Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 2, 271–278

Persistent URL: `http://dml.cz/dmlcz/128326`

ATTRACTIVE PROPERTIES VIA GENERALIZED DERIVATIVES OF CONTINUOUS FUNCTIONS

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(Received November 16, 1989)

1. INTRODUCTION

Let f be a continuous function mapping a closed interval I into itself and let x_0 be a fixed point of f. An elementary theorem asserts that if $|f'(x_0)| < 1$, then there is a neighborhood U of x_0 such that for each $x \in U$, the sequence $\{f^n(x)\}$ of iterates of x under f converges to x_0. Here $f^0(x) = x$ and $f^n(x) = f^{n-1}(x)$, $n = 1, 2, \ldots$.

A variant of this theorem can be obtained by replacing the condition on the derivative by a condition on the difference-quotient: if there is a number $d \in (0,1)$ such that

$$-d < \frac{f(x) - f(x_0)}{x - x_0} < d \tag{1}$$

for all $x \in U$, then $\lim\limits_{n \to \infty} f^n(x) = x_0$ for all $x \in U$.

Suppose, now, that we replace the full neighborhood U of x_0 with some other set E. We can ask whether the condition $|f'_E(x_0)| < 1$ where $f'_E(x_0)$ denotes the derivative of f with respect to E at x_0, or the condition (1) holding for all $x \in E$, still implies some sort of attraction to the fixed point x_0. The strength of the conclusion will depend, of course, on the set E. It is, perhaps surprising that porosity conditions, rather than density conditions, are the relevant ones.

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* This paper was prepared while the second listed author visited University of Carolina at Davis.

Suppose x_0 is a fixed point of f and $0 < f'(x_0) < 1$. It is then clear that for x sufficiently close to x_0 the sequence $\{f^n(x)\}$ approaches x_0 monotonically. Suppose we weaken the assumption to $0 < f'_{ap}(x_0) < 1$. This means that there is a set E having x_0 as a point of density that

$$0 < \lim_{\substack{x \to x_0 \\ x \in E}} \frac{f(x) - f(x_0)}{x - x_0} < 1.$$

If $x \in E$ and $f^n(x) \in E$ for all $n = 1, 2, \ldots$, then $f^n(x) \to x_0$. Perhaps the fact that E has x_0 as a density point suggests there must be an x attracted to x_0 whose orbit lies entirely in E. We shall see that this is not necessarily the case, although there will be a point not necessarily in E, whose orbit approaches x_0. Our goal in this section is to obtain two theorems. The first provides a condition on a set E that guarantees that any continuous function such that $0 < \lim\limits_{x \to x_0;\, x \in E} \frac{f(x)-f(x_0)}{x-x_0} = d < 1$ has some point x (not necessarily in E) whose orbit is attracted to the fixed point x_0 without landing on x_0. The second theorem provides a condition that guarantees that if $0 < \lim\limits_{x \to x_0;\, x \in E} \frac{f(x)-f(x_0)}{x-x_0} < 1$ then there is a point $x \in E$ whose orbit is attracted to x_0 without landing on x_0. Actually each theorem reveals a bit more.

We begin with an example that may be instructive.

Example 2.1. Assume that $0 \in \operatorname{int}(I)$, where $\operatorname{int}(I)$ denotes the interior of the interval I. Then there exists a continuous function $f\colon I \to I$ and a set E such that $f(0) = 0$, 0 is a right point of E, $0 < \lim\limits_{x \searrow 0;\, x \in E} \frac{f(x)}{x} = \frac{1}{2}$, and $|f^n(x)| > \delta_x > 0$, $n = 1, 2, \ldots$, for every $x \in E$, $x > 0$. The notation $x \searrow y$ means that x converges to y from the right.

Verification of Example 2.1. Before turning to the details we want to sketch the idea of our example. We split each interval $[2^{-n}, 2^{-n+1}]$ into 2^n subintervals $I_{n,m}$, $n = 1, 2, \ldots$, $m = 1, 2, \ldots, 2^n$ and we choose small subintervals $J_{n,m}$ about the endpoints of $I_{n,m}$. The set E will be $(0, 1] \setminus \bigcup\limits_{n,m} J_{n,m}$ and the function f will be constant on each set $E \cap I_{n,m}$. These constants will be choosen so that we can obtain $0 < \lim\limits_{x \searrow 0;\, x \in E} \frac{f(x)}{x} = \frac{1}{2}$ and each constant will equal a fixed point of f. Hence for any $x \in E$ we have $f^k(x) = f(x)$, $k = 1, 2, \ldots$, that is, no point of E will be attracted to 0.

Put $I_{n,m} = [2^{-n} + (m-1)4^{-n}, 2^{-n} + m4^{-n}]$ and $J_{n,m} = \big(2^{-n} + (m - 4^{-n})4^{-n}, 2^{-n} + (m + 4^{-n})4^{-n}\big)$ for $n = 1, 2, \ldots$, $m = 1, 2, \ldots, 2^n$. Put $E = (0, 1] \setminus \bigcup\limits_{n,m} J_{n,m}$ and $f(x) = \frac{1}{2}(2^{-n} + m4^{-n})$ for $x \in I_{n,m} \cap E$, $n = 1, 2, \ldots$, $m = 1, 2, \ldots, 2^n$. We also put $f(2^{-n}+m4^{-n})=2^{-n}+m4^{-n}$ for $n=1,2,\ldots, m=1,2,\ldots,2^n$. Otherwise define f arbitrarily so as to be continuous. Then for every $x\in E$ there exist n and m such that $x\in I_{n,m}$. Thus $f(x)=\frac{1}{2}(2^{-n}+m4^{-n})=2^{-n-1}+2m4^{-n-1}$ and hence $f^k(f(x))=f(x)$, $k=1,2,\ldots$. As the reader easily can verify 0 is a right density point of E and $\lim\limits_{x\searrow 0;\, x\in E}\frac{f(x)}{x}=\frac{1}{2}$.

We remark that it will follow from Theorem 2.1 below that there is one point x not in E whose orbit is attracted to 0.

Lemma 2.1. *Suppose that $f(0)=0\in int(I)$, $0<f(x)\leqslant x$ for $x>0$, $f\colon I\to I$ is continuous, there exists $a_n\searrow 0$ such that $\frac{f(a_n)}{a_n}<d\in(0,1)$, and $a_{n+1}/a_n>d$. We also assume that an interval $[c_0,b_0]$ is given such that $0<c_0<d\cdot b_0\leqslant a_1$, $f([c_0,b_0])\cap[c_0,b_0]\neq\emptyset$. Then $0\in\operatorname{cl}\bigl(\cup\{f^n([c_0,b_0])\colon n=1,2,\ldots\}\bigr)$.*

Proof. Since f is continuous $f^n([c_0,b_0])$ is an interval denoted by $[c_n,b_n]$. From $0<f(x)\leqslant x$ it follows that $0\leqslant\ldots\leqslant c_n\leqslant c_{n-1}\leqslant\ldots<c_0$, and $0\leqslant\ldots\leqslant b_n\leqslant b_{n-1}\leqslant\ldots\leqslant b_0$. From $f([c_0,b_0])\cap[c_0,b_0]\neq\emptyset$ it follows that there exist $z,w\in[c_0,b_0]$ such that $f(w)=z$. Then $f^{n+1}(w)=f^n(z)\in[c_{n+1},b_{n+1}]\cap[c_n,b_n]$ for every $n\geqslant 1$. Thus $\bigcup\limits_{k=0}^{n}f^k([c_0,b_0])=[c_n,b_0]$.

To obtain a contradiction suppose that $\lim\limits_{n\to\infty}c_n=\gamma>0$. Then $\bigcup\limits_{k=0}^{\infty}([c_k,b_k])=\bigcup\limits_{k=0}^{\infty}f^k([c_0,b_0])\supset(\gamma,b_0]$. We also have $f(\gamma)=\gamma$ since otherwise $f(\gamma)<\gamma$ and by the continuity of f there would exist a c_n such that $f(c_n)<\gamma$ which is clearly impossible. If there is $x\in(\gamma,b_0]$ such that $f(x)<\gamma$ then from $(\gamma,b_0]\subset\bigcup\limits_{k=0}^{\infty}[c_k,b_k]$ it follows that $x\in[c_k,b_k]$ for a $k\in\mathbb{N}$. This would imply $c_{k+1}\leqslant f(x)<\gamma$ which again contradicts the definition of γ. Therefore $f(x)\geqslant\gamma$ for $x\in(\gamma,b_0]$. Since $a_{n+1}/a_n>d$ and $b_0<a_1$ there exists an n such that $\gamma<a_n\leqslant\gamma/d$. By the assumption of this lemma $f(a_n)<d\cdot a_n$. On the other hand also by the assumption of this lemma $\gamma\leqslant c_0<d\cdot b_0$ and hence $\gamma/d\in(a_n,b_0]$. Thus $f(a_n)\geqslant\gamma=d\cdot\gamma/d\geqslant d\cdot a_n$, a contradiction proving $\lim\limits_{n\to\infty}c_n=0$, which implies $0\in\operatorname{cl}\bigl(\cup\{f^n([c_0,b_0])\colon n=1,2,\ldots\}\bigr)$. □

Lemma 2.2. *If $f(0)=0$, $0<f(x)\leqslant x$ for $x>0$, f is continuous, and there exist $0<c_0<b_0$ such that $0\in\operatorname{cl}\bigl(\cup\{f^n([c_0,b_0])\colon n=1,2,\ldots\}\bigr)$ then there exists $y\in[c_0,b_0]$ such that $\lim\limits_{n\to\infty}f^n(y)=0$ and $f^n(y)>0$ for $n=1,2,\ldots$.*

Proof. From the continuity of f^n it follows that $f^n([c_0,b_0])$ is an interval $[c_n,b_n]$. Since $f(x)\leqslant x$ we have $c_0\geqslant c_1\geqslant\ldots\geqslant c_n\geqslant\ldots>0$, $b_0\geqslant b_1\geqslant\ldots\geqslant b_n\geqslant\ldots>0$. The assumption of this lemma implies that $\lim\limits_{n\to\infty}c_n=0$. If there exists c_n such that $c_n=c_{n+1}$ then $f([c_n,b_n])=[c_{n+1},b_{n+1}]=[c_n,b_{n+1}]\subset[c_n,b_n]$ and hence $f^m([c_n,b_n])\subset f^{m-1}([c_n,b_n])\subset f([c_n,b_n])\subset[c_n,b_n]$ contradicting the fact that

$\lim_{n\to\infty} c_n = 0$. Thus $c_0 > c_1 > \ldots > c_n > \ldots > 0$. Put $F_n = f^{-n}([c_n, c_{n-1}]) \cap [c_0, b_0]$. If we have finitely many F_n's say $F_{n_1}, F_{n_2}, \ldots, F_{n_m}$, $n_1 > n_2 > \ldots > n_m$ then we show that their intersection is nonempty. Choose a $z \in [c_{n_m}, c_{n_m-1}) \cap [c_{n_m}, b_{n_m}]$. Then there exists a $u \in F_{n_m}$ such that $f^{n_m}(u) = z$. Suppose that there exists a j such that $u \notin F_{n_j}$, that is, $f^{n_j}(u) \notin [c_{n_j}, c_{n_j-1}]$. Thus $f^{n_j}(u) \in (c_{n_j-1}, b_{n_j}] \subset [c_{n_j-1}, b_{n_j-1}]$ and hence there exists a $w \in [c_0, b_0]$ such that $f^{n_j}(u) = f^{n_j-1}(w)$ and hence $z = f^{n_j+(n_m-n_j)}(u) = f^{n_j-1+(n_m-n_j)}(w) = f^{n_m-1}(w) \in [c_{n_m-1}, b_{n_m-1}]$ contradicting to $z \in [c_{n_m}, c_{n_m-1})$. Thus by compactness there exists a $y \in \cap_{n=1}^{\infty} F_n$. Since $f^n(y) \in [c_n, c_{n-1}]$ we obtain $\lim_{n\to\infty} f^n(y) = 0$. □

Lemma 2.3. *Suppose that $f(0) = 0$, f is continuous, and there exist $q_n, r_n \to 0$, $q_1 > r_1 > q_2 > r_2 > \ldots > q_k > r_k > \ldots > 0$, such that $f(q_k) > 0$, and $f(r_k) = 0$, $k = 1, 2, \ldots$ then there exists a nonempty perfect set P such that $\lim_{n\to\infty} f^n(y) = 0$, $f^n(y) > 0$, for every $y \in P$, $n = 1, 2, \ldots$.*

Proof. Put $J_\emptyset = [r_1, q_1]$. Suppose that s is a finite zero-one sequence of length $m \in \{0, 1, 2, \ldots\}$ (when $m = 0$ then $s = \emptyset$) and $J_s = \left[r_{n(s)}, q_{n(s)}\right]$. Since $f(q_{n(s)}) > 0$ we can choose $n(s0) > n(s1) > n(s)$ such that $J_{s0} = \left[r_{n(s0)}, q_{n(s0)}\right] \subset \left[0, f(q_{n(s)})\right]$, and $J_{s1} = \left[r_{n(s1)}, q_{n(s1)}\right] \subset \left[0, f(q_{n(s)})\right]$.

Put $F_\emptyset = J_\emptyset$. Suppose that s is of the length $n-1$, the closed set F_s is defined, and $f^{n-1}(F_s) = J_s$. Then put $F_{s0} = f^{-n}(J_{s0}) \cap F_s$ and $F_{s1} = f^{-n}(J_{s1}) \cap F_s$. From $J_{s0} \subset \left[0, f(q_{n(s)})\right]$ it follows that for every $x \in J_{s0}$ there exists a $z \in J_s$ such that $f(z) = x$. Since $f^{n-1}(F_s) = J_s$ we can find a $w \in F_s$ such that $f^{n-1}(w) = z$. Thus $f^n(w) = x$ and $w \in F_{s0}$. This implies that $f^n(F_{s0}) = J_{s0}$. A similar argument shows that $f^n(F_{s1}) = J_{s1}$.

Put $H_m = \cup\{F_s : s$ is a zero-one sequence of length $m\}$ and $H = \cap_{m=1}^{\infty} H_m$. Since the sets F_s are closed and there are 2^m zero-one sequences of length m the set H_m is closed. Therefore H is also closed.

Suppose that $y \in H$. Then $f^m(y) \in \left[r_{n(s)}, q_{n(s)}\right]$, where s is of lenght m. It is easy to see that $n(s) > m$. Thus $f^m(y) \to 0$ and $f^m(y) > 0$.

If φ is an infinitive zero-one sequence then denote by $\varphi|m$ the first m terms of φ.

For arbitrary zero-one sequence φ choose a $y(\varphi) \in \cap_{m=1}^{\infty} F_{\varphi|m} \in H$. We remark that $y(\varphi)$ is well defined since $F_{\varphi|m}$, $m = 1, 2, \ldots$ is a nested sequence of closed sets. If φ, ψ are different zero-one sequences then there exists an m such that $\varphi|m = \psi|m = s$ and $\varphi|m+1 \neq \psi|m+1$. By symmetry we may assume that $\varphi|m+1 = s0$ and $\psi|m+1 = s1$. Then $y(\varphi) \in F_{s0}$ and $y(\psi) \in F_{s1}$. Since $f^{m+1}\left(y(\varphi)\right) \in J_{s0}$, $f^{m+1}\left(y(\psi)\right) \in J_{s1}$ and $J_{s0} \cap J_{s1} = \emptyset$ we obtain $y(\varphi) \neq y(\psi)$. Therefore the cardinality of H is that of the continuum and there exists a nonempty perfect subset $P \subset H$. This proves Lemma 2.3. □

It will be convenient to state our results in the language of porosity. Let S be a set, x_0 a point and $h > 0$. Let $\ell(h)$ denote the length of the longest interval in

$[x_0, x_0 + h] \setminus S$. The right porosity of S at x_0 is defined as

$$p_+(x_0, S) = \limsup_{h \to 0} \frac{\ell(h)}{h}.$$

If $p_+(x_0, S) = 0$, we say S is *nonporous from the right* at x_0. If $p_+(x_0, S) = 1$ we say S is *strongly porous from the right at* x_0. The notions of left porosity and (bilateral) porosity are now defined in the obvious way.

Theorem 2.1. *If x_0 is a fixed point of the continuous function f and*

$$0 < \frac{f(x) - f(x_0)}{x - x_0} < d < 1$$

for all x in set E such that $p_+(E, x_0) < 1 - d$, then there exists y such that $\lim_{n\to\infty} f^n(y) = x_0$ and for every n, $f^n(y) \neq x_0$.

Proof. Replacing f by $f(x + x_0) - x_0$ we may assume that $x_0 = 0$. The porosity condition now implies the existence of a strictly decreasing sequence $\{a_n\}$ such that $a_n \to 0$, $0 < f(a_n)/a_n < d$, and $a_{n+1}/a_n > d$.

If there exist $x_n \searrow 0$ such that $f(x_n) \leqslant 0$ then using $0 < f(a_n)$ one can choose $q_n, r_n \to 0$, $q_1 > r_1 > \ldots > q_k > r_k > \ldots > 0$, such that $f(q_k) > 0$, $f(r_k) = 0$ for $k = 1, 2, \ldots$. In this case applying Lemma 2.3 we can complete the proof of Theorem 2.1.

Thus we may assume that there exists $\delta > 0$ such that $0 < f(x)$ for $x \in (0, \delta]$. Put

$$f_1(x) = \begin{cases} f(x), & \text{if } \; f(x) \leqslant x, \quad \text{and} \quad 0 \leqslant x < \delta; \\ x, & \text{if } \; f(x) > x, \quad \text{and} \quad 0 \leqslant x < \delta; \\ f(x), & \text{if } \; x < 0; \\ \min(\delta, f(\delta)), & \text{if } \; \delta \leqslant x. \end{cases}$$

Since $a_n \searrow 0$ there exists $m \in \mathbb{N}$ such that $a_m < \delta$. Put $a'_n = a_{n+m}$ for $n = 1, 2, \ldots$.

Obviously $0 < f_1(x) \leqslant x$ for $x > 0$. Put $b_0 = a'_1$ and choose a $c_0 \in \big(0, f_1(b_0)\big]$. Then $f_1([c_0, b_0]) \cap [c_0, b_0] \neq \emptyset$. Applying Lemma 2.1 for f_1, a'_n and $[c_0, b_0]$ we obtain that $0 \in \mathrm{cl}\big(\cup\{f_1^n([c_0, b_0]) \colon n = 1, 2, \ldots\}\big)$. Therefore we can apply Lemma 2.2 for f_1 and hence there exists $y \in [c_0, b_0]$ such that $\lim_{n\to\infty} f_1^n(y) = 0$ and $f_1^n(y) > 0$ for $n = 1$, $2, \ldots$. From $0 < f_1^n(x) \leqslant x$ it follows that $0 < \ldots \leqslant f_1^{n+1}(y) \leqslant f_1^n(y) \ldots \leqslant y$. If $f_1^{n+1}(y) = f_1^n(y)$ for an $n \in \mathbb{N}$ then $f_1^{\ell+1}(y) = f_1^{n+1+(\ell-n)}(y) = f_1^{n+(\ell-n)}(y) = f_1^{\ell}(y)$ for every $\ell \geqslant n$ and this would contradict the fact that $\lim_{n\to\infty} f_1^n(y) = 0$. Therefore $0 < f_1\big(f_1^n(y)\big) < f_1^n(y)$ for $n = 0, 1, \ldots$. Since $0 < \ldots < f_1^n(y) < \ldots < y < b_0 = a_{m+1} \leqslant a_m < \delta$ we obtain that $0 < f_1^n(y) < \delta$ and $f_1(f_1^n)(y) < f_1(y)$ $n = 1, 2, \ldots$. From the definition of f_1 it follows that $f^n(y) = f_1^n(y)$ for $n = 1, 2, \ldots$. Thus $\lim_{n\to\infty} f^n(y) = 0$ and this completes the proof of Theorem 2.1. □

Example 2.2 below shows that the porosity condition in Theorem 2.1 cannot be weakened.

Example 2.2. Suppose that $0 < \varepsilon < \frac{1}{2}$. Put $f(x) = 2^{-n}$ for $x \in \left[2^{-n}, \frac{2^{-n+1}}{1+\varepsilon}\right]$, $n = 1, 2, \ldots$. If $\frac{2^{-n+1}}{1+\varepsilon} < x < 2^{-n+1}$, $n = 1, 2, \ldots$ choose $2^{-n} \leqslant f(x) \leqslant 2^{-n+1}$ so that it continuously connects $f(\frac{2^{-n+1}}{1+\varepsilon}) = 2^{-n}$ and $f(2^{-n+1}) = 2^{-n+1}$. If $x \leqslant 0$ or $x > \frac{2}{1+\varepsilon}$ choose f so that $f(0) = 0$, $f(\frac{2}{1+\varepsilon}) = 1$ and f is continuous. Then the porosity of $E = \left\{x : f(x) < \left(\frac{1}{2} + \varepsilon\right)x\right\}$ at 0 is no greater than $\frac{1}{2}$ since $f(\frac{2^{-n+1}}{1+\varepsilon}) = \frac{1+\varepsilon}{2}\left(\frac{1}{1+\varepsilon} \cdot 2^{-n+1}\right) < \left(\frac{1}{2} + \varepsilon\right)\frac{2^{-n+1}}{1+\varepsilon}$ $n = 1, 2, \ldots$. It is also clear that each interval $[2^{-n}, 2^{-n+1}]$ is mapped into itself and hence no point is attracted from the right to 0.

We saw Example 2.1 that even when E is nonporous at x_0 and $f'_E(x_0) = \frac{1}{2}$ there may be no point in E whose orbit is attracted from the right to x_0. Observe that $\mathbb{R} \setminus E$ was also nonporous from the right at x_0 in that example. One can rectify this flaw by requiring $\mathbb{R} \setminus E$ to be strongly porous from the right at x_0.

Theorem 2.2. *Suppose E is nonporous from the right at x_0 and $\mathbb{R} \setminus E$ is strongly porous from the right at x_0. If f is a continuous function having x_0 as a fixed point and satisfies an inequality*

$$0 < d_1 < \frac{f(x) - f(x_0)}{x - x_0} < d_2 < 1$$

for all x in E then there exists $y \in E$ such that $\lim_{n\to\infty} f^n(y) = x_0$ and $f^n(y) > 0$ for $n = 1, 2, \ldots$. In particular, if $f'_E(x_0)$ exists and $0 < f'_E(x_0) < 1$, the conclusion follows.

Proof. Replacing f by $f(x + x_0) - x_0$ we can reduce the proof to the case that $x_0 = 0$. The assumptions of the theorem imply that there exist $c_0 < b_0$, and a sequence $\{a_n\}$ converging monotone decreasingly to 0 such that for each $n, a_n \in E$, $a_{n+1}/a_n > d_2$, $[c_0, b_0] \subset E \cap [0, a_1]$ and $\frac{c_0}{b_0} < d_1$. Put

$$f_1(x) = \begin{cases} f(x), & \text{if} \quad f(x) \leqslant x, \quad \text{and} \quad 0 \leqslant x; \\ x, & \text{if} \quad f(x) > x, \quad \text{and} \quad 0 \leqslant x; \\ f(x), & \text{if} \quad x < 0. \end{cases}$$

Observe that $f_1(a_n)/a_n < d_2$, $c_0 < d_1 b_0 < d_2 b_0 < b_0 \leqslant a_1$ and finally $f_1(b_0)/b_0 > d_1 > c_0/b_0$. Therefore $f_1(b_0) > c_0$, and hence $f_1([c_0, b_0]) \cap [c_0, b_0] \neq \emptyset$. Thus we can apply Lemma 2.1 and Lemma 2.2 for f_1 with $\{a_n\}$, d_2, c_0, b_0 and find a point $y \in [c_0, b_0] \subset E$ such that $\lim_{n\to\infty} f_1^n(y) = 0$, $f_1^n(y) > 0$, $n = 0, 1, \ldots$. If there exists an $n \geqslant 1$ such that $f_1^n(y) \neq f^n(y)$ then $f_1(f_1^{n-1}(y)) = f_1^{n-1}(y)$ and hence $f_1^m(y) = f_1^{m-n+1}(f_1^{n-1}(y)) = f_1^{n-1}(y)$ which contradicts to $\lim_{n\to\infty} f_1^n(y) = 0$. Thus $f_1^n(y) = f^n(y)$ and this proves Theorem 2.2. □

E x a m p l e 2.3. In this example we want to show that in Theorem 2.2 the condition $0 < d_1 < \frac{f(x)-f(x_0)}{x-x_0} < d_2 < 1$ cannot be replaced by $0 < \frac{f(x)-f(x_0)}{x-x_0} < 1$. Put

$$[c_n, b_n] = \left[\frac{1}{n}\frac{1}{(n+1)!}, \frac{1}{n}\frac{1}{n!}\right]$$

and $E = \cup_{n=1}^{\infty}[c_n, b_n]$. Then E is non porous from the right at 0 and $\mathbb{R} \setminus E$ is strongly porous from the right at 0. Put $f(\frac{b_{n+1}+c_n}{2}) = \frac{b_{n+1}+c_n}{2}$, $n = 1, 2, \ldots$, and $f(x) = \frac{b_{n+1}+c_n}{2}$ for $x \in [c_n, b_n]$, $n = 1, 2, \ldots$. Otherwise choose f so that it is continuous on $\mathbb{R}$. Then, obviously no point from E is attracted to 0.

3. Results for "typical" continuous functions

The previous results show that adequate behavior of the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$ on sets satisfying certain porosity conditions with respect to a fixed point x_0 will guarantee certain attractive behavior to x_0. But "must" continuous functions will not exhibit this behavior with respect to any fixed point [BH]. In this section, we make this statement precise and show that, nonetheless, most continuous functions have the property that to each fixed point x_0 corresponds a nonempty perfect set $P(x_0)$ such that for each $y \in P(x_0)$, $\lim_{n\to\infty} f^n(y) = x_0$ with $f^n(y) \neq x_0$ for each $n = 1, 2, \ldots$.

Let $\mathcal{C}$ denote the class of continuous functions mapping I into I. When $\mathcal{C}$ is furnished with the sup norm it becomes a complete metric space.

Theorem 3.1. a) *Let A consist of those functions f in $\mathcal{C}$ such that for each $x_0 \in I$ the set*

$$S(x_0) = \left\{x : \left|\frac{f(x) - f(x_0)}{x - x_0}\right| < 1\right\}$$

is bilaterally strongly porous at x_0. Then A is a residual subset of $\mathcal{C}$.

b) *Let B consist of those functions f in $\mathcal{C}$ such that to every fixed point x_0 of f corresponds a nonempty perfect set $P(x_0)$ such that $\lim_{n\to\infty} f^n(y) = x_0$ for $y \in P(x_0)$ and $f^n(y) \neq x_0$ for all n. Then B is a residual subset of $\mathcal{C}$.*

P r o o f. a) In [BH] one finds that for f in some residual subset D of $\mathcal{C}$, the set $S = \{x : f(x) = g(x)\}$ is bilaterally strongly porous at each point of S whenever f satisfies a Lipschitz condition. If for some $x_0 \in I$ the set $S(x_0)$ in part a) of the statement fails to be bilaterally strongly porous at x_0, we can argue as in the proof of [BH] Thm. 3.2, that is, we can find a Lipschitz function g that agrees with f on a subset of $S(x_0)$ that is also not strongly porous at x_0. It follows that $D \subset A$ so A is residual in $\mathcal{C}$.

To prove b) assume that $I = [a_0, b_0]$ and put $\mathcal{C}_1 = \{f \in \mathcal{C} : f(a_0) \neq a_0$ and $f(b_0) \neq b_0\}$. Obviously $\mathcal{C}_1$ is a residual subset of $\mathcal{C}$.

Assume that $J = [a, b] \subset I$ is given. Denote by $M(J)$ the set of functions $f \in \mathcal{C}_1$ such that the maximum of f on J is achieved at a fixed point of f and this fixed point is different from b. First we show that for a given J the set $M(J)$ is nowhere dense in $\mathcal{C}_1$. Let $N(f, \varepsilon)$ be the sphere in $\mathcal{C}_1$ that is centered at f and has radius $\varepsilon > 0$. We find a sphere $N(g, \delta)$ inside $N(f, \varepsilon)$ whose intersection with $M(J)$ is empty. Without loss of generality, suppose $f \in M(J)$ with $f(c) = c = \max\limits_{x \in J} f(x)$, $a \leqslant c < b$. We may also assume $\varepsilon < b - c$ and choose $d \in (c, b)$ with $d - c < \varepsilon/2$. Let $g \in N(f, \varepsilon/2)$ satisfy $d < g(c) < f(c) + \varepsilon/2$, $g(x) \leqslant c$ for $x \geqslant d$. This is possible since $f \leqslant c$ on J. Let u be the maximum fixed point of g in J. Then $c < u < d$. Choose $\delta \in (0, \varepsilon/2)$ such that $g(c) - \delta > d$ and $g(x) + \delta < d$ for $x \geqslant u$. If $h \in N(g, \delta)$ then $h(c) > d$ and h has no fixed points greater than $d \in J$. Thus $N(g, \delta) \cap M(J)$ is empty and $N(g, \delta) \subset N(f, \varepsilon)$. It follows that $M(J)$ is nowhere dense in $\mathcal{C}_1$.

By applying the above result to each interval with rational endpoints and by applying a similar argument for "minimum" instead of "maximum", we find that the set of functions E in $\mathcal{C}_1$ that achieve a local extremum at a fixed point is first category in $\mathcal{C}_1$ and hence these functions form a first category subset of $\mathcal{C}$.

Now according to a theorem of Jarnik [B: p. 213], each function in some residual subset S of $\mathcal{C}$ has every real number as a derived number at every point. This means that for $f \in S$, $x_0 \in I$ and $t \in \mathbb{R}$, there exists a sequence x_n converging to x_0 such that $\big(f(x_n) - f(x_0)\big)/(x_n - x_0)$ converges to t. Let f be in $(\mathcal{C}_1 \cap S) \setminus E$. Applying Lemma 2.3 to f (with respect to x_0 and $f(x_0)$ instead of 0 and $f(0)$) we find f in the required set B. Thus the residual set $(\mathcal{C}_1 \cap S) \setminus E$ is contained in B so B is residual in $\mathcal{C}$. This concludes our proof. $\square$

Finally we remark that using similar techniques to those in Chapter VIII of [B] one can show that for the typical $f \colon I \to I$ the fixed point set is a nowhere dense non-empty perfect set.

References

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