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))]\|_{\mathrm{BMO}} \\ &\quad= \lambda^\alpha \|J^{-\alpha} f * (\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \leqslant \lambda^\alpha \|J^{-\alpha} f\|_{\mathrm{BMO}} \|(\lambda^{-n} G_\alpha(\lambda^{-1}\cdot))\|_1.\end{aligned} \tag{20}$$

Thus (19) and (20) tend to zero as $\lambda \to 0$ and the last two terms on the right-hand side of (18) tend to zero (in BMO). Next, we will prove that the first two terms on the right-hand side of (18) also tend to zero (in BMO). In order to do that let us pick $B_1 = B(0,1)$ the unit ball centered at the origin, and $\delta > 0$, then we have

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|J^{-\alpha} f(x)|}{(1+|x|)^{n+\delta}}\,\mathrm{d}x &\leqslant \int_{\mathbb{R}^n} \frac{|J^{-\alpha} f(x) - (J^\alpha f)_{B_1}|}{(1+|x|)^{n+\delta}}\,\mathrm{d}x + \int_{\mathbb{R}^n} \frac{|(J^{-\alpha} f)_{B_1}|}{(1+|x|)^{n+\delta}}\,\mathrm{d}x \\ &\leqslant C\|J^{-\alpha} f\|_{\mathrm{BMO}} + C_1.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^n} \frac{|J^{-\alpha} f(x)|}{(1+|x|)^{n+\delta}} \, dx < \infty. \tag{21}$$

Next, we need to prove that the convolution of the function $J^{-\alpha} f$ with the function $\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot)$ is defined as an absolutely convergent integral. Observe that from (21) the function $J^{-\alpha} f * g$ exists as an absolutely convergent integral for any function $g \in L_1(\mathbb{R}^n)$ satisfying the estimate

$$|g(x)| \leqslant \frac{C}{|x|^{n+\delta}}, \tag{22}$$

for all $|x| > 1$ and some $\delta > 0$.

Next, we will show that the function Φ_α satisfies the condition (22). The following formula holds (see [3])

$$\Phi_\alpha(x) = -\sum_{m=1}^{\infty} A_{m,\alpha} G_{2m}(x). \tag{23}$$

Using formula (3) for the Bessel potential kernel G_{2m}, we obtain that for $|x| \geqslant 1$ and $0 < \delta \leqslant 1$,

$$\begin{aligned} G_{2m}(x) &= \frac{1}{(2\pi)^{n/2} 2^m (m-1)!} \int_0^\infty t^{(2m+\delta)/2} t^{-(n+\delta)/2} e^{-|x|^2/2t} e^{-t/2} \frac{dt}{t} \\ &\leqslant \frac{1}{(2\pi)^{n/2} 2^m (m-1)!} \sup_{t>0} \{t^{-(n+\delta)/2} e^{-|x|^2/2t}\} \int_0^\infty t^{(2m+\delta)/2} e^{-t/2} \frac{dt}{t} \\ &\leqslant \frac{2^{(2m+\delta)/2} \Gamma(\frac{1}{2}(2m+\delta))}{(2\pi)^{n/2} 2^m (m-1)!} \sup_{t>0} \{t^{-(n+\delta)/2} e^{-|x|^2/2t}\} \\ &\leqslant a_n \frac{\Gamma\left(\frac{1}{2}(2m+\delta)\right)}{(m-1)!} |x|^{-(n+\delta)}. \end{aligned} \tag{24}$$

Using Stirling's formula (see [4]) for the Gamma function, we have

$$\frac{\Gamma\left(\frac{1}{2}(2m+\delta)\right)}{(m-1)!} \leqslant c m^{\delta/2}, \tag{25}$$

for all $m \geqslant m_0$. In (25) the constant c does not depend on δ and m, and the constant m_0 does not depend on δ. It follows from (24) and (25) that there exists a constant $\widetilde{c}_{n,\delta}$ such that

$$G_{2m}(x) \leqslant \widetilde{c}_{n,\delta} m^{\delta/2} |x|^{-(n+\delta)}, \tag{26}$$

for all $m \geqslant 1$ and $|x| \geqslant 1$. Next (10), (23) and (26) show that the function Φ_α satisfies condition (22). Hence, the function given by

$$\lambda^{-n}|J^{-\alpha}f| * |\Phi_\alpha(\lambda^{-1}\cdot)|(x) = \int_{\mathbb{R}^n} \lambda^{-n}|J^{-\alpha}f(y)|\Big|\Phi_\alpha\Big(\frac{x-y}{\lambda}\Big)\Big|\,\mathrm{d}y$$

is well defined. Therefore, using Lemma 1, we have

$$\|J^{-\alpha}f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \leqslant \|J^{-\alpha}f\|_{\mathrm{BMO}}\|\Phi_\alpha\|_1.$$

Since

$$\int_{\mathbb{R}^n} \Phi_\alpha(y)\,\mathrm{d}y = F(\Phi_\alpha)(0) = 1,$$

we have

$$\begin{aligned}(27)\qquad &\|J^{-\alpha}f - J^{-\alpha}f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \\ &= \Big\|\int_{\mathbb{R}^n} (J^{-\alpha}f(\cdot) - J^{-\alpha}f(\cdot - y))\lambda^{-n}\Phi_\alpha\Big(\frac{y}{\lambda}\Big)\,\mathrm{d}y\Big\|_{\mathrm{BMO}} \\ &\leqslant \int_{\mathbb{R}^n} \lambda^{-n}\Big|\Phi_\alpha\Big(\frac{y}{\lambda}\Big)\Big|\|J^{-\alpha}f(\cdot) - J^{-\alpha}f(\cdot - y)\|_{\mathrm{BMO}}\,\mathrm{d}y,\end{aligned}$$

where we have used the estimate (1). Since $J^{-\alpha}f \in \mathrm{VMO}$ and

$$\lim_{\lambda\to 0^+} \lambda^{-n}\int_{|y|>\varepsilon} |\Phi_\alpha\Big(\frac{y}{\lambda}\Big)|\,\mathrm{d}y = 0,$$

for every $\varepsilon > 0$, we get from (27)

$$(28)\qquad \lim_{\lambda\to 0^+} \|J^{-\alpha}f - J^{-\alpha}f * (\lambda^{-n}\Phi_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} = 0.$$

Finally, from (18), (19), (20) and (28) we have

$$\lambda^\alpha\|J^{-\alpha}f\|_{\mathrm{BMO}} = \lambda^\alpha\|f * (\lambda^{-n}G_\alpha(\lambda^{-1}\cdot))\|_{\mathrm{BMO}} \to 0,$$

as $\lambda \to 0$ and the proof of Theorem 2 is complete. □

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