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DECOMPOSITION OF VECTOR SPACES AND APPLICATION TO THE STOKES PROBLEM IN ARBITRARY DIMENSION

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1. Introduction and preliminaries

Let Ω be bounded domain of $\mathbb{R}^d$ $(d \geqslant 2)$ with boundary Γ; let f be a given function defined in Ω and φ and g two given functions defined on Γ, φ and g satisfying adequate compatibility conditions. Recall that the Stokes problem consists in finding a pair of functions (u, p) solution of:

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega, \\ \operatorname{div} u &= \varphi \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma. \end{aligned}$$

In two dimensions, this system is fairly simple because it can be reduced to a biharmonic equation. In higher dimensions, this problem is substantially more difficult; it has been studied by many authors, from different points of view, and it would be too long to list them all here. But to our knowledge, Cattabriga [13] was the first to establish complete results of existence and regularity of the solution in the case of an open subset of $\mathbb{R}^3$; he achieved this by using techniques of integral representations. Yudovich [40] and later on Solonnikov [33] obtained similar results by other approaches. About the same period, Geymonat [17] solved general elliptic systems, that are extensions of the Stokes problem. More recently, Ghidaglia [18] studied, in arbitrary dimensions, another generalization of the Stokes problem by means of differential quotients.

The purpose of this paper is to propose a new approach to establish existence, uniqueness and regularity of the solution of the Stokes problem in $\mathbf{W}^{m,r}(\Omega) \times W^{m-1,r}(\Omega)$ for $m \geqslant 0$, namely by linking these results to a Helmholtz decomposition of vector fields. Besides the fact that Helmholtz decompositions are very interesting as such (particularly, in practical applications), this approach has the advantage of leading to a well-constructed theory, the arguments developed here being not only straightforward, but also applicable to higher-order problems. Another advantage is that it applies to domains of arbitrary dimensions, in many cases with optimal regularity assumptions on the boundary and also to singular data. A sketch of this theory has been announced in a note by the authors [5] and the complete proofs are given in the report [6].

The regularity of the solution of the continuity equation:

$$\begin{aligned} \operatorname{div} u = \varphi \quad \text{in } \Omega, \\ u = g \quad \text{on } \Gamma, \end{aligned}$$

plays a fundamental part in studying the Helmholtz decomposition of vector fields. The best-known contribution in this area is that of Bogovskii in [9] and [10], who established regularity by considering an integral representation of the solutions. In this paper we use a different argument, that does not require integral representations. Instead, we use a powerful equivalence of norms proved by Nečas in [27], on which this work is based, and theorems on the trace of the divergence, proved by Héron in [22]. One advantage of proceeding thus is that it extends, the results of Bogovskii, and that of latter authors such as Borchers & Sohr [11], to singular data on the boundary.

An outline of the paper is as follows. Paragraph 2 derives a simplified version of De Rham's theorem [30] which has here two important applications. First, it permits to characterize a family of "divergence-free" function spaces. The lower order spaces are associated with the Stokes operator and the higher order spaces are associated with "generalized Stokes" operators, where the Laplace operator is replaced by the biharmonic (or higher-order operators) and boundary conditions on the normal derivative (or higher-order normal derivatives) are added to the standard Dirichlet boundary condition. We refer to [7] for a study of "generalized Stokes" operators. A second very useful application is a characterization of some distributions by means of their gradient and is an extension of Nečas' theorem [27]. Some of the results of this paragraph have been announced in a note by Amrouche [4]. We also refer to a recent work of Simon [31] for a different proof of De Rham's theorem and its consequences.

Paragraph 3 uses the results of Paragraph 2 to show that the divergence operator is an isomorphism between adequate spaces, which is a generalization of the well-known "inf-sup" condition of Babuška [8] and Brezzi [12]. This, combined with an important result of Héron [22] (that expresses the traces of the divergence of functions in $\mathbf{W}^{m,r}(\Omega)$), allows to construct functions in $\mathbf{W}^{m,r}(\Omega)$ with prescribed divergence and trace.

Paragraph 4 is devoted to the Stokes problem. For $m \geqslant 2$, we first establish a Helmholtz decomposition of the space $\mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$ by applying a result of Agmon, Douglis & Nirenberg [3] valid for elliptic systems. From this we derive the existence and uniqueness of the solution of Stokes problem in $\mathbf{W}^{m,r}(\Omega) \times W^{m-1,r}(\Omega)$. By a duality argument introduced by Lions & Magenes [25], this result carries over to $m = 0$ and by interpolating between $m = 2$ and $m = 0$, we complete the case $m = 1$ and arbitrary r. In turn, this permits to establish a Helmholtz decomposition of the space $\mathbf{W}_0^{1,r}(\Omega)$.

We end with a short Paragraph 5 that decouples the pressure from the velocity by a penalty method.

In the sequel, r denotes a real number such that $1 < r < \infty$ and r' stands for its conjugate: $1/r + 1/r' = 1$. Recall that $\mathscr{D}(\Omega)$ is the space of $\mathscr{C}^\infty$ functions with compact support in Ω and $\mathscr{D}'(\Omega)$ is its dual space. For any multi-index k in $\mathbb{N}^d$, we denote by ∂^k the differential operator of order k:

$$\partial^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_d^{k_d}}, \quad \text{with} \quad |k| = k_1 + k_2 + \dots k_d.$$

Then for m in $\mathbb{N}$, $W^{m,r}(\Omega)$ is the standard Sobolev space:

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega);\, \forall k \in \mathbb{N}^d, 1 \leqslant |k| \leqslant m,\ \partial^k v \in L^r(\Omega)\},$$

and $W_0^{m,r}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ in $W^{m,r}(\Omega)$. It can be shown that for the domains Ω in which we shall work and for $m \geqslant 1$, this space is characterized by:

$$W_0^{m,r}(\Omega) = \{v \in W^{m,r}(\Omega);\, \forall k \in \mathbb{N}, 0 \leqslant k \leqslant m-1,\ \gamma_k v = 0 \text{ on } \Gamma\},$$

where γ_k denotes the normal trace operator of order k. The dual space of $W_0^{m,r}(\Omega)$ is denoted by $W^{-m,r'}(\Omega)$, and this extends the definition of $W^{m,r}(\Omega)$ to all integer values of m. If $m = 0$, $W_0^{m,r}(\Omega)$ reduces to $L^r(\Omega)$. When $r = 2$, the space $W^{m,r}(\Omega)$ (resp. $W_0^{m,r}(\Omega)$) is usually denoted by $H^m(\Omega)$ (resp. $H_0^m(\Omega)$). The reader can refer to Nečas [28] or Adams [1] for other properties of the above spaces and to Grisvard [21] for a careful study of the effect of nonsmooth boundaries.

2. A simplified version of De Rham's Theorem

For any nonnegative integer m and any real number r such that $1 < r < \infty$, we introduce the following spaces, closely related to the Stokes operator:

$$\mathscr{V} = \{v \in \mathscr{D}(\Omega);\ \operatorname{div} v = 0\}, \tag{2.1}$$

$$V_{m,r} = \text{closure of } \mathscr{V} \text{ in } \mathbf{W}^{m,r}(\Omega), \tag{2.2}$$

where $\mathbf{X}$ denotes the space X^d. For each integer $m \geqslant 1$, we also define the space:

$$U_{m,r} = \{v \in \mathbf{W}_0^{m,r}(\Omega);\ \operatorname{div} v = 0\}, \tag{2.3}$$

and for $m = 0$,

$$U_{0,r}(\Omega) = \{v \in \mathbf{L}^r(\Omega);\ \operatorname{div} v = 0,\ v \cdot n = 0 \text{ on } \Gamma\}. \tag{2.4}$$

When $r = 2$, it is proved in Temam [37] (cf. also Girault & Raviart [20]) that the normal trace $v \cdot n$ belongs to $H^{-\frac{1}{2}}(\Gamma)$. The arguments extend readily to the case where $r \neq 2$.

The theorem below states De Rham's Theorem, not with all its generality, since De Rham established it for flows on a variety (cf. [30], Théorème 17′, p. 114).

Theorem 2.1. (De Rham) *Let Ω be any open subset of $\mathbb{R}^d$ and let f be a distribution of $\mathscr{D}'(\Omega)$ that satisfies:*

$$\forall v \in \mathscr{V}, \quad \langle f, v\rangle = 0. \tag{2.5}$$

Then there exists a distribution p in $\mathscr{D}'(\Omega)$ such that

$$f = \nabla p. \tag{2.6}$$

This theorem has an immediate application to the Stokes problem (cf. the approach of Temam in [37] and Lions in [24]), but De Rham's Theorem is a very powerful and difficult result because it deals with arbitrary distributions, whereas Stokes problem involves in fact distributions for which we have much more information. In the case where f belongs to $\mathbf{H}^{-1}(\Omega)$ and satisfies (2.5), Tartar arrives in [35] at the conclusion of Theorem 2.1, but with a much simpler proof; this approach is also developed by Girault & Raviart in [20]. In this paragraph, we propose to extend the argument of Tartar to the case where the distribution f belongs to $\mathbf{W}^{-m,r}(\Omega)$. Previously, we require two basic results. The first one is an abstract algebraic result which is known as the "Peetre-Tartar Lemma", (cf. Peetre [29] and Tartar [34]), and has many valuable applications.

Theorem 2.2. (Peetre & Tartar) *Let E_1, E_2, E_3 be three Banach spaces, A an operator in $\mathscr{L}(E_1; E_2)$ and B a compact operator in $\mathscr{L}(E_1; E_3)$ satisfying:*

$$(2.7) \qquad \forall u \in E_1, \quad \|u\|_{E_1} \simeq \|Au\|_{E_2} + \|Bu\|_{E_3}.$$

Then the following properties hold:

i) *The dimension of* Ker $A = \{v \in E_1 \,;\, Av = 0\}$ *is finite. The range space $R(A)$ of the operator A is a closed subspace of E_2 and the mapping $A\colon E_1/\operatorname{Ker} A \mapsto R(A)$ is an isomorphism.*

ii) *If G is a Banach space and $M \in \mathscr{L}(E_1; G)$ satisfies*

$$\forall u \in \operatorname{Ker} A \setminus \{0\}, \quad Mu \neq 0,$$

then,

$$(2.8) \qquad \forall u \in E_1, \quad \|u\|_{E_1} \simeq \|Au\|_{E_2} + \|Mu\|_{G}.$$

And the second result we shall use is an important equivalence of norms due to Nečas [27].

Theorem 2.3. (Nečas) *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m an arbitrary integer and r any real number with $1 < r < \infty$. There exists a constant $C > 0$, depending only on Ω, m and r, such that:*

$$(2.9) \qquad \forall f \in W^{m,r}(\Omega), \quad \|f\|_{W^{m,r}(\Omega)} \leqslant C(\|\nabla f\|_{\mathbf{W}^{m-1,r}(\Omega)} + \|f\|_{W^{m-1,r}(\Omega)}).$$

Remark 2.4. Nečas' Theorem is difficult to establish because the boundary of Ω is only Lipschitz-continuous. When the boundary of Ω is smoother, Tartar gives in [35] a far simpler proof in the case where $r = 2$ and $m = 0$.

As a first application of these two theorems, the following corollary derives some properties of the gradient operator.

Corollary 2.5. *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m an arbitrary integer and r any real number with $1 < r < \infty$. We have:*

i) *The range space of the operator* grad $\in \mathscr{L}(W^{m,r}(\Omega);\, \mathbf{W}^{m-1,r}(\Omega))$ *is a closed subspace of* $\mathbf{W}^{m-1,r}(\Omega)$.

ii) *If in addition* Ω *is connected, there exists a constant* $C > 0$, *depending only on* Ω, m *and* r *such that:*

$$\text{(2.10)} \qquad \forall \dot{p} \in W^{m,r}(\Omega)/\mathbb{R} \qquad \|\dot{p}\|_{W^{m,r}(\Omega)/\mathbb{R}} \leqslant C\|\nabla p\|_{\mathbf{W}^{m-1,r}(\Omega)}.$$

iii) *For any open subset* ω *of* Ω, *with positive measure, there exists a constant* $C_\omega > 0$, *depending only on* ω, Ω, m *and* r, *such that:*

$$\text{(2.11)} \qquad \forall p \in W^{m,r}(\Omega), \quad \|p\|_{W^{m,r}(\Omega)} \leqslant C_\omega(\|p\|_{W^{m,r}(\omega)} + \|\nabla p\|_{\mathbf{W}^{m-1,r}(\Omega)}).$$

Proof. The proof consists in applying Theorem 2.2 with the following correspondence: $E_1 = W^{m,r}(\Omega)$, $E_2 = \mathbf{W}^{m-1,r}(\Omega)$, $E_3 = W^{m-1,r}(\Omega)$, $A = \text{grad}$, $B = i$, the identity operator. As the domain is bounded, the canonical imbedding i of E_1 into E_3 is compact. Besides, it is easy to show that

$$\text{(2.12)} \qquad \forall p \in W^{m,r}(\Omega), \quad \|p\|_{W^{m-1,r}(\Omega)} + \|\nabla p\|_{\mathbf{W}^{m-1,r}(\Omega)} \leqslant C\|p\|_{W^{m,r}(\Omega)}.$$

Then, with the above correspondence, (2.9) and (2.12) yield the equivalence of norms (2.7) and part i) follows from part i) of Theorem 2.2.

Next, Ker(grad) $= \mathbb{R}$ when Ω is connected. Thus part i) of Theorem 2.2 also states that the operator grad is an isomorphism from $W^{m,r}(\Omega)/\mathbb{R}$ onto $R(\text{grad})$ and hence there exists a constant $C > 0$ such that:

$$\forall \dot{p} \in W^{m,r}(\Omega)/\mathbb{R}, \quad \|\dot{p}\|_{W^{m,r}(\Omega)/\mathbb{R}} = \inf_{p \in \dot{p}} \|p\|_{W^{m,r}(\Omega)} \leqslant C\|\nabla p\|_{\mathbf{W}^{m-1,r}(\Omega)}.$$

Finally, let $G = W^{m,r}(\omega)$ and let $M\colon W^{m,r}(\Omega) \mapsto W^{m,r}(\omega)$ be the identity mapping; as ω has positive measure, then $\|M\lambda\|_{W^{m,r}(\omega)} = \|\lambda\|_{W^{m,r}(\omega)} > 0$ for all $\lambda \in \mathbb{R}^*$ and (2.11) follows directly from the equivalence of norms (2.8). □

Part ii) of Corollary 2.5 has an interesting consequence concerning the space:

$$X_{m,r} = \left\{ f \in W^{m,r}_{\text{loc}}(\Omega);\, \nabla f \in \mathbf{W}^{m-1,r}(\Omega) \right\}.$$

Let us fix an arbitrary subset ω of Ω with positive measure, and define:

$$|||p||| = \|p\|_{W^{m,r}(\omega)} + \|\nabla p\|_{\mathbf{W}^{m-1,r}(\Omega)}.$$

Because ω has positive measure, $|||\cdot|||$ is a norm on $X_{m,r}$. Moreover, owing to (2.11), $|||\cdot|||$ and $\|\cdot\|_{W^{m,r}(\Omega)}$ are equivalent norms on $W^{m,r}(\Omega)$ so that $W^{m,r}(\Omega)$ is a Banach space for $\|\!| \cdot |\!\|$. The following corollary states that $X_{m,r}$ coincides in fact with $W^{m,r}(\Omega)$. Its proof consists in showing that $W^{m,r}(\Omega)$ is dense in $X_{m,r}$ for the norm $\|\!| \cdot |\!\|$; we skip the proof because it is entirely similar to that of Corollary 2.2, Chapter I of [20], established in the case where $m = 0$ and $r = 2$.

Corollary 2.6. *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m an arbitrary integer and r any real number with $1 < r < \infty$. The following topological and algebraic identity holds:*

$$X_{m,r} = W^{m,r}(\Omega).$$

In addition to the spaces defined by (2.1)–(2.4), we shall also use the polar spaces $U^{\circ}_{m,r}$ and $V^{\circ}_{m,r}$ defined by:

$$U^{\circ}_{m,r} = \left\{ y \in \mathbf{W}^{-m,r'}(\Omega);\ \langle y, v\rangle = 0,\ \forall v \in U_{m,r} \right\}, \tag{2.13}$$

with a similar definition for $V^{\circ}_{m,r}$.

The next lemma establishes a first simplification of De Rham's Theorem.

Lemma 2.7. *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m a nonnegative integer, r any real number with $1 < r < \infty$ and r' its conjugate: $1/r + 1/r' = 1$. A distribution $f \in \mathbf{W}^{-m,r}(\Omega)$ satisfies:*

$$\forall \varphi \in U_{m,r'}, \quad \langle f, \varphi\rangle = 0, \tag{2.14}$$

if and only if there exists $p \in W^{-m+1,r}(\Omega)$ such that $f = \nabla p$. If in addition the set Ω is connected, then p is defined uniquely, up to an additive constant, by f and there exists a positive constant C, independent of f, such that

$$\|\dot{p}\|_{W^{-m+1,r}(\Omega)/\mathbb{R}} \leqslant C\|f\|_{\mathbf{W}^{-m,r}(\Omega)}. \tag{2.15}$$

Proof. Consider first the case where $m \geqslant 1$. Observe that the operator $-\operatorname{grad}$ that belongs to $\mathscr{L}(W^{-m+1,r}(\Omega);\ \mathbf{W}^{-m,r}(\Omega))$ is the dual operator of div in $\mathscr{L}(\mathbf{W}_0^{m,r'}(\Omega);\ W_0^{m-1,r'}(\Omega))$. But according to Corollary 2.5, the range space, $R(\text{grad})$ is a closed subspace of $\mathbf{W}^{-m,r}(\Omega)$. Then the Closed Range Theorem of Banach (cf. Yosida [39]) implies that:

$$R(\text{grad}) = (\operatorname{Ker}(\operatorname{div}))^{\circ} = U^{\circ}_{m,r'}.$$

This is precisely the statement of the necessary condition of the lemma. Then the upper bound (2.15) is an immediate application of (2.10). The sufficient condition is obvious.

When $m = 0$, a function f that satisfies (2.14) also satisfies it for $m = 1$. Therefore, the above proof with $m = 1$ shows that $f = \nabla p$ for some p in $L^r(\Omega)$. Thus, $p \in W^{1,r}(\Omega)$ and Corollary 2.5 yields the bound (2.15) when Ω is connected. $\square$

With Corollary 2.6, this lemma can be refined and gives a second simplified version of De Rham's Theorem.

Theorem 2.8. *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m a nonnegative integer, r any real number with $1 < r < \infty$ and r' its conjugate: $1/r + 1/r' = 1$. Let $f \in \mathbf{W}^{-m,r}(\Omega)$ satisfy:*

$$\forall \varphi \in \mathscr{V}, \quad \langle f, \varphi \rangle = 0.$$

Then the conclusion of Lemma 2.7 holds, i.e. there exists $p \in W^{-m+1,r}(\Omega)$ such that $f = \nabla p$. If in addition the set Ω is connected, then p is defined uniquely, up to an additive constant, by f and there exists a positive constant C, independent of f, such that (2.15) holds.

Proof. Let us assume that Ω is connected (otherwise, we can apply the argument below to each connected component of Ω). In view of Corollary 2.6, it suffices to show that $f = \nabla p$, for some $p \in W_{\text{loc}}^{-m+1,r}(\Omega)$. To this end, consider an increasing sequence $(\Omega_k)_{k \geqslant 1}$ of Lipschitz-continuous, connected open sets such that $\overline{\Omega_k} \subset \Omega$ and $\cup_k \Omega_k = \Omega$. Take any divergence-free function v in $\mathbf{W}_0^{m,r'}(\Omega_k)$ if $m \geqslant 1$ or v in $\mathbf{L}^{r'}(\Omega_k)$ with $v \cdot n = 0$ on the boundary of Ω_k if $m = 0$ and let us extend it by zero outside Ω_k. Then the extended function, still denoted by v, belongs to $V_{m,r'}$. For any $\varepsilon > 0$, let (ϱ_ε) be a sequence of mollifiers, i.e. $\varrho_\varepsilon \in \mathscr{D}(\mathbb{R}^d)$ and

$$\varrho_\varepsilon(x) \geqslant 0, \ \int_{\mathbb{R}^d} \varrho_\varepsilon \, dx = 1, \ \operatorname{supp} \varrho_\varepsilon \subset \overline{B}(0,\varepsilon), \ \lim_{\varepsilon \to 0} \varrho_\varepsilon = \delta \text{ in } \mathscr{D}'(\mathbb{R}^d).$$

Then, for all sufficiently small $\varepsilon > 0$, we have:

$$\varrho_\varepsilon * v \in \mathscr{D}(\Omega), \quad \operatorname{div}(\varrho_\varepsilon * v) = \varrho_\varepsilon * \operatorname{div} v = 0.$$

As $\varrho_\varepsilon * v \in \mathscr{V}$, the assumption on f yields:

$$\langle f, v \rangle = \lim_{\varepsilon \to 0} \langle f, \varrho_\varepsilon * v \rangle = 0.$$

Then Lemma 2.7 applied in Ω_k to $f_{|\Omega_k}$ implies that there exists $p_k \in W^{-m+1,r}(\Omega_k)$ such that $f_{|\Omega_k} = \nabla p_k$; and since $p_{k+1} - p_k$ is constant in Ω_k, this constant can be chosen so that $p_{k+1} = p_k$ in Ω_k, and hence $f = \nabla p$ with $p \in W^{-m+1,r}(\omega)$ for any proper subset ω of Ω, i.e. $p \in W_{\rm loc}^{-m+1,r}(\Omega)$. Therefore, by virtue of Corollary 2.6, p belongs to $W^{-m+1,r}(\Omega)$ and (2.15) follows again from (2.10). $\square$

Now we are in a position to prove that the spaces $U_{m,r}$ and $V_{m,r}$ are the same.

Theorem 2.9. *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m a nonnegative integer and r any real number with $1 < r < \infty$. The space $V_{m,r}$ defined by (2.2) coincides with the space $U_{m,r}$ defined by (2.3) or (2.4).*

P r o o f. It suffices to prove that $\mathscr{V}$ is dense in $U_{m,r}$. Let l be an element of $(U_{m,r})'$ that vanishes on $\mathscr{V}$ and let us prove that $l = 0$. As $U_{m,r}$ is a closed subspace of $\mathbf{W}_0^{m,r}(\Omega)$, l has a (non unique) extension $\tilde{l} \in \mathbf{W}^{-m,r'}(\Omega)$. Thus $\langle l, v\rangle = \langle \tilde{l}, v\rangle$ for all $v \in U_{m,r}$ and in particular, $\langle \tilde{l}, \varphi\rangle = 0$ for all $\varphi \in \mathscr{V}$. Then, according to Theorem 2.8, there exists $p \in W^{-m+1,r'}(\Omega)$ such that $\tilde{l} = \nabla p$. Hence, $\langle \tilde{l}, v\rangle = \langle \nabla p, v\rangle = 0$ for any $v \in U_{m,r}$ and therefore $\langle l, v\rangle = \langle \tilde{l}, v\rangle = 0$ for all $v \in U_{m,r}$. $\square$

We finish this paragraph with an important application of Theorem 2.8 showing that distributions are determined by their gradient.

Proposition 2.10. *Let Ω be a bounded Lipschitz-continuous domain of $\mathbb{R}^d$, m any integer and r any real number with $1 < r < \infty$.*

i) *If $p \in \mathscr{D}'(\Omega)$ has its gradient in $\mathbf{W}^{m-1,r}(\Omega)$, then p belongs to $W^{m,r}(\Omega)$. If in addition Ω is connected, then p satisfies (2.10). If Ω is arbitrary (not necessarily bounded nor Lipschitz-continuous), then p belongs to $W_{\rm loc}^{m,r}(\Omega)$.*

ii) *When $m \geqslant 0$ and Ω is connected, there exists a constant $C > 0$ such that all distributions p in $\mathscr{D}'(\Omega)$ with ∇p in $W^{m-1,r}(\Omega)$ and $\int_\Omega p \,\mathrm{d}x = 0$ satisfy the bound*

$$\|p\|_{W^{m,r}(\Omega)} \leqslant C\|\nabla p\|_{\mathbf{W}^{m-1,r}(\Omega)}.$$

P r o o f. The proof of part i) depends upon the value of m.

a) Let $m \leqslant 0$. Observe that

$$\forall v \in \mathscr{V}, \quad \langle \nabla p, v\rangle = -\langle p, \operatorname{div} v\rangle = 0.$$

Therefore, according to Theorem 2.8, there exists a function $q \in W^{m,r}(\Omega)$ such that $\nabla p = \nabla q$. Hence, the difference $p - q$ is constant in each connected component of Ω and as Ω is bounded this implies that p belongs to $W^{m,r}(\Omega)$. When Ω is connected, Corollary 2.5 shows that p satisfies (2.10).

b) Let $m > 0$. The argument of part i) shows that p belongs at least to $L^r(\Omega)$ and therefore p is in $W^{m,r}(\Omega)$ and (2.10) follows again from Corollary 2.5.

ii) We already know that that p belongs at least to $L^r(\Omega)$ and (2.10) holds. It suffices to prove that there exists a constant $C_1 > 0$ such that for all $\dot{p}$ in $L^r(\Omega)/\mathbf{R}$, the representative p with mean-value zero satisfies:

$$||p||_{L^r(\Omega)} \leqslant C_1 ||\dot{p}||_{L^r(\Omega)/\mathbf{R}}.$$

On one hand, $\dot{p}$ has exactly one representative $\bar{p}$ with mean value zero. On the other hand, $\dot{p}$ has a representative $\tilde{p}$ such that

$$||\tilde{p}||_{L^r(\Omega)} = ||\dot{p}||_{L^r(\Omega)/\mathbf{R}}.$$

Then $\bar{p} = \tilde{p} - \frac{1}{\text{meas}(\Omega)} \int_\Omega \tilde{p}\, dx$ and it satisfies

$$||\bar{p}||_{L^r(\Omega)} \leqslant (1 + \text{meas}(\Omega)^{-1/r}) ||\dot{p}||_{L^r(\Omega)/\mathbf{R}}.$$

□

Remark 2.11. This proposition is an extension of Theorem 2.3. It was proved by Magenes & Stampacchia [26] in a domain of class $\mathscr{C}^{1,1}$. Moreover, part ii) is a generalized Poincaré-type inequality for functions with mean-value zero (cf. for instance Dautray & Lions [14, vol. 3]).

Theorems 2.9 and Proposition 2.10 have been proved by Borchers & Sohr in [11]. Their proofs are based on the results of Bogovskii [9] and [10].

3. Properties of the divergence operator

Recall the following result, valid for two reflexive Banach spaces M and X (cf. Taylor [36]): let Y be a closed subspace of X and B a linear operator from X/Y into M', then its adjoint operator B' is an isomorphism from M onto the polar space Y° if and only if B is an isomorphism from X/Y onto M'.

The next corollary is a direct consequence of Lemma 2.7 and this result.

Corollary 3.1. *Let Ω be a bounded, connected, Lipschitz-continuous domain of $\mathbf{R}^d$, m a nonnegative integer, r any real number with $1 < r < \infty$, and r' its conjugate.*

i) *The gradient operator is an isomorphism from $W^{-m,r'}(\Omega)/\mathbf{R}$ onto $V^\circ_{m+1,r}$;*

ii) *The divergence operator is an isomorphism from $\mathbf{W}_0^{m+1,r}(\Omega)/V_{m+1,r}$ onto $W_0^{m,r}(\Omega) \cap L_0^r(\Omega)$, where $L_0^r(\Omega)$ denotes the space of functions of $L^r(\Omega)$ with mean-value zero.*

As mentioned in the introduction, the second part of this corollary has been established by Bogovskii in [9] and [10], who constructed explicitly this isomorphism by means of integral representations.

The theorem of Babuška & Brezzi (cf. Babuška [8], Brezzi [12] or [20]) implies that the statement of Corollary 3.1 with $m = 0$ is equivalent to an "inf-sup" condition in the spaces $\mathbf{W}_0^{1,r} \times L_0^{r'}$. This condition is used for instance in solving nonlinear problems with divergence constraint.

Corollary 3.2. ("Inf-sup" condition.) *Let Ω be a bounded, connected, Lipschitz-continuous domain of $\mathbb{R}^d$ and let r be any real number with $1 < r < \infty$, and r' its conjugate. There exists a constant $\beta > 0$ such that:*

$$\inf_{\mu \in L_0^{r'}(\Omega)} \sup_{v \in \mathbf{W}_0^{1,r}(\Omega)} \frac{\int_\Omega \mu \operatorname{div} v \, \mathrm{d}x}{\|v\|_{\mathbf{W}_0^{1,r}(\Omega)} \|\mu\|_{L^{r'}(\Omega)}} \geqslant \beta.$$

Lemma 3.3. *Let Ω be a bounded, Lipschitz-continuous, connected domain of $\mathbb{R}^d$ and r any real number with $1 < r < \infty$. Let $g \in \mathbf{W}^{1-1/r,r}(\Gamma)$ and $\varphi \in L^r(\Omega)$ satisfy the compatibility condition:*

$$\int_\Gamma g \cdot n \, \mathrm{d}\sigma = \int_\Omega \varphi \, \mathrm{d}x \,. \tag{3.1}$$

Then there exists $u \in \mathbf{W}^{1,r}(\Omega)$, unique up to an additive function of $V_{1,r}$, such that

$$\operatorname{div} u = \varphi \text{ in } \Omega, \quad u = g \text{ on } \Gamma. \tag{3.2}$$

Furthermore, there exists a constant $C > 0$, independent of u, φ and g, with

$$\inf_{v \in V_{1,r}} \|u + v\|_{\mathbf{W}^{1,r}(\Omega)} \leqslant C(\|\varphi\|_{L^r(\Omega)} + \|g\|_{\mathbf{W}^{1-1/r,r}(\Gamma)}). \tag{3.3}$$

Proof. Let w be a function in $\mathbf{W}^{1,r}(\Omega)$ such that $w = g$ on Γ. Green's formula and the compatibility condition (3.1) yield:

$$\int_\Omega \operatorname{div} w \, \mathrm{d}x = \int_\Gamma g \cdot n \, \mathrm{d}\sigma = \int_\Omega \varphi \, \mathrm{d}x,$$

so that $\operatorname{div} w - \varphi \in L_0^r(\Omega)$. Then Corollary 3.1 with $m = 0$ implies that there exists a function $u_0 \in \mathbf{W}_0^{1,r}(\Omega)$, unique up to additive functions of $V_{1,r}$, such that

$$\operatorname{div} u_0 = \operatorname{div} w - \varphi,$$

and satisfying

$$\inf_{v\in V_{1,r}} \|u_0 + v\|_{\mathbf{W}^{1,r}(\Omega)} \leqslant C_1 \| \operatorname{div} w - \varphi\|_{L^r(\Omega)}.$$

The function $u = w - u_0$ satisfies both conditions (3.2) and for any v in $V_{1,r}$, we have

$$\|u + v\|_{\mathbf{W}^{1,r}(\Omega)} \leqslant \|w\|_{\mathbf{W}^{1,r}(\Omega)} + \|u_0 - v\|_{\mathbf{W}^{1,r}(\Omega)}.$$

By taking the infimum of both sides of this relation, we derive

$$\inf_{v\in V_{1,r}} \|u + v\|_{\mathbf{W}^{1,r}(\Omega)} \leqslant C_2(\|\varphi\|_{L^r(\Omega)} + \|g\|_{\mathbf{W}^{1-1/r,r}(\Gamma)}),$$

where $C_2 > 0$ is a constant that depends only on Ω and r. □

This lemma generalizes the standard result with $\varphi = 0$ and $r = 2$ (cf. for instance [20] Lemma 2.2, p. 24). In addition, it covers Simon's Lemma [32] stated for $r = 2$ and constant φ. More precisely, we have:

Corollary 3.4. *Let Ω be a bounded, connected, Lipschitz-continuous domain of $\mathbf{R}^d$ and r any real number with $1 < r < \infty$. For any $g \in \mathbf{W}^{1-1/r,r}(\Gamma)$, there exists $u \in \mathbf{W}^{1,r}(\Omega)$, unique up to an additive function of $V_{1,r}$, such that*

$$\operatorname{div} u = \frac{1}{\operatorname{meas}(\Omega)} \int_\Gamma g \cdot n \, \mathrm{d}\sigma \ \text{ in } \Omega, \quad u = g \ \text{ on } \Gamma.$$

Furthermore, there exists a constant $C > 0$, independent of u and g, such that

$$\inf_{v\in V_{1,r}} \|u + v\|_{\mathbf{W}^{1,r}(\Omega)} \leqslant C\|g\|_{\mathbf{W}^{1-1/r,r}(\Gamma)}.$$

Lemma 3.3 extends to higher order traces by applying the fundamental result below, proved by Héron [22] (Lemme 3.3, p. 1316) in the case of H^m spaces.

Lemma 3.5. (Héron) *Let Ω be a bounded, connected domain of $\mathbf{R}^d$ of class $\mathscr{C}^{1,1}$.*

i) *Every function $u \in \mathbf{H}^2(\Omega)$ satisfies:*

$$\gamma_0(\operatorname{div} u) = \operatorname{div}_\Gamma(\gamma_0(u)_t) + \gamma_1(u) \cdot n - 2K\gamma_0(u) \cdot n, \tag{3.4}$$

where K denotes the mean curvature of Γ, $\operatorname{div}_\Gamma$ is the surface divergence, $v_t = v - (v \cdot n)n$ is the tangential component of v and $\gamma_i = \frac{\partial^i}{\partial n^i}$ are the normal derivatives of order i.

ii) *Let $m \in \mathbb{N}^*$ and Ω be of class $\mathscr{C}^{m+1,1}$; every function $u \in \mathbf{H}^{m+2}(\Omega)$ satisfies:*

$$
(3.5) \qquad \begin{aligned}
\gamma_m(\operatorname{div} u) = {} & \operatorname{div}_\Gamma S_m + \sum_{j=0}^{m-1} \binom{m}{j} S_j \cdot \operatorname{grad} B_{m-j} \\
& + \gamma_{m+1} u \cdot n + \sum_{j=0}^{m-1} \binom{m}{j} B_{m+1-j}(\gamma_j u \cdot n),
\end{aligned}
$$

where

$$
\forall i = 0, \ldots, m, \quad S_i = i! \sum_{j=0}^{i} \frac{1}{j!} (\gamma_j u)_t \Big(- \frac{\partial n}{\partial \xi} \Big)^{i-j},
$$

and the functions B_i are expressed in terms of the curvature tensor of Γ.

Remark 3.6. Héron derives (3.4) (respectively (3.5)), by assuming that the domain Ω is of class $\mathscr{C}^3$ (respectively $\mathscr{C}^{m+3}$). In fact, it can be checked that (3.4) (respectively (3.5)) is satisfied in the sense of $H^{1/2}(\Gamma)$ (respectively $H^{m+1/2}(\Gamma)$) whenever Ω is of class $\mathscr{C}^{1,1}$ (respectively $\mathscr{C}^{m+1,1}$). The equation (3.4) is also proved by Grisvard [21] when Ω is $\mathscr{C}^{1,1}$. In addition, the proof of Héron is easily transposed to the spaces $W^{m+2,r}(\Omega)$ with $m \geqslant 0$ and $r > 1$.

Corollary 3.7. *Let $m \geqslant 0$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded, connected domain of $\mathbb{R}^d$ of class $\mathscr{C}^{m+1,1}$. For every $g \in \mathbf{W}^{m+2-1/r,r}(\Gamma)$ and $\psi_i \in W^{m+2-i-1/r,r}(\Gamma)$, $i = 1, \ldots, m+1$, there exists a function $u \in \mathbf{W}^{m+2,r}(\Omega)$ such that*

$$
\gamma_{i-1}(\operatorname{div} u) = \psi_i, \ i = 1, \ldots, m+1, \quad u = g \ \text{ on } \Gamma,
$$

and a constant $C > 0$ that depends only on Ω, m and r such that

$$
\|u\|_{\mathbf{W}^{m+2,r}(\Omega)} \leqslant C \Big(\|g\|_{\mathbf{W}^{m+2-1/r,r}(\Gamma)} + \sum_{i=1}^{m+1} \|\psi_i\|_{W^{m+2-i-1/r,r}(\Gamma)} \Big).
$$

Proof. To simplify the discussion, we shall only write the proof for $m = 1$; the proof is simpler when $m = 0$ and pretty similar when $m \geqslant 2$. Set $\psi_1 = \psi$ et $\psi_2 = \theta$. In view of the regularity of Ω, there exists $u \in \mathbf{W}^{3,r}(\Omega)$ such that

$$
\begin{aligned}
& u = g \ \text{ on } \Gamma, \\
& \gamma_1(u) = 2Kg - n \operatorname{div}_\Gamma(g_t) + \psi n \ \text{ on } \Gamma, \\
& \gamma_2(u) = \theta n - (\operatorname{div}_\Gamma S_1 + S_0 \operatorname{grad} B_1) n - B_2 g \ \text{ on } \Gamma.
\end{aligned}
$$

Owing to the assumptions, the right-hand sides above have enough regularity to guarantee that u belongs indeed to $\mathbf{W}^{3,r}(\Omega)$ and u can be chosen so that it depends continuously upon the data g, ψ and θ. The functions S_i and B_i have the expression:

$$S_0 = g_t,\ S_1 = -\frac{\partial n}{\partial \xi} g_t + 2K g_t,$$
$$B_1(\xi) = -2K(\xi),\ B_2(\xi) = 2\det\left(\frac{\partial n}{\partial \xi}\right) - 4K^2(\xi).$$

In view of (3.4) and (3.5), it can be easily checked that u satisfies

$$\gamma_0(\operatorname{div} u) = \psi,\ \gamma_1(\operatorname{div} u) = \theta,\quad u = g \ \text{ on } \Gamma.$$

□

As a consequence, we can extend the statement of Lemma 3.3 to $\mathbf{W}^{m,r}(\Omega)$.

Corollary 3.8. *Let m belong to $\mathbb{N} \cup \{-1\}$, r be any real number with $1 < r < \infty$ and let Ω be a bounded, connected domain of $\mathbb{R}^d$ of class $\mathscr{C}^{m+1,1}$ (i.e. Lipschitz-continuous if $m = -1$). For any $g \in \mathbf{W}^{m+2-1/r,r}(\Gamma)$ and $\varphi \in W^{m+1,r}(\Omega)$ satisfying the compatibility condition:*

$$\int_\Gamma g \cdot n \,\mathrm{d}\sigma = \int_\Omega \varphi \,\mathrm{d}x, \tag{3.6}$$

there exists $u \in \mathbf{W}^{m+2,r}(\Omega)$, unique up to additive functions of $V_{m+2,r}$, such that

$$\operatorname{div} u = \varphi \ \text{in } \Omega, \quad u = g \ \text{on } \Gamma. \tag{3.7}$$

Moreover, there exists a constant $C > 0$ independent of u, φ and g with

$$\inf_{v \in V_{m+2,r}} \|u + v\|_{\mathbf{W}^{m+2,r}(\Omega)} \leqslant C(\|\varphi\|_{W^{m+1,r}(\Omega)} + \|g\|_{\mathbf{W}^{m+2-1/r,r}(\Gamma)}). \tag{3.8}$$

Proof. When $m = -1$, this is the statement of Lemma 3.3; so we can assume that $m \geqslant 0$. Applying Corollary 3.7, we can find a function $w \in \mathbf{W}^{m+2,r}(\Omega)$ such that

$$\gamma_i(\operatorname{div} w) = \gamma_i(\varphi),\ i = 0, \ldots, m, \quad w = g \ \text{ on } \Gamma.$$

Green's formula and the compatibility condition (3.6) yield:

$$\int_\Omega \operatorname{div} w \,\mathrm{d}x = \int_\Gamma g \cdot n \,\mathrm{d}\sigma = \int_\Omega \varphi \,\mathrm{d}x,$$

so that $\operatorname{div} w - \varphi \in L^r_0(\Omega) \cap W^{m+1,r}_0(\Omega)$. Then, according to Corollary 3.1, there exists a function $u_0 \in \mathbf{W}^{m+2,r}_0(\Omega)$ (unique up to additive functions of $V_{m+2,r}$) such that

$$\operatorname{div} u_0 = \operatorname{div} w - \varphi,$$

and

$$\inf_{v \in V_{m+2,r}} \|u_0 + v\|_{\mathbf{W}^{m+2,r}(\Omega)} \leqslant C \|\operatorname{div} w - \varphi\|_{W^{m+1,r}(\Omega)}.$$

The function $u = w - u_0$ satisfies (3.7) and we readily derive (3.8). □

4. Stokes Problem

As mentioned in the introduction, we propose to solve the Stokes problem by relating it to a Helmholtz decomposition. This approach has the advantage of being straightforward, it is valid in arbitrary dimensions, and in nearly every case it applies to domains with minimum regularity.

Recall that Stokes problem consists in finding a pair of functions (u, p) solution of:

$$-\Delta u + \nabla p = f \text{ in } \Omega, \tag{4.1}$$

$$\operatorname{div} u = \varphi \text{ in } \Omega, \tag{4.2}$$

$$u = g \text{ on } \Gamma, \tag{4.3}$$

for given functions f, φ, g satisfying the compatibility condition (3.6). The homogeneous case corresponds to $\varphi = 0$ and $g = 0$. In two dimensions, this problem can be easily solved by reducing it to a biharmonic problem. In three dimensions, the salient result is Cattabriga's [13] famous theorem:

Theorem 4.1. (Cattabriga) *Let m be a strictly positive integer, r any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^3$, of class $\mathscr{C}^m$ if $m \geqslant 2$ or of class $\mathscr{C}^2$ if $m = 1$. Assume that the data have the regularity:*

$$f \in \mathbf{W}^{m-2,r}(\Omega), \ \varphi \in W^{m-1,r}(\Omega), \ g \in \mathbf{W}^{m-1/r,r}(\Gamma),$$

and that the compatibility condition (3.6) between g and φ holds. Then the non-homogeneous Stokes problem (4.1)–(4.3) has a unique solution $u \in \mathbf{W}^{m,r}(\Omega)$ and $p \in W^{m-1,r}(\Omega)/\mathbb{R}$ and they satisfy the bound,

$$\|u\|_{\mathbf{W}^{m,r}(\Omega)} + \|p\|_{W^{m-1,r}(\Omega)/\mathbb{R}} \leqslant C(\|f\|_{\mathbf{W}^{m-2,r}(\Omega)} + \|\varphi\|_{W^{m-1,r}(\Omega)} + \|g\|_{\mathbf{W}^{m-1/r,r}(\Gamma)}),$$

with a constant $C > 0$ that depends only on m, r and Ω.

To illustrate the close relationship that exists between the solution of the Stokes problem and the Helmholtz decomposition, let us start with the following classical result that we prove by a new argument instead of the standard proof.

Proposition 4.2. *Let Ω be a bounded and connected domain of $\mathbb{R}^d$. For every $f \in \mathbf{H}^{-1}(\Omega)$, the homogeneous Stokes problem has a unique solution $u \in \mathbf{H}^1_0(\Omega)$ and $p \in L^2(\Omega)/\mathbb{R}$ and there exists a constant $C > 0$ that depends only on Ω such that:*

$$\|u\|_{\mathbf{H}^1(\Omega)} + \|p\|_{L^2(\Omega)/\mathbb{R}} \leqslant C\|f\|_{\mathbf{H}^{-1}(\Omega)}. \tag{4.4}$$

Proof. To begin with, recall the decomposition (cf. for instance [20])

$$\mathbf{H}^1_0(\Omega) = V \oplus V^{\perp}, \tag{4.5}$$

where for the sake of simplicity, V stands for $V_{1,2}$ and the characterisation of $V^{\perp}$:

$$V^{\perp} = \left\{(-\Delta)^{-1}\text{grad } q,\ q \in L^2_0(\Omega)\right\}. \tag{4.6}$$

Now, let us first solve the Laplace system of equations:

$$-\Delta w = f \quad \text{in } \Omega,$$
$$w = 0 \quad \text{on } \Gamma.$$

It has a unique solution $w \in \mathbf{H}^1_0(\Omega)$ and there exists a constant $C > 0$ that depends only on Ω such that:

$$\|w\|_{\mathbf{H}^1(\Omega)} \leqslant C_1\|f\|_{\mathbf{H}^{-1}(\Omega)}.$$

In addition, since

$$\int_\Omega \operatorname{div} w \,\mathrm{d}x = \int_\Gamma w \cdot n \,\mathrm{d}\sigma = 0,$$

it follows from Corollary 3.1 with $m = 0$ and $r = 2$, that there exists a unique function $v \in V^{\perp}$ such that $\operatorname{div} v = \operatorname{div} w$ (note that in the Hilbert case, $\mathbf{H}^1_0(\Omega)/V$ can be identified with $V^{\perp}$), and

$$\|v\|_{\mathbf{H}^1(\Omega)} \leqslant C_2\|\operatorname{div} w\|_{L^2(\Omega)} \leqslant C_3\|f\|_{\mathbf{H}^{-1}(\Omega)}.$$

Then, the characterization (4.6) of $V^{\perp}$ shows that there exists a unique $p \in L^2_0(\Omega)$ such that $-\Delta v = \nabla p$ and

$$\|p\|_{L^2_0(\Omega)} \leqslant C_4\|\nabla p\|_{\mathbf{H}^{-1}(\Omega)} \leqslant C_5\|v\|_{\mathbf{H}^1(\Omega)} \leqslant C_6\|f\|_{\mathbf{H}^{-1}(\Omega)}.$$

Thus, the pair $(u = w - v, p)$ is a solution of the homogeneous Stokes problem and it satisfies (4.4). Since obviously the Stokes problem has at most one solution, it is the only solution. $\square$

The above proof shows that the Stokes problem reduces to a standard Laplace's equation, provided a decomposition of the form (4.5)–(4.6) exists. When $m \geqslant 2$, there is a direct proof of this decomposition and therefore we divide the remainder of this paragraph into three sections according to the value of m.

4.1. THE CASE $m \geqslant 2$

In order to prove for $\mathbf{W}^{m,r}(\Omega)$ a decomposition analogous to (4.5)–(4.6), we require the following result.

Proposition 4.3. *Let $m \geqslant 2$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$, of class $\mathscr{C}^{m-1,1}$. Suppose that $u \in \mathbf{W}^{2,r}(\Omega)$ and $p \in W^{1,r}(\Omega)$ is a solution of the homogeneous Stokes problem with right-hand side $f \in \mathbf{W}^{m-2,r}(\Omega)$. Then $u \in \mathbf{W}^{m,r}(\Omega)$, $p \in W^{m-1,r}(\Omega)$ and there exists a constant $C > 0$ that depends only on m, r and Ω such that*

$$\|u\|_{\mathbf{W}^{m,r}(\Omega)} + \|p\|_{W^{m-1,r}(\Omega)/\mathbb{R}} \leqslant C\|f\|_{\mathbf{W}^{m-2,r}(\Omega)}. \tag{4.7}$$

P r o o f. The proof is similar to that of Proposition 2.2 of Temam (cf. [37] p. 33). It can be easily checked that the homogeneous Stokes system can be expressed as an elliptic system in the sense of Agmon, Douglis & Nirenberg [3] (pp. 38–39 and 42–43). Indeed, take $p = u_{d+1}$ and $f_{d+1} = 0$. The system (4.1)–(4.2), with $\varphi = 0$, reads

$$\sum_{j=1}^{d+1} l_{ij}(D)u_j = f_i, \quad 1 \leqslant i \leqslant d+1, \tag{4.8}$$

where the matrix $(l_{ij}(\xi))$, for $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$, is defined by:

$$\begin{aligned}
&l_{ij} = |\xi|^2\delta_{ij} \quad \text{if } 1 \leqslant i, j \leqslant d,\\
&l_{i,d+1} = l_{d+1,i} = -\xi_i \quad \text{if } 1 \leqslant i \leqslant d,\\
&l_{d+1,d+1} = 0,\\
&|\xi|^2 = \xi_1^2 + \ldots + \xi_d^2.
\end{aligned}$$

Take $s_i = 0, t_j = 2$ for $1 \leqslant i, j \leqslant d$ and $s_{d+1} = -1, t_{d+1} = 1$; the degree of the polynomial $l_{ij}(\xi)$ is less than or equal to $s_i + t_j$. With the same notations, it is clear that the determinant $L(\xi) = |\xi|^{2d}$ vanishes if and only if $\xi = 0$, in other words, if and only if the ellipticity condition (1.5) p. 39 of reference [3] holds as well as the uniform ellipticity condition (1.7) with $m = d$. The additional condition on L also holds: indeed, it is obvious that the polynomial $\tau \to L(\xi + \tau\xi')$ has exactly d roots, with positive imaginary part, all equal to

$$\tau^+(\xi,\xi') = (-\xi \cdot \xi' + \mathrm{i}(|\xi|^2|\xi'|^2 - |\xi \cdot \xi'|^2)^{1/2})/|\xi'|^2.$$

As far as the boundary conditions are concerned, the matrix $(B_{hj}) = (B'_{hj})$ is given by

$$B_{hj} = \delta_{hj}, \quad 1 \leqslant h \leqslant d,\ 1 \leqslant j \leqslant d+1.$$

Take $r_h = -2$; the degree of the polynomial B_{hj} is not larger than $r_h + t_j$ (considering that $B_{hj} = 0$ if $r_h + t_j < 0$). It remains to show that the complementary boundary condition (2.3) of reference [3] holds. Let ξ be a tangent vector and n a normal vector at a point of Γ. Denote by $\tau_h^+(\xi,n)$ the d roots of $L(\xi+\tau n)$; then $M^+(\xi,\tau) = (\tau - \tau_h^+(\xi,n))^d$ and $\tau_h^+(\xi,n) = \mathrm{i}|\xi|/|n|$. Let (L^{jk}) be the adjoint matrix of $(l'_{ij}) = (l_{ij})$; we readily check that the matrix $\sum_{j=1}^{d+1} B'_{hj}(\xi+\tau n)L^{jk}(\xi+\tau n)$ is, modulo $M^+(\xi,\tau)$, a matrix of order d, and this shows that condition (2.3) holds.

Now, we are in a position to apply Theorem 10.5 of reference [3] p. 78; if the domain Ω is of class $\mathscr{C}^m$, we obtain a weaker estimate than (4.7), namely:

$$||u||_{\mathbf{W}^{m,r}(\Omega)} + ||p||_{W^{m-1,r}(\Omega)/\mathbb{R}} \leqslant C_1(||f||_{\mathbf{W}^{m-2,r}(\Omega)} + d_r||u||_{\mathbf{L}^r(\Omega)}),$$

where the constant $C_1 > 0$ depends only on m, r and Ω and where $d_r = 0$ if $r \geqslant 2$, $d_r = 1$ if $1 < r < 2$. In addition, by applying the material of Grisvard [21], we verify that the estimate of Theorem 10.5 of reference [3] is valid for domains of class $\mathscr{C}^{m-1,1}$.

Finally, since the domain is bounded, we can take $d_r = 0$ when the solution is unique (cf. Remark 2 pp. 668–669 of reference [2]). The uniqueness is obvious from the above inequality when the domain is $\mathscr{C}^\infty$. When the domain is at least $\mathscr{C}^{1,1}$, uniqueness can be proved by induction, using an idea of Kozono & Sohr [23]. First, the solution u is unique for all $r \geqslant r_0 = \frac{2d}{d+2}$, because in this case, $W^{2,r}(\Omega)$ is imbedded in $H^1(\Omega)$. If $r_0 \leqslant 1$, the solution is unique for all r; otherwise, the conclusions of the remainder of this section are valid for all real numbers r in the interval $[r_0, r'_0]$ and in particular, the statement of Proposition 4.11 shows that the solution u is unique if it belongs to $\mathbf{L}^r(\Omega)$ for some r in this interval. Hence, repeating the above argument, we obtain uniqueness for all $r_0 \geqslant r_1 = \frac{2d}{d+6}$, because in this case, $W^{2,r}(\Omega)$ is imbedded in $L^{r_0}(\Omega)$. In this fashion, we generate a sequence of real numbers $r_k = \frac{2d}{d+2+4k}$ and the solution is unique for all r whenever $d \leqslant 2+4k$; in a finite number of steps, this proves uniqueness for any dimension d. □

Remark 4.4. The above proposition is not an existence result: it does not prove that a solution satisfying the estimate (4.7) exists. It merely says that whenever a solution exists in the adequate spaces, then this solution has necessarily the regularity (4.7).

With Proposition 4.3, we can establish the analogue of the decomposition (4.5)–(4.6).

Proposition 4.5. *Let $m \geqslant 2$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$, of class $\mathscr{C}^{m-1,1}$. The following decomposition holds:*

$$\mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega) = (\mathbf{W}^{m,r}(\Omega) \cap V_{1,r}) \oplus (\mathbf{W}^{m,r}(\Omega) \cap G_{1,r}), \tag{4.9}$$

where

$$G_{1,r} = \left\{ v \in \mathbf{W}_0^{1,r}(\Omega);\ -\Delta v = \nabla q, q \in L_0^r(\Omega) \right\}.$$

Proof. Denote by E (respectively, F) the left-hand side (respectively, right-hand side) of (4.9). Clearly, $F \subset E$. The proof of the equality proceeds in two steps.

i) First, let us show that F is dense in E. Observe that $\mathbf{W}^{m,r}(\Omega) \cap V_{1,r}$ is dense in $V_{1,r}$ for the topology of $\mathbf{W}_0^{1,r}(\Omega)$ (by virtue of the density of $\mathscr{V}$ in $V_{1,r}$). Similarly, $\mathbf{W}^{m,r}(\Omega) \cap G_{1,r}$ is dense in $G_{1,r}$ for the topology of $\mathbf{W}_0^{1,r}(\Omega)$. Indeed, let $v \in G_{1,r}$; there exists $q \in L^r(\Omega)$ such that $-\Delta v = \nabla q$. As $\mathscr{D}(\Omega)$ is dense in $L^r(\Omega)$, there exists $q_n \in \mathscr{D}(\Omega)$ such that $q_n \to q$ in $L^r(\Omega)$. This implies the existence of $v_n \in \mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$ solution of $-\Delta v_n = \nabla q_n$ and such that v_n converges to v in $\mathbf{W}_0^{1,r}(\Omega)$.

Next, let L be a continuous linear functional on E such that

$$\langle L, v \rangle = 0 \quad \forall v \in F,$$

and let us prove that $L = 0$. As E is dense in $\mathbf{W}_0^{1,r}(\Omega)$, L has a unique extension $\tilde{L} \in \mathbf{W}^{-1,r'}(\Omega)$. Therefore

$$\left\langle \tilde{L}, v \right\rangle = 0 \quad \forall v \in V_{1,r} \oplus G_{1,r}.$$

In particular, $\left\langle \tilde{L}, v \right\rangle = 0$ for all $v \in V_{1,r}$, so that according to Lemma 2.7 and Theorem 2.9, there exists $q \in L^{r'}(\Omega)$ such that $\tilde{L} = \nabla q$. Let $z \in \mathbf{W}_0^{1,r'}(\Omega)$ be the solution of the problem $-\Delta z = \nabla q$ in Ω, $z = 0$ on Γ. Then, for all $w \in G_{1,r}$ we have,

$$0 = \langle \nabla q, w \rangle = \langle -\Delta z, w \rangle = \langle z, -\Delta w \rangle,$$

which means that $\langle z, \nabla p\rangle = 0$ for all $p \in L^{r'}(\Omega)$. In other words, div $z = 0$, and hence $z = 0$ and $\tilde{L} = 0$.

ii) Now, to establish (4.9), it suffices to prove that F is closed in E. To this end, let $u_n = v_n + w_n$ be a sequence such that $v_n \in \mathbf{W}^{m,r}(\Omega) \cap V_{1,r}$, $w_n \in \mathbf{W}^{m,r}(\Omega) \cap G_{1,r}$ and $u_n \to u$ in $\mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$, and let us show that u belongs to F. The sequence w_n satisfies $-\Delta w_n = \nabla q_n$, where $q_n \in W^{m-1,r}(\Omega)$. Set $-\Delta v_n + \nabla q_n = -\Delta u_n = f_n$. The pair (v_n, q_n) is a solution of Stokes problem (since div $v_n = 0$ in Ω and $v_n = 0$ on Γ) with v_n in $\mathbf{W}^{m,r}(\Omega)$ and q_n in $W^{m-1,r}(\Omega)$ and right-hand side f_n in $\mathbf{W}^{m-2,r}(\Omega)$. Then Proposition 4.3 implies that

$$\|v_n\|_{\mathbf{W}^{m,r}(\Omega)} + \|q_n\|_{W^{m-1,r}(\Omega)/\mathbb{R}} \leqslant C\|f_n\|_{\mathbf{W}^{m-2,r}(\Omega)}.$$

Since f_n is bounded in $\mathbf{W}^{m-2,r}(\Omega)$, the sequences v_n and q_n are bounded respectively in $\mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$ and $W^{m-1,r}(\Omega)/\mathbb{R}$. As a consequence, w_n is also bounded in $\mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$ (because the Laplace operator is an isomorphism of $W^{m,r}(\Omega) \cap W_0^{1,r}(\Omega)$ onto $W^{m-2,r}(\Omega)$). Hence,

$$v_n \rightharpoonup v, \quad w_n \rightharpoonup w \text{ in } \mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega) \text{ weakly},$$

and

$$q_n \rightharpoonup q \quad \text{in } W^{m-1,r}(\Omega)/\mathbb{R} \text{ weakly}.$$

Moreover, it is easy to check that $v \in \mathbf{W}^{m,r}(\Omega) \cap V_{1,r}$ and

$$-\Delta w = \nabla q,$$

which means that $w \in \mathbf{W}^{m,r}(\Omega) \cap G_{1,r}$. Therefore $u = v + w$ belongs to F. □

This decomposition permits to solve the homogeneous Stokes problem.

Theorem 4.6. *Under the hypotheses of Proposition 4.5, for any $f \in \mathbf{W}^{m-2,r}(\Omega)$, the homogeneous Stokes problem has a unique solution $u \in \mathbf{W}^{m,r}(\Omega)$ and $p \in W^{m-1,r}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on m, r and Ω, such that*

$$\|u\|_{\mathbf{W}^{m,r}(\Omega)} + \|p\|_{W^{m-1,r}(\Omega)/\mathbb{R}} \leqslant C\|f\|_{\mathbf{W}^{m-2,r}(\Omega)}. \tag{4.10}$$

Proof. In view of Proposition 4.3, it suffices to establish the existence of $u \in \mathbf{W}^{2,r}(\Omega)$ and $p \in W^{1,r}(\Omega)$, but in fact, it will be just as easy to prove the existence of $u \in \mathbf{W}^{m,r}(\Omega)$ and $p \in W^{m-1,r}(\Omega)$.

The idea of the proof is very similar to that of Proposition 4.2. We first solve the Laplace's system of equations: $-\Delta v = f$ in Ω, $v = 0$ on Γ; owing to the regularity assumption on f, this system has a unique solution $v \in \mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$. Then, applying the decomposition (4.9), v can be split uniquely into a sum $v = u + w$, with $u \in \mathbf{W}^{m,r}(\Omega) \cap V_{1,r}$ and $w \in \mathbf{W}^{m,r}(\Omega) \cap G_{1,r}$. Since $-\Delta w = \nabla p$, with $p \in W^{m-1,r}(\Omega)$, we infer that u and p satisfy $-\Delta u + \nabla p = f$, whence the result. □

Remark 4.7. It can be readily checked by interpolation that if $s = m + \alpha$ where $0 < \alpha < 1$ and if the domain Ω is of class $\mathscr{C}^{m,1}$, then $u \in \mathbf{W}^{s,r}(\Omega)$ and $p \in W^{s-1,r}(\Omega)/\mathbb{R}$ whenever $f \in \mathbf{W}^{s-2,r}(\Omega)$. However, this result is probably not optimal in the sense that the above assumption on Ω is a little too strong. It is probably sufficient to suppose that Ω is of class $\mathscr{C}^{m,\alpha}$.

Now, we turn to the nonhomogeneous problem.

Theorem 4.8. *Let $m \geqslant 2$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$, of class $\mathscr{C}^{m-1,1}$. Let*

$$f \in \mathbf{W}^{m-2,r}(\Omega), \quad \varphi \in W^{m-1,r}(\Omega) \text{ and } g \in \mathbf{W}^{m-1/r,r}(\Gamma), \quad 1 < r < \infty,$$

be given with g and φ satisfying the compatibility condition (3.6). Then the nonhomogeneous Stokes problem has a unique solution $u \in \mathbf{W}^{m,r}(\Omega)$ and $p \in W^{m-1,r}(\Omega)/\mathbb{R}$. In addition, there exists a constant $C > 0$ depending only on m, r and Ω, such that

$$\begin{aligned}(4.11) \qquad &\|u\|_{\mathbf{W}^{m,r}(\Omega)} + \|p\|_{W^{m-1,r}(\Omega)/\mathbb{R}} \\ &\qquad \leqslant C(\|f\|_{\mathbf{W}^{m-2,r}(\Omega)} + \|\varphi\|_{W^{m-1,r}(\Omega)} + \|g\|_{\mathbf{W}^{m-1/r,r}(\Gamma)}).\end{aligned}$$

Proof. Corollary 3.8 reduces the nonhomogeneous to the homogeneous case. Indeed, by fixing an adequate representative, we can find $u_0 \in \mathbf{W}^{m,r}(\Omega)$, such that

$$\operatorname{div} u_0 = \varphi \text{ in } \Omega, \quad u_0 = g \text{ on } \Gamma,$$

and

$$(4.12) \qquad \|u_0\|_{\mathbf{W}^{m,r}(\Omega)} \leqslant C(\|\varphi\|_{W^{m-1,r}(\Omega)} + \|g\|_{\mathbf{W}^{m-1/r,r}(\Gamma)}),$$

where $C > 0$ is a constant independent of u, φ and g. Thus the nonhomogeneous problem amounts to find $u - u_0 \in \mathbf{W}^{m,r}(\Omega)$ and $p \in W^{m-1,r}(\Omega)/\mathbb{R}$ such that:

$$\begin{aligned} &-\Delta(u - u_0) + \nabla p = f + \Delta u_0 \ \text{ in } \Omega, \\ &\operatorname{div}(u - u_0) = 0 \ \text{ in } \Omega, \\ &u - u_0 = 0 \ \text{ on } \Gamma. \end{aligned}$$

Theorem 4.6 guarantees that this problem has a unique solution $(u-u_0, p)$ and (4.12) and (4.10) yield (4.11). $\square$

Remark 4.9. The same argument as in Remark 4.7 shows that the statement of Theorem 4.8 extends to $W^{s,r}$ spaces for real s.

4.2. THE AUXILIARY CASE $m = 0$

Unfortunately, the proof of Proposition 4.3 is not valid for $m = 1$ and, in this case, we do not know how to apply directly the preceding approach. To turn this difficulty, we shall first solve an auxiliary Stokes problem with $m = 0$, by formulating it as the dual of a Stokes problem that corresponds to $m = 2$ and to which Theorem 4.8 applies. This idea has been introduced by Lions & Magenes in [25] to solve a Laplace's equation with singular data and it has been applied by Giga in [19] to solve a Stokes problem with singular data on the boundary. Once an adequate result is obtained for $m = 0$, we shall derive a similar result for $m = 1$ by interpolating between $m = 0$ and $m = 2$. The reader can refer to Galdi & Simader [16] for a different proof. They handle the case $m = 1$ in a domain of class $\mathscr{C}^2$ by considering fundamental solutions in $\mathbb{R}^d$.

Before describing this technique, we must give a meaning to singular data for a Stokes problem and we require some preliminary spaces. More precisely, we want to show that, for the solution of the Stokes problem, a boundary condition of the form $u_{|\Gamma} \in \mathbf{W}^{-1/r,r}(\Gamma)$ makes sense. To this end, we introduce the space

$$Y(\Omega) = \left\{u \in \mathbf{W}^{2,r'}(\Omega)\,;\, u = 0, \operatorname{div} u = 0 \text{ on } \Gamma\right\},$$

where, as before, $1 < r < \infty$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Equation (3.4) of Lemma 3.5 shows that this space is also equal to:

$$Y(\Omega) = \left\{u \in \mathbf{W}^{2,r'}(\Omega)\,;\, u = 0, \frac{\partial u}{\partial n} \cdot n = 0 \text{ on } \Gamma\right\},$$

and the range space of the normal derivative $\gamma_1 : Y(\Omega) \to \mathbf{W}^{1/r,r'}(\Gamma)$ is

$$Z(\Gamma) = \left\{w \in \mathbf{W}^{1/r,r'}(\Gamma)\,;\, w \cdot n = 0\right\}.$$

Now, we define the space

$$X_r(\Omega) = \left\{v \in \mathbf{W}_0^{1,r}(\Omega)\,;\; \operatorname{div} v \in W_0^{1,r}(\Omega)\right\},$$

which is a reflexive Banach space for the norm

$$\|v\|_{X_r(\Omega)} = \|v\|_{\mathbf{W}_0^{1,r}(\Omega)} + \|\operatorname{div} v\|_{W_0^{1,r}(\Omega)}.$$

We skip the proof of the next lemma because it uses standard tools (cf. Temam [37] or Girault & Raviart [20]).

Lemma 4.10. *The space $\mathscr{D}(\Omega)$ is dense in $X_r(\Omega)$.*

In the sequel, we shall also make use of the spaces

$$T_r(\Omega) = \{v \in \mathbf{L}^r(\Omega);\ \Delta v \in (X_{r'}(\Omega))'\}\,,\ T_{r,0}(\Omega) = \{v \in T_r(\Omega);\ \operatorname{div}\, v = 0\}\,,$$

that are reflexive Banach spaces for the norm

$$\|v\|_{T_r(\Omega)} = \|v\|_{\mathbf{L}^r(\Omega)} + \|\Delta v\|_*,$$

where $\|\cdot\|_*$ denotes the dual norm of the space $(X_{r'}(\Omega))'$. A technique of proof similar, although more intricate, than that of Lemme 4.10, allows to prove that

$$\mathscr{D}(\overline{\Omega}) \text{ is dense in } T_r(\Omega) \text{ and } \left\{v \in \mathscr{D}(\overline{\Omega});\ \operatorname{div} v = 0\right\} \text{ is dense in } T_{r,0}(\Omega).$$

Now, on one hand, the functions v of $T_{r,0}(\Omega)$ are such that their normal trace $v \cdot n$ on Γ belongs to $W^{-1/r,r}(\Gamma)$. On the other hand, for all v in $\mathscr{D}(\overline{\Omega})$, we have the following Green's formula

$$\forall \varphi \in Y(\Omega), \quad \langle \Delta v, \varphi\rangle = \langle v, \Delta\varphi\rangle \ - \ \left\langle v, \frac{\partial\varphi}{\partial n}\right\rangle.$$

But recall that $\frac{\partial\varphi}{\partial n}$ sweeps $Z(\Gamma)$ when φ sweeps $Y(\Omega)$ and observe that the dual space $Z'(\Gamma)$ of $Z(\Gamma)$ can be identified with the space $\left\{g \in \mathbf{W}^{-1/r,r}(\Gamma);\ g \cdot n = 0\right\}$. Therefore, by means of the density of $\mathscr{D}(\overline{\Omega})$ in $T_r(\Omega)$, we can show that if Γ is of class $\mathscr{C}^{1,1}$, then the tangential trace of functions of $T_{r,0}(\Omega)$ also belongs to $\mathbf{W}^{-1/r,r}(\Gamma)$ (i.e. the complete trace of v belongs to $\mathbf{W}^{-1/r,r}(\Gamma)$) and

$$\text{(4.13)} \qquad \forall \varphi \in Y(\Omega),\ \forall v \in T_{r,0}(\Omega), \quad \langle \Delta v, \varphi\rangle = \langle v, \Delta\varphi\rangle - \left\langle v, \frac{\partial\varphi}{\partial n}\right\rangle.$$

The following proposition is partly due to Giga [19] (who considered the case of a domain Ω of class $\mathscr{C}^\infty$). We shall prove it in detail.

Proposition 4.11. *Let r be any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$, of class $\mathscr{C}^{1,1}$. Let the boundary data g satisfy*

$$g \in \mathbf{W}^{-1/r,r}(\Gamma), \;\; g \cdot n = 0.$$

Then, the Stokes problem

$$-\Delta v + \nabla q = 0 \textit{ in } \Omega, \tag{4.14}$$

$$\operatorname{div} v = 0 \textit{ in } \Omega, \tag{4.15}$$

$$v = g \textit{ on } \Gamma, \tag{4.16}$$

has exactly one solution $v \in \mathbf{L}^r(\Omega)$ and $q \in W^{-1,r}(\Omega)/\mathbb{R}$. Moreover, there exists a constant $C > 0$ depending only on r and Ω such that

$$\|v\|_{\mathbf{L}^r(\Omega)} + \|q\|_{W^{-1,r}(\Omega)/\mathbb{R}} \leqslant C\|g\|_{\mathbf{W}^{-1/r,r}(\Gamma)}. \tag{4.17}$$

Proof. We shall first prove that if the pair (v, q), with v in $\mathbf{L}^r(\Omega)$ and q in $W^{-1,r}(\Omega)/\mathbb{R}$, satisfies (4.14) and (4.15), then v belongs to $T_{r,0}(\Omega)$ and thus the boundary condition (4.16) makes sense.

For this, observe that, in view of the density stated by Lemma 4.10, if a function q belongs to $W^{-1,r}(\Omega)/\mathbb{R}$ then its gradient ∇q belongs to $(X_{r'}(\Omega))'$ and

$$\|\nabla q\|_* \leqslant \|q\|_{W^{-1,r}(\Omega)/\mathbb{R}}.$$

Hence, if the pair (v, q), with v in $\mathbf{L}^r(\Omega)$ and q in $W^{-1,r}(\Omega)/\mathbb{R}$, satisfies (4.14) and (4.15), then $\Delta v \in (X_{r'}(\Omega))'$; therefore $v \in T_{r,0}(\Omega)$ and its trace belongs to $\mathbf{W}^{-1/r,r}(\Gamma)$.

Next, let us prove that problem (4.14)–(4.16) is equivalent to the variational formulation (cf. [19]): find $v \in \mathbf{L}^r(\Omega)$ and $q \in W^{-1,r}(\Omega)/\mathbb{R}$ such that:

$$\forall u \in Y(\Omega),\ \forall p \in W^{1,r'}(\Omega), \quad \int_\Omega [v(-\Delta u + \nabla p) - q \operatorname{div} u]\, dx = -\int_\Gamma g \frac{\partial u}{\partial n}\, d\sigma \tag{4.18}$$

where of course the integral signs denote adequate dualities. Indeed, let (v, q) be a solution of (4.14)–(4.16); Green's formula (4.13) yields for all $u \in Y(\Omega)$:

$$-\langle \Delta v, u\rangle + \langle \nabla q, u\rangle = -\langle v, \Delta u\rangle + \left\langle g, \frac{\partial u}{\partial n}\right\rangle - \langle q, \operatorname{div} u\rangle = 0.$$

In addition, for all $p \in W^{1,r'}(\Omega)$, we have

$$\langle v, \nabla p\rangle = -\langle \operatorname{div} v, p\rangle + \langle v \cdot n, p\rangle = 0.$$

(Here, we use the density of the functions of $\mathscr{D}(\overline{\Omega})$ with divergence zero in $T_{r,0}(\Omega)$.) This shows that the pair (v, q) is a solution of the variational formulation (4.18). Conversely, we readily prove that if (v, q) satisfies the variational formulation (4.18), then (v, q) is a solution of problem (4.14)–(4.16).

Now, let us solve problem (4.18). According to Theorem 4.8 applied with $m = 2$, for each $f \in \mathbf{L}^{r'}(\Omega)$ and $\varphi \in W_0^{1,r'}(\Omega) \cap L_0^{r'}(\Omega)$, there exists a unique $u \in Y(\Omega)$ and $p \in W^{1,r'}(\Omega)/\mathbb{R}$ solution of:

$$\begin{aligned} -\Delta u + \nabla p &= f \text{ in } \Omega, \\ \operatorname{div} u &= \varphi \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma. \end{aligned}$$

Furthermore, by virtue of the continuity of the mapping $\gamma_1 : W^{2,r'}(\Omega) \longrightarrow W^{1/r,r'}(\Gamma)$ and the estimate (4.11), for any pair (f, φ) in $\mathbf{L}^{r'}(\Omega) \times [W_0^{1,r'}(\Omega) \cap L_0^{r'}(\Omega)]$, we have:

$$\begin{aligned} \left| \int_\Gamma g \frac{\partial u}{\partial n} \, \mathrm{d}\sigma \right| &\leqslant C_1 \|g\|_{\mathbf{W}^{-1/r,r}(\Gamma)} \|u\|_{\mathbf{W}^{2,r'}(\Omega)} \\ &\leqslant C_2 \|g\|_{\mathbf{W}^{-1/r,r}(\Gamma)} (\|f\|_{\mathbf{L}^{r'}(\Omega)} + \|\varphi\|_{W^{1,r'}(\Omega)}). \end{aligned}$$

In other words, the mapping $(f, \varphi) \longmapsto \int_\Gamma g \frac{\partial u}{\partial n} \, \mathrm{d}\sigma$ defines an element of the dual space of $\mathbf{L}^{r'}(\Omega) \times [W_0^{1,r'}(\Omega) \cap L_0^{r'}(\Omega)]$, with norm bounded by $C_2 \|g\|_{\mathbf{W}^{-1/r,r}(\Gamma)}$.

Finally, observing that $(\mathbf{L}^{r'}(\Omega))' = \mathbf{L}^r(\Omega)$ and the dual of $W_0^{1,r'}(\Omega) \cap L_0^{r'}(\Omega)$ is $W^{-1,r}(\Omega)/\mathbb{R}$, we infer by Riesz' representation theorem that there exists a unique $v \in \mathbf{L}^r(\Omega)$ and $q \in W^{-1,r}(\Omega)/\mathbb{R}$ satisfying (4.14)–(4.16) and the bound (4.17). $\square$

The next corollary relaxes the constraint on the data g.

Corollary 4.12. *We retain the assumptions of Proposition* 4.11, *but here we suppose only that $g \in \mathbf{W}^{-1/r,r}(\Gamma)$ satisfies the compatibility condition $\langle g \cdot n, 1 \rangle = 0$. Then problem* (4.14)–(4.16) *has a unique solution $v \in \mathbf{L}^r(\Omega)$ and $q \in W^{-1,r}(\Omega)/\mathbb{R}$ and the bound* (4.17) *holds.*

P r o o f. Let us solve the Neumann problem

$$\begin{aligned} \Delta \theta &= 0 \text{ in } \Omega, \\ \frac{\partial \theta}{\partial n} &= g \cdot n \text{ on } \Gamma. \end{aligned}$$

It has a unique solution $\theta \in W^{1,r}(\Omega)$. Set $w = \nabla \theta$; then w belongs to $T_{r,0}(\Omega)$ and

$$\|w\|_{T_r(\Omega)} = \|w\|_{\mathbf{L}^r(\Omega)} \leqslant C \|g\|_{\mathbf{W}^{-1/r,r}(\Gamma)}.$$

Let $\tilde{g} = g - \gamma_0(w)$, $\tilde{g}$ satisfies the hypotheses of Proposition 4.11, and therefore there exist v_0 and q_0 solution of problem (4.14)–(4.16) with g replaced by $\tilde{g}$. Then the pair of functions $v = v_0 + w$ and q_0 is the required solution. $\square$

Remark 4.13. The above arguments also allow to solve problem (4.14)–(4.16) with a nonzero right-hand side f in $(X_{r'}(\Omega))'$, a space just a little smaller than $\mathbf{W}^{-1,r}(\Omega)$. Thus, for f given in $(X_{r'}(\Omega))'$ and g given in $\mathbf{W}^{-1/r,r}(\Gamma)$ satisfying $\langle g \cdot n, 1\rangle = 0$, there exists a unique u in $\mathbf{L}^r(\Omega)$ and p in $W^{-1,r}(\Omega)/\mathbb{R}$ such that

$$\begin{aligned} &-\Delta u + \nabla p = f \text{ in } \Omega, \quad \operatorname{div} u = 0 \text{ in } \Omega, \\ &u = g \text{ on } \Gamma. \end{aligned}$$

This can be used to establish a Helmholtz decomposition analogous to (4.9) for functions in $T_r(\Omega)$.

Remark 4.14. There is another more familiar Helmholtz decomposition that can be derived directly for functions of $\mathbf{L}^r(\Omega)$ (cf. Fujiwara & Morimoto [15] and von Wahl [38]). With the notation of Paragraph 2, we have

$$\mathbf{L}^r(\Omega) = V_{0,r} \oplus Y_r,$$

where $Y_r = \left\{\nabla q;\, q \in W^{1,r}(\Omega)\right\}$. Indeed, let $u \in \mathbf{L}^r(\Omega)$ and let $p \in W^{1,r}(\Omega)/\mathbb{R}$ be the unique solution of the nonhomogeneous Neumann problem:

$$\forall q \in W^{1,r'}(\Omega), \quad (\nabla p, \nabla q) = (u, \nabla q).$$

Then $u - \nabla p \in V_{0,r}$; set $v = u - \nabla p$; therefore, $u = v + \nabla p$ with $v \in V_{0,r}$ and $p \in W^{1,r}(\Omega)$. Moreover, assuming that the boundary is of class $\mathscr{C}^{1,1}$, there exists a constant $C > 0$ independent of u such that:

$$\|p\|_{W^{1,r}(\Omega)/\mathbb{R}} \leqslant C\|u\|_{\mathbf{L}^r(\Omega)}.$$

Clearly, this is a direct sum, because if $v \in V_{0,r}$ is of the form $v = \nabla p$ with $p \in W^{1,r}(\Omega)$, then $\Delta p = 0$ in Ω and $\frac{\partial p}{\partial n} = 0$ on Γ. Hence p is constant and $v = 0$.

4.3. The case $m = 1$

By interpolating between Theorem 4.8 with $m = 2$ and Corollary 4.12, we derive our next result.

Corollary 4.15. *Let r be any real number with $1 < r < \infty$, let Ω be a bounded and connected domain of $\mathbb{R}^d$ of class $\mathscr{C}^{1,1}$ and let the data g satisfy*

$$g \in \mathbf{W}^{1-1/r,r}(\Gamma)$$

and the compatibility condition $\int_\Gamma g \cdot n \, \mathrm{d}\sigma = 0$. *Then, problem* (4.14)–(4.16) *has a unique solution* $v \in \mathbf{W}^{1,r}(\Omega)$ *and* $q \in L^r(\Omega)/\mathbb{R}$ *and there exists a constant* $C > 0$ *depending only on* r *and* Ω *such that*

$$\|v\|_{\mathbf{W}^{1,r}(\Omega)} + \|q\|_{L^r(\Omega)/\mathbb{R}} \leqslant C\|g\|_{\mathbf{W}^{1-1/r,r}(\Gamma)}.$$

Lemma 4.16. *Let* r *be any real number with* $1 < r < \infty$, *let* Ω *be a bounded and connected domain of* $\mathbb{R}^d$ *of class* $\mathscr{C}^{1,1}$ *and let* φ *be given in* $L^r_0(\Omega)$. *Then, the problem*

$$-\Delta w + \nabla\theta = 0 \text{ in } \Omega, \tag{4.19}$$

$$\operatorname{div} w = \varphi \text{ in } \Omega, \tag{4.20}$$

$$w = 0 \text{ on } \Gamma, \tag{4.21}$$

has a unique solution $w \in \mathbf{W}^{1,r}_0(\Omega)$ *and* $\theta \in L^r(\Omega)/\mathbb{R}$ *and there exists a constant* $C > 0$ *depending only on* r *and* Ω *such that*

$$\|w\|_{\mathbf{W}^{1,r}(\Omega)} + \|\theta\|_{L^r(\Omega)/\mathbb{R}} \leqslant C\|\varphi\|_{L^r(\Omega)}. \tag{4.22}$$

Proof. The uniqueness is obvious. To prove existence, we solve the homogeneous Neumann problem

$$\Delta\psi = \varphi \text{ in } \Omega,$$
$$\frac{\partial\psi}{\partial n} = 0 \text{ on } \Gamma.$$

It has a unique solution $\psi \in W^{2,r}(\Omega)$ and setting $w_0 = \nabla\psi$, we have

$$\|w_0\|_{\mathbf{W}^{1,r}(\Omega)} \leqslant C_1\|\varphi\|_{L^r(\Omega)}.$$

Now, problem (4.19)–(4.21) is equivalent to finding w and θ such that

$$-\Delta(w - w_0) + \nabla(\theta - \Delta\psi) = 0 \text{ in } \Omega,$$
$$\operatorname{div}(w - w_0) = 0 \text{ in } \Omega,$$
$$w - w_0 = -w_0 \text{ on } \Gamma.$$

As $w_0 \cdot n = 0$ on Γ, the existence follows from Corollary 4.15; the continuity of the trace $\gamma_0 \colon \mathbf{W}^{1,r}(\Omega) \mapsto \mathbf{W}^{1-1/r,r}(\Gamma)$ gives:

$$\|w - w_0\|_{\mathbf{W}^{1,r}(\Omega)} + \|\theta - \Delta\psi\|_{L^r(\Omega)/\mathbb{R}} \leqslant C_2\|w_0\|_{\mathbf{W}^{1-1/r,r}(\Gamma)} \leqslant C_3\|\varphi\|_{L^r(\Omega)}.$$

This proves (4.22). □

With this lemma, we can prove the decomposition of vector fields for $m = 1$.

Proposition 4.17. *Let r be any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$ of class $\mathscr{C}^{1,1}$. The following decomposition holds:*

$$\mathbf{W}_0^{1,r}(\Omega) = V_{1,r} \oplus G_{1,r}.$$

P r o o f. Let $u \in \mathbf{W}_0^{1,r}(\Omega)$. Then, $\operatorname{div} u \in L_0^r(\Omega)$ and therefore, by virtue of Lemma 4.16, the problem

$$\begin{aligned} -\Delta w + \nabla \theta &= 0 \text{ in } \Omega, \\ \operatorname{div} w &= \operatorname{div} u \text{ in } \Omega, \\ w &= 0 \text{ on } \Gamma, \end{aligned}$$

has a unique solution $w \in \mathbf{W}_0^{1,r}(\Omega)$ and $\theta \in L^r(\Omega)/\mathbb{R}$ and it satisfies

$$\|w\|_{\mathbf{W}^{1,r}(\Omega)} + \|\theta\|_{L^r(\Omega)/\mathbb{R}} \leqslant C\|\operatorname{div} u\|_{L^r(\Omega)}.$$

Take $v = u - w$; then v belongs to $V_{1,r}$; and since $u = v + w$, with $w \in G_{1,r}$, this is the desired decomposition. $\square$

This decomposition enables us to establish, with exactly the same proof, the analogue of Theorem 4.6 with $m = 1$.

Theorem 4.18. *Let r be any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$ of class $\mathscr{C}^{1,1}$. For each right-hand side $f \in \mathbf{W}^{-1,r}(\Omega)$, the homogeneous Stokes problem has a unique solution $u \in \mathbf{W}_0^{1,r}(\Omega)$ and $p \in L^r(\Omega)/\mathbb{R}$. It satisfies the bound*

$$\|u\|_{\mathbf{W}^{1,r}(\Omega)} + \|p\|_{L^r(\Omega)/\mathbb{R}} \leqslant C\|f\|_{\mathbf{W}^{-1,r}(\Omega)},$$

with a constant $C > 0$ depending only on r and Ω.

Remark 4.19. Once the homogeneous Stokes problem is solved, the nonhomogeneous one can be solved as for $m \geqslant 2$.

Remark 4.20. The regularity assumption on the domain is probably not optimal when $m = 1$. This comes from the method of proof which makes use (because of the interpolation) of the regularity required by the case where $m = 2$.

Remark 4.21. Let m belong to $\mathbb{N} \cup \{-1\}$, r be any real number with $1 < r < \infty$ and let Ω be a bounded, connected domain of $\mathbb{R}^d$ of class $\mathscr{C}^{m+1,1}$. It follows from Corollary 3.8 that the divergence operator maps $\mathbf{W}^{m+2,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$ onto $W^{m+1,r}(\Omega) \cap L_0^r(\Omega)$. Then, it follows from Proposition 4.5 if $m \geqslant 0$ or Proposition 4.17 if $m = -1$ (assuming in this case that Ω is of class $\mathscr{C}^{1,1}$) that the divergence operator is an isomorphism from $\mathbf{W}^{m+2,r}(\Omega) \cap G_{1,r}$ onto $W^{m+1,r}(\Omega) \cap L_0^r(\Omega)$.

5. The penalty method

This short paragraph is devoted to the penalty method introduced by Temam in [37] to decouple the computation of the velocity from that of the pressure (cf. also Dautray & Lions [14, vol. 7] or Girault & Raviart [20]).

Let (u, p) be the solution of the homogeneous Stokes problem with right-hand side f. Recall that the penalty method replaces the Stokes system by:

$$-\Delta u_\varepsilon - \frac{1}{\varepsilon}\nabla \operatorname{div} u_\varepsilon = f \quad \text{in } \Omega, \tag{5.1}$$

$$u_\varepsilon = 0 \quad \text{on } \Gamma, \tag{5.2}$$

where ε is a positive parameter that will tend to zero. The pressure is approximated by setting $p_\varepsilon = -\frac{1}{\varepsilon}\operatorname{div}\, u_\varepsilon$ and obviously, u_ε is intended to approximate u. The next theorem, proved by Temam in [37] (cf. also [14] or [20]), establishes the convergence of this method.

Theorem 5.1. *Let Ω be a bounded, Lipschitz-continuous, connected domain of $\mathbb{R}^d$ and let f belong to $\mathbf{H}^{-1}(\Omega)$. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u \text{ in } \mathbf{H}_0^1(\Omega), \quad p_\varepsilon \to p \text{ in } L^2(\Omega),$$

where (u, p) is the solution of the homogeneous Stokes problem with right-hand side f.

The following theorem extends this convergence to $\mathbf{W}^{m,r}(\Omega)$.

Theorem 5.2. *Let $m \geqslant 1$ be an integer, r any real number with $1 < r < \infty$ and let Ω be a bounded and connected domain of $\mathbb{R}^d$, of class $\mathscr{C}^{m-1,1}$ if $m \geqslant 2$ or $\mathscr{C}^{1,1}$ if $m = 1$. For f given in $\mathbf{W}^{m-2,r}(\Omega)$, let (u, p) be the solution of the homogeneous Stokes problem. Then, as $\varepsilon \to 0$, we have the convergence*

$$u_\varepsilon \to u \text{ in } \mathbf{W}^{m,r}(\Omega), \quad p_\varepsilon = -\frac{1}{\varepsilon}\operatorname{div} u_\varepsilon \to p \text{ in } W^{m-1,r}(\Omega).$$

Proof. System (5.1) is elliptic and has a unique solution u_ε in $\mathbf{W}^{m,r}(\Omega)$ that satisfies the bound

$$\|u_\varepsilon\|_{\mathbf{W}^{m,r}(\Omega)} \leqslant C\|f\|_{\mathbf{W}^{m-2,r}(\Omega)},$$

with a constant C independent of ε. Hence the sequence ∇p_ε is bounded in $\mathbf{W}^{m-2,r}(\Omega)$ and, since p_ε has mean-value zero (because u_ε vanishes on Γ), it follows from Proposition 2.10, that the sequence p_ε is bounded in $W^{m-1,r}(\Omega)$. Therefore, on one hand

$$\operatorname{div} u_\varepsilon \to 0 \ \text{ strongly in } W^{m-1,r}(\Omega).$$

On the other hand, u_ε tends weakly to u in $\mathbf{W}^{m,r}(\Omega) \cap \mathbf{W}_0^{1,r}(\Omega)$ and p_ε tends weakly to p in $W^{m-1,r}(\Omega)$ and it is easy to prove that the pair (u,p) is the solution of the homogeneous Stokes problem with right-hand side f. By taking the difference between this limit system and system (5.1) and applying Theorem 4.8 when $m \geqslant 2$ or Lemma 4.16 when $m = 1$, we obtain:

$$\|u_\varepsilon - u\|_{\mathbf{W}^{m,r}(\Omega)} + \|p_\varepsilon - p\|_{W^{m-1,r}(\Omega)/\mathbb{R}} \leqslant C\|\operatorname{div} u_\varepsilon\|_{W^{m-1,r}(\Omega)}.$$

Therefore $u_\varepsilon - u$ and $p_\varepsilon - p$ both tend to zero at the same rate as $\operatorname{div} u_\varepsilon$, i.e. at the same rate as ε. □

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