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Dao Rong Tong

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THE STRUCTURE OF A COMPLETE l-GROUP

DAO-RONG TON,* Nanjing

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1. PRELIMINARIES

We will use the standard notation for l-groups, cf. [1, 4, 7, 8]. Throughout the whole paper G is an l-group, R is the real group, Q is the rational group, Z is the integer group and N is the set of all natural numbers. Let $\{G_\alpha \mid \alpha \in A\}$ be a system of l-groups and $\prod_{\alpha\in A} G_\alpha$ their direct product. For an element $g \in \prod_{\alpha\in A} G_\alpha$, we denote by g_α the α component of g. An l-group G is said to be a subdirect sum of l-groups G_α, in symbols $G \subseteq' \prod_{\alpha\in A} G_\alpha$, if G is an l-subgroup of $\prod_{\alpha\in A} G_\alpha$ such that for each $\alpha \in A$ and each $g' \in G_\alpha$ there exists $g \in G$ with the property $g_\alpha = g'$. An l-group G is said to be an ideal subdirect sum of l-groups G_α, in symbols $G \subseteq^* \prod_{\alpha\in A} G_\alpha$, if $G \subseteq' \prod_{\alpha\in A} G_\alpha$ and G is an l-ideal of $\prod_{\alpha\in A} G_\alpha$. We denote the l-subgroup of $\prod_{\alpha\in A} G_\alpha$ consisting of the elements with only finitely many non-zero components by $\sum_{\alpha\in A} G_\alpha$. An l-group G is said to be a completely subdirect sum, if G is an l-subgroup of $\prod_{\alpha\in A} G_\alpha$ and $\sum_{\alpha\in A} G_\alpha \subseteq G$.

A subset $D \subseteq G$ with $0\bar{\in} D$ is said to be disjoint, if $g_1 \wedge g_2 = 0$ for any pair of distinct elements $g_1, g_2 \in D$. For any $X \subset G$ we write $X^\perp = \{g \in G \mid |g| \wedge |x| = 0$ for each $x \in X\}$. For $g \in G$, $[g]$ is the convex l-subgroup of G generated by g, (g) is the polar subgroup of G generated by g. Clearly, $[g] \subset (g)$. We denote the least cardinal α such that $|A| \leqslant \alpha$ for each bounded disjoint subset A of G by vG, where $|A|$ denotes the cardinal of A. G is said to be v-homogeneous if $vH = vG$ for any convex l-subgroup $H \neq \{0\}$ of the l-group G. A v-homogeneous l-group G is said to be v-homogeneous of α type if $vG = \alpha$. An l-group G is said to be ic-homogeneous of

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β type if any nontrivial interval in G has the same cardinality β. Let α and β be two cardinal numbers. An l-group G is said to be of (α, β) type if G is v-homogeneous of α type and ic-homogeneous of β type. For example, R is an l-group of $(1, 2^{\aleph_0})$ type. The goal of this paper is to prove that any complete l-group G is l-isomorphic to an ideal subdirect sum of the integer groups Z and complete l-groups of $(\alpha, \aleph_j)$ type. Consequently, we can give a structure character for a complete l-group.

In [10] Jakubík proved that any complete l-group is a completely subdirect sum of v-homogeneous l-groups. Now we can strengthen this result.

Lemma 1.1. *Any complete l-group is l-isomorphic to an ideal subdirect sum of complete v-homogeneous l-groups.*

Proof. Let G be a complete l-group. Without loss of generality, by virtue of Theorem 3.7 in [10] we may assume that

$$\sum_{\delta\in\Delta} T_\delta \subseteq G \subseteq' \prod_{\delta\in\Delta} T_\delta, \tag{1.1}$$

where each T_δ $(\delta \in \Delta)$ is a v-homogeneous l-group.

(1) First we prove that each T_δ $(\delta \in \Delta)$ is complete. For each $\delta \in \Delta$ we put $\bar{T}_\delta = \{g \in G \mid \delta' \neq \delta \Rightarrow g_{\delta'} = 0\}$. It is easy to verify that each $\bar{T}_\delta$ is a direct factor of G and it is a folklore that each direct factor of a complete l-group is again complete. Hence $\bar{T}_\delta$ is complete and thus T_δ is complete as well.

(2) We prove that G is an ideal subdirect sum of T_δ $(\delta \in \Delta)$. Let $0 < g \in \prod\limits_{\delta\in\Delta} T_\delta$, then $g_{\delta_0} > 0$ for some $\delta_0 \in \Delta$. Let $\bar{g}_{\delta_0}$ be the element in $\prod\limits_{\delta\in\Delta} T_\delta$ whose δth component is g_{δ_0} and all other components are zero. Then it follows from (1.1) that $\bar{g}_{\delta_0} \in G$, and so $0 < \bar{g}_{\delta_0} \leqslant g$, therefore $G \subseteq' \prod\limits_{\delta\in\Delta} T_\delta$ is a dense l-subgroup of $\prod\limits_{\delta\in\Delta} T_\delta$. Let $\{x^\alpha \mid \alpha \in A\} \subset \prod\limits_{\delta\in\Delta} T_\delta$ and $x \in \prod\limits_{\delta\in\Delta} T_\delta$. Suppose that $x^\alpha \leqslant x$ for all $\alpha \in A$, then there exists $x'_\delta = \bigvee\limits_{\alpha\in A} {}^{(T_\delta)} x^\alpha_\delta$ for any $\delta \in \Delta$. Put $x' = (\ldots x'_\delta \ldots)$, then $x^\alpha \leqslant x'$ for all $\alpha \in A$. Assuming that y is any upper bound of $\{x^\alpha \mid \alpha \in A\}$, we have $x^\alpha_\delta \leqslant y_\delta$ $(\alpha \in A)$ for any $\delta \in \Delta$. Thus $x'_\delta \leqslant y_\delta$ and $x' \leqslant y$. Therefore $x' = \bigvee\limits_{\alpha\in A}^{(\prod_{\delta\in\Delta} T_\delta)} x^\alpha$. On the other hand, G is complete. So it follows from Lemma 2.3 in [3] that G is an l-ideal of $\prod\limits_{\delta\in\Delta} T_\delta$, i.e.

$$G \subseteq^* \prod_{\delta\in\Delta} T_\delta.$$

□

The following lemma is an immediate consequence of Theorem 1 of the fourth chapter in [7].

Lemma 1.2. *Any non-zero complete totally ordered group is l-isomorphic to the real group or the integer group.*

2. v-HOMOGENEOUS l-GROUP OF $\aleph_i$ TYPE

R and Z are complete v-homogeneous l-groups of 1 type. In this section we will discuss the character of a non-totally ordered complete v-homogeneous l-group. First of all we have

Lemma 2.1. *Let G be v-homogeneous and non-totally ordered. Then $vG \geqslant \aleph_0$.*

Proof. Since G is not totally ordered, there exist incomparable elements $a, b \in G$. Put $a_1 = a-(a\wedge b)$, $b_1 = b-(a\wedge b)$ and $g = a_1\vee b_1$. Then the set $\{a_1, b_1\}$ is disjoint and the convex l-subgroup $[g]$ is not totally ordered. Since G is v-homogeneous, $[b_1]$ is not totally ordered, either. Thus $[0, b_1]$ is not a chain by 4.3 in [10]. Hence there exists a disjoint subset $\{a_2, b_2\} \subseteq [0, b_1]$ and $\{a_1, a_2\}$ is clearly a disjoint set. Analogously we can construct disjoint sets $\{a_1, a_2, \ldots, a_n\}$ $(n = 1, 2, \ldots)$. Then the set $\{a_n\}_{n=1}^{\infty}$ is disjoint as well, it is a subset of $[0, g]$. Hence $vG \geqslant \aleph_0$. □

Thus, if G is a v-homogeneous and non-totally ordered l-group, then there exists an infinite cardinal $\aleph_i$ such that G is a v-homogeneous l-group of $\aleph_i$ type.

From Lemma 1.1, Lemma 1.2 and Lemma 2.1 we get

Proposition 2.2. *Any complete l-group G is l-isomorphic to an ideal subdirect sum of real groups, integer groups and complete v-homogeneous l-groups of $\aleph_i$ type.*

Proposition 2.3. *Let G be an Archimedian v-homogeneous l-group of $\aleph_i$ type and $G \neq \{0\}$. Then G has the following properties:*

(1) G has no basic element,

(2) G has no basic,

(3) the radical $R(G) = G$,

(4) G is not completely distributive,

(5) the distributive radical $D(G) = G$.

Moreover, every non-trivial convex l-subgroup of G enjoys the same five properties.

Proof. By Theorem 5.10 in [4] we need only to prove (1). For any $0 < a \in G$, $v[a] = vG > 1$. So $[a]$ is not totally ordered, and by 4.3 in [10], $[0, a]$ is not totally ordered, either. □

An l-group G is said to be continuous, if for any $0 < x \in G$ we have $x = x_1 + x_2$ and $x_1 \wedge x_2 = 0$, where $x_1 \neq 0$, $x_2 \neq 0$. An l-group G is said to be of countable type, if $vG \leqslant \aleph_0$.

E x a m p l e. Let S be the set of all real, mesurable, almost everywhere finite functions $x(t)$ on a closed interval $[a, b] \subseteq R$. The algebraic operations are introduced in S in the usual way. The class of positive elements is selected in S with the aid of the following definition: we define $x > 0$ ($x \in S$) if $x(t) \geqslant 0$ almost everywhere, but in this connection $x(t) > 0$ on a set of positive measure. Mutually equivalent functions are identified, i.e., they are viewed as the same element of the set S. It is easy to see that S is a complete vector lattice of countable type [12], and it is also easy to see that S is continuous.

Lemma 2.4. *A complete l-group G is continuous if and only if G has no basic element.*

P r o o f. The necessity is clear. Suppose that G has no basic element and $0 < x \in G$. Then $[0, x]$ is not totally ordered. By a standard argument there exist $a_1, b_1 \in [0, x]$ such that $a_1 \wedge b_1 = 0$. Since G is complete, $[x]$ is also complete. From the Riesz decomposition theorem of a complete l-group we have

$$[x] = a_1^{\perp} \boxplus a_1^{\perp\perp}. \tag{2.1}$$

Further, $a_1 \in a_1^{\perp\perp}$ and $b_1 \in a_1^{\perp}$, so $a_1^{\perp} \neq 0$, $a_1^{\perp\perp} \neq 0$. From (2.1) we have

$$x = x_1 + x_2, \qquad 0 < x_1 < x,\ 0 < x_2 < x \text{ and } x_1 \wedge x_2 = 0.$$

Hence G is continuous. □

Lemma 2.5. *Let G be a projectable and non-totally ordered l-group. Then G is directly decomposable.*

P r o o f. Since G is not totally ordered, there exist $a_1, b_1 \in G$ such that $0 < a_1$, $0 < b_1$ and $a_1 \wedge b_1 = 0$. G is projectable, so

$$G = a_1^{\perp} \boxplus a_1^{\perp\perp},$$

where $a_1 \in a_1^{\perp\perp}$, $b_1 \in a_1^{\perp}$. □

An l-group is said to be ideal subdirect irreducible if G cannot be expressed as an ideal of an ideal subdirect sum of non-zero l-groups.

Lemma 2.6. *A complete l-group G is directly indecomposable if and only if G is ideal subdirect irreducible.*

Proof. Necessity. Suppose that $G \neq \{0\}$ is directly indecomposable. If $G \subseteq^* \prod_{\delta\in\Delta} G_\delta$, then $\sum_{\delta\in\Delta} G_\delta \subseteq G$. Put $\overline{G}_\delta = \{g \in G \mid \delta' \neq \delta \Rightarrow g_{\delta'} = 0\}$ for $\delta \in \Delta$. Then there exists $\delta \in \Delta$ with $\overline{G}_\delta \neq \{0\}$ and

$$G = \overline{G}_\delta \boxplus \overline{G}_\delta^{\perp},$$

where $\overline{G}_\delta^{\perp} = \{g \in G \mid g_\delta = 0\}$.

The sufficiency is obvious. □

Lemma 2.7. *An Archimedean l-group G is subdirectly irreducible if and only if the Dedekind completion $G^\wedge$ of G is ideal subdirect irreducible.*

Proof. Necessity. Suppose that G is subdirectly irreducible. If $G^\wedge \subseteq^* \prod_{\delta\in\Delta} G_\delta$ then

$$G \subseteq' \prod_{\delta\in\Delta} G'_\delta,$$

where $G'_\delta = G\varrho_\delta$ and ϱ_δ is the projection from $G^\wedge$ onto G_δ for $\delta \in \Delta$. So $G^\wedge$ must be ideal subdirect irreducible.

Sufficiency. Suppose that $G^\wedge$ is ideal subdirect irreducible. Since any non-zero complete l-group is l-isomorphic to an ideal subdirect sum of real groups, integer groups and complete v-homogeneous l-groups of $\aleph_i$ type, by Lemma 2.5 any complete v-homogeneous l-group of $\aleph_i$ type is directly decomposable. So $G^\wedge = R$ or Z and G is a subgroup of reals. □

Now from Lemma 2.4, Lemma 2.5 and Lemma 2.6 we have

Proposition 2.8. *Let G be a complete v-homogeneous l-group of $\aleph_i$ type. Then*
(1) *G is continuous,*
(2) *G is directly decomposable,*
(3) *G is not ideal subdirect irreducible,*
(4) *G has a closed l-ideal.*
Moreover, each nontrivial convex l-subgroup of G enjoys the same four properties.

From Lemma 2.7 and Proposition 2.8 we obtain

Corollary 2.9. *An Archimedean v-homogeneous l-group of $\aleph_i$ type is not subdirectly irreducible.*

Now let G be an Archimedean v-homogeneous l-group of $\aleph_i$ type. Then the divisible hull G^d of G is a vector space over Q. If $\{x_\alpha \mid \alpha \in A\}$ is a disjoint subset in G^d, then $\{x_\alpha \mid \alpha \in A\}$ is linearly independent. In fact, suppose that there exists a finite subset $\{x_{\alpha_1}, \dots, x_{\alpha_n}\}$ in $\{x_\alpha \mid \alpha \in A\}$ which is linearly dependent. That is, there exist $\lambda_i \in Q$ $(i = 1, \dots, n)$ (not all 0) such that

$$\lambda_1 x_{\alpha_1} + \dots + \lambda_n x_{\alpha_n} = 0.$$

Then we have

$$x_{\alpha_i} = \sum_{k \neq i} \left(- \frac{\lambda_k}{\lambda_i} \right) x_{\alpha_k}$$

for some $\lambda_i \neq 0$. But in this case $x_{\alpha_i} \wedge x_{\alpha_k} \neq 0$ for some $k \neq i$, a contradiction. Conversely, if $\{x_\alpha \mid \alpha \in A\}$ is linearly independent, then $\{x_\alpha \mid \alpha \in A\}$ need not be a disjoint subset. In particular, we have

Proposition 2.10. *Let G be an Archimedean v-homogeneous l-group of $\aleph_i$ type. If $\{x_\alpha \mid \alpha \in A\}$ is a maximal linearly independent subset in G^d, then $\{x_\alpha \mid \alpha \in A\}$ is not disjoint.*

Proof. Assume that $\{x_\alpha \mid \alpha \in A\}$ is a maximal linearly independent subset in G^d. If $\{x_\alpha \mid \alpha \in A\}$ is disjoint, take some x_{α_0} $(\alpha_0 \in A)$. Then $x_{\alpha_0} = x'_{\alpha_0}/n$ with $x'_{\alpha_0} \in G$ and $n \in N$. Since G is v-homogeneous l-group of $\aleph_i$ type, $v[x'_{\alpha_0}] = vG > 1$. So there exist $0 < y_{\beta_1}, y_{\beta_2} \leqslant x'_{\alpha_0}$ such that $y_{\beta_1} \wedge y_{\beta_2} = 0$. Since $\{x_\alpha \mid \alpha \in A\}$ is maximal linearly independent, there exists a finite subset $\{x_{\alpha_i} \mid i = 1, \dots, n\}$ of $\{x_\alpha \mid \alpha \in A\}$ such that $\{y_{\beta_1}, x_{\alpha_i} \mid i = 1, \dots, n\}$ is linearly dependent. Hence $y_{\beta_1} = \sum_{i=0}^{n} \lambda_i x_{\alpha_i}$. It is easy to see that $y_{\beta_1} > 0$ implies $\lambda_i \geqslant 0$ $(i = 0, 1, \dots, n)$ by the Bernau representation of an Archimedean l-group (see Theorem 3.3 in [5]). It is also easy to see that $x_1 \wedge x_2 = 0$ if and only if $\lambda_1 x_1 \wedge \lambda_2 x_2 = 0$ for $x_1, x_2 \in G^d$ and $\lambda_1, \lambda_2 \in Q$. Hence, if $\alpha_i \neq \alpha_0$, then

$$0 = y_{\beta_1} \wedge x_{\alpha_i} = \left(\sum_{j=0}^{n} \lambda_j x_{\alpha_j} \right) \wedge x_{\alpha_i} = \left(\bigvee_{j=0}^{n} \lambda_j x_{\alpha_j} \right) \wedge x_{\alpha_i}$$
$$= \lambda_i x_{\alpha_i} \wedge x_{\alpha_i}.$$

So $\lambda_i = 0$ $(i = 1, \dots, n)$ if $\alpha_i \neq \alpha_0$. Thus there exists $j \in \{1, 2, \dots, n\}$ such that $\alpha_j = \alpha_0$. Then

$$y_{\beta_1} = y_{\beta_1} \wedge x_{\alpha_j}$$

and by an analogous method as above we get

$$y_{\beta_1} \wedge x_{\alpha_j} = \lambda_j x_{\alpha_j} \wedge x_{\alpha_j} = (\lambda_j + 1) x_{\alpha_j},$$

hence $\lambda_j + 1 > 0$. Put $= \lambda_j - 1 = \lambda_0$. Thus $y_{\beta_1} = \lambda_0 x_{\alpha_0}$. Similarly, $y_{\beta_2} = \mu_0 x_{\alpha_0}$. But in this case

$$y_{\beta_1} \wedge y_{\beta_2} = \lambda_0 x_{\alpha_0} \wedge \mu_0 x_{\alpha_0} \neq 0,$$

a contradiction. □

3. Complete *ic*-homogeneous *l*-group of $\aleph_j$ type

In this section we will discuss properties of a complete *ic*-homogeneous *l*-group of $\aleph_j$ type.

Proposition 3.1. *Let G be a complete ic-homogeneous l-group of α type and $vG = \aleph_i$. Then $\alpha = \alpha^{\aleph_j}$ for any $\aleph_j < \aleph_i$ if i is a limit ordinal, and $\alpha = \alpha^{\aleph_i}$ if i is not a limit ordinal or $\aleph_i = \aleph_0$.*

Proof. Suppose i is a limit ordinal and $\aleph_j < \aleph_i$. Then there exists a bounded disjoint subset $\{x_\alpha \mid \alpha \in A\}$ in G with $|A| = \aleph_j$. Put

$$x = \bigvee_{\alpha \in A} x_\alpha.$$

Consider the mapping $\varphi\colon y \to \{y \wedge x_\alpha\}$ of the lattice $[0, x]$ onto $\prod\limits_{\alpha \in A} [0, x_\alpha]$. By the infinite distributivity of $[0, x]$ it is easy to show that φ is an isomorphism. Hence $\alpha = \alpha^{\aleph_j}$. If i is not a limit ordinal or $\aleph_i = \aleph_0$, then there exists a bounded disjoint subset $\{x_\alpha \mid \alpha \in A\}$ in G such that $|A| = \aleph_i$. So we have $\alpha = \alpha^{\aleph_i}$ similarly as before. □

Proposition 3.2. *Let G be an ic-homogeneous l-group of $\aleph_j$ type. Then the divisible hull G^d of G is also an ic-homogeneous l-group of $\aleph_j$ type.*

Proof. Suppose $0 < g \in G$. Then $\operatorname{card}[0, g]^G = \aleph_j$. If $g' \in [0, g]^{G^d}$, then $g' = \bar{g}/m$ with $\bar{g} \in G$ and $\bar{g} = mg' \in [0, mg]^G$. Hence

$$\aleph_j = \operatorname{card}[0, g]^G \leqslant \operatorname{card}[0, g]^{G^d} \leqslant \operatorname{card}\left(\bigcup_{m=1}^{\infty} [0, mg]^G\right) \leqslant \aleph_0 \cdot \aleph_j = \aleph_j.$$

So

$$\operatorname{card}[0, g]^{G^d} = \aleph_j.$$

Now assume $0 < g \in G^d$. Then $g = g'/n$ with $g' \in G$, and so

$$\operatorname{card}[0, g]^{G^d} = \operatorname{card}[0, ng]^{G^d} = \operatorname{card}[0, g']^{G^d} = \aleph_j.$$

□

Lemma 3.3. *Let $\{G_\delta \mid \delta \in \Delta\}$ be a collection of ic-homogeneous l-groups of $\aleph_j$ type. If $|\Delta| < \max\{vG_\delta \mid \delta \in \Delta\}$, then any subdirect sum $G \subseteq' \prod_{\delta\in\Delta} G_\delta$ of $\{G_\delta \mid \delta \in \Delta\}$ is also ic-homogeneous of $\aleph_j$ type.*

Proof. For any $0 < x \in G \subseteq' \prod_{\delta\in\Delta} G_\delta$, let $x = (\ldots x_\delta \ldots)$. Consider some $\delta_0 \in \Delta$. For any $y_{\delta_0} \in G_{\delta_0}$ with $0 \leqslant y_{\delta_0} \leqslant x_{\delta_0}$ there exists $z \in G$ such that $z_{\delta_0} = y_{\delta_0}$. Then

$$(z \vee 0) \wedge x \in [0, x]^G.$$

So there exists a one-to-one mapping from $[0, x_{\delta_0}]^{G_{\delta_0}}$ into $[0, x]^G$. Hence

$$\operatorname{card}[0, x_{\delta_0}]^{G_{\delta_0}} \leqslant \operatorname{card}[0, x]^G \leqslant \operatorname{card} \prod_{\delta\in\Delta} [0, x_\delta]^{G_\delta}.$$

By Proposition 3.1 we have

$$\aleph_j \leqslant \operatorname{card}[0, x]^G \leqslant \aleph_j^{|\Delta|} = \aleph_j.$$

That is,

$$\operatorname{card}[0, x]^G = \aleph_j.$$

For any nontrivial interval $[a, b]$ in G we have

$$\operatorname{card}[a, b]^G = \operatorname{card}[0, b - a]^G = \aleph_j.$$

□

4. The structure character of a complete l-group

In this section we first give some properties of an l-group of $(\aleph_i, \aleph_j)$ type.

Lemma 4.1. *Let G be a complete l-group of $(\aleph_i, \aleph_j)$ type. Then*

(1) $\aleph_j^{\aleph_l} = \aleph_j$ for any $\aleph_l < \aleph_i$ if i is a limit ordinal and $\aleph_j^{\aleph_i} = \aleph_j$ if i is not a limit ordinal or $\aleph_i = \aleph_0$.

(2) $\aleph_i \leqslant \aleph_j$. If i is not a limit ordinal or $\aleph_i = \aleph_0$, then $2^{\aleph_i} \leqslant \aleph_j$.

Proof. (1) It follows from Proposition 3.1.

(2) Let G be a complete l-group of $(\aleph_i, \aleph_j)$ type and $[0, g]$ a nontrivial interval in G. Assume that neither i is a limit ordinal nor $\aleph_i = \aleph_0$. Since $v[g] = vG = \aleph_i$, there exists a disjoint subset $\{x_\alpha \mid \alpha \in A\}$ in $[g]$ such that $|A| = \aleph_i$. Then $\{x_\alpha \wedge g \mid \alpha \in A\}$

is also a disjoint subset in $[0,g]$. For a subset A_β of A, put $z_\beta = \bigvee_{\alpha\in A_\beta} (x_\alpha \wedge g)$. Then $z_\beta \in [0,g]$. Using the Bernau representation of a complete l-group, it is easy to see that $A_\beta \neq A_{\beta'}$ implies $z_\beta \neq z_{\beta'}$. (In fact, $[g]$ is a complete l-group. There exists a maximal disjoint subset M in $[g]$ such that $M \supseteq \{x_\alpha \wedge g \mid \alpha \in A\}$. By Theorem 3.3 in [5], we can choose an l-isomorphism π such that $M\pi$ is a set of characteristic functions of a family of pairwise disjoint clopen subsets of the Stone space X whose union is dense in X.) Let B be the set of all subsets of A. Then

$$\aleph_j = \operatorname{card}[0,g] \geqslant |B| = 2^{\aleph_i} > \aleph_i.$$

If i is a limit ordinal, for any $\aleph_l < \aleph_i$ there exists a disjoint subset $\{x_\alpha \mid \alpha \in A\}$ in $[g]$ such that $|A| = \aleph_l$. Similarly we have $\aleph_j \geqslant 2^{\aleph_l} > \aleph_l$. So $\aleph_j \geqslant \aleph_i$. □

Lemma 4.2. *An ideal subdirect sum of finitely many complete l-groups of $(\aleph_i, \aleph_j)$ type is also a complete l-group of $(\aleph_i, \aleph_j)$ type.*

Proof. Suppose

$$G \subseteq^* \prod_{i=1}^{n} G_i,$$

where G_i $(i = 1, \ldots, n)$ is a complete l-group of $(\aleph_i, \aleph_j)$ type. Then $G = \prod_{i=1}^{n} G_i$. Let G' be a convex l-subgroup of G. Then

$$vG' \leqslant vG = v(\prod_{i=1}^{n} G_i) \leqslant \aleph_i^n = \aleph_i. \tag{4.1}$$

On the other hand, let ϱ_i be the projection to G_i. Then $G'\varrho_i$ is a convex l-subgroup in G_i. Put

$$\overline{G}_i = \{g \in G \mid j \neq i \Rightarrow g_j = 0,\ g_i \in G'\varrho_i\}.$$

Then $\overline{G}_i$ is a convex l-subgroup in G' and so

$$vG' \geqslant v\overline{G}_i = vG'\varrho_i = \aleph_i. \tag{4.2}$$

Combining (4.1) and (4.2) we get $vG' = \aleph_i$ for any convex l-subgroup of G. Hence G is a v-homogeneous l-group of $\aleph_i$ type. Now let $[a,b]$ be any nontrivial interval in G. Then

$$\aleph_j \leqslant \operatorname{card}[a,b] \leqslant \aleph_j^n = \aleph_j.$$

So $\operatorname{card}[a,b] = \aleph_j$, and G is also an ic-homogeneous l-group of $\aleph_j$ type. □

Proceeding similarly as in the proof of Lemma 1.1, from Theorem 3.7 in [11] we obtain

Proposition 4.3. *Any complete l-group G is l-isomorphic to an ideal subdirect sum of integer groups and complete ic-homogeneous l-groups.*

Let G be a complete v-homogeneous l-group of $\aleph_i$ type. Then no direct summand of G is Z or R. Further, every direct summand of a complete v-homogeneous l-group of $\aleph_i$ type is also a complete v-homogeneous l-group of $\aleph_i$ type. So Proposition 4.3 yields

Lemma 4.4. *A complete v-homogeneous l-group G of $\aleph_i$ type is l-isomorphic to an ideal subdirect sum of complete l-groups of $(\aleph_i, \aleph_j)$ type.*

Theorem 4.5. *Any complete l-group G is l-isomorphic to an ideal subdirect sum of integer groups and complete l-groups of $(\alpha, \aleph_j)$ type.*

Proof. By Proposition 2.2, without loss of generality, we have

$$G \subseteq^* \prod_{\delta\in\Delta} G_\delta, \tag{4.3}$$

where each $G_\delta = Z$ or R or a complete v-homogeneous l-group of $\aleph_i$ type for $\delta \in \Delta$. If G_δ is a complete v-homogeneous l-group of $\aleph_i$ type, then, by Lemma 4.4, we have

$$G_\delta \subseteq^* \prod_{\lambda\in\Lambda_\delta} G_{\lambda\delta}, \tag{4.4}$$

where each $G_{\lambda\delta}$ is a complete l-group of $(\aleph_i, \aleph_j)$ type. Because an ideal subdirect sum of ideal subdirect sums of complete l-groups is still an ideal subdirect sum of complete l-groups, so substituting (4.4) into (4.3) we get

$$G \subseteq^* \prod_{\lambda\in\Lambda} G_\lambda, \tag{4.5}$$

where each G_λ is either Z or a complete l-group of $(\alpha, \aleph_j)$ type. □

In this section we deal with the essential closure of a complete l-group. Let G be a complete l-group and $0 < x \in G$. Put

$$P(x) = \{x_1 \in [x] \mid x = x_1 + x_1',\ x_1 \wedge x_1' = 0\}.$$

For example, if $G = R$ and $0 < x \in G$, then $P(x) = \{0, x\}$. If G is a complete v-homogeneous l-group of $\aleph_i$ type and $0 < x \in G$, then G is continuous by Proposition 2.8 and it is easy to verify that $P(x)$ is infinite.

Lemma 5.1. *Let G be a complete l-group and $0 < x \in G$. Then $P(x)$ is a complete Boolean algebra.*

Proof. For any $x_1 \in P(x)$ we have $x = x_1 + x_1'$ with $x_1 \wedge x_1' = 0$. So $x = x_1 \vee x_1'$. Hence $P(x)$ is a Boolean algebra. Let $x_\alpha \in P(x)$ $(\alpha \in A)$. Then

$$x = x_\alpha + x_\alpha',\ x_\alpha \wedge x_\alpha' = 0$$

for $\alpha \in A$. Since G is complete and $0 \leqslant x_\alpha \leqslant x$ $(\alpha \in A)$, there exist $y = \bigvee_{\alpha \in A} x_\alpha$ and $z = \bigwedge_{\alpha \in A} x_\alpha$. By elementary calculations we obtain

$$y \wedge z = 0,\ y \vee z = x.$$

Hence $P(x)$ is a complete Boolean algebra. □

Let G be an l-group, let $P(G)$ denote the Boolean algebra of all polars in G. Let $P_p(G) = \{g^{\perp\perp} \mid g \in G\}$ be the set of all principal polars of G, and let $\operatorname{Co} P_p(G) = \{g^\perp \mid g \in G\}$. The map $a^{\perp\perp} \to a^\perp$ is a lattice anti-isomorphism between $P_p(G)$ and $\operatorname{Co} P_p(G)$. From Theorem 5.2.9 in [8] we obtain

Lemma 5.2. *Let G be an l-group. Then for any $0 < x_1, x_2 \in G$,*

$$x_1^\perp \wedge x_2^\perp = (x_1 \vee x_2)^\perp,\ x_1^\perp \Cup x_2^\perp = (x_1 \wedge x_2)^\perp,$$

where $\Cup$ is the polar join.

From Lemma 5.2 we have

Lemma 5.3. *Let G be an l-group and $0 < x \in G$. Then for any $0 < x_1, x_2 \in P(x)$,*

$$x_{1[x]}^\perp \wedge x_{2[x]}^\perp = (x_1 \vee x_2)_{[x]}^\perp,\ x_{1[x]}^\perp \Cup x_{2[x]}^\perp = (x_1 \wedge x_2)_{[x]}^\perp$$

and

$$x^{\perp}_{1x^{\perp\perp}} \wedge x^{\perp}_{2x^{\perp\perp}} = (x_1 \vee x_2)^{\perp}_{x^{\perp\perp}}, \ x^{\perp}_{1x^{\perp\perp}} \Cup x^{\perp}_{2x^{\perp\perp}} = (x_1 \wedge x_2)^{\perp}_{x^{\perp\perp}},$$

where $x^{\perp}_{1[x]}$ and $x^{\perp}_{1x^{\perp\perp}}$ denote the principal polars in $[x]$ and $x^{\perp\perp}$, respectively, and similarly for x_2.

Lemma 5.4. *Let G be a complete l-group and $0 < x \in G$. Then $P([x])$ $(P(x^{\perp\perp}))$ and $P(x)$ are anti-isomorphic, and $P([x])$ and $P(x^{\perp\perp})$ are isomorphic as Boolean algebras.*

Proof. First we show that there exist 1–1 correspondences between $P(x^{\perp\perp})$, $P([x])$ and $P(x)$. Consider the Bernau representation of G

$$\begin{gathered}\pi\colon G \to \hat{G} \subseteq D(X_G),\\ x \to \hat{x} \in \hat{G}.\end{gathered}$$

By Theorem 3.3 in [2] the l-isomorphism π can be chosen such that $\hat{x}$ is the characteristic function of a clopen subset S of the Stone space X_G. Suppose $M_1 \in P(\hat{x}^{\perp\perp})$. Since G and $\hat{G}$ are complete, $\hat{x}^{\perp\perp}$ is also complete. So

$$\hat{x}^{\perp\perp} = M_1 \boxplus M_2.$$

Then

$$[\hat{x}] = M_1' \boxplus M_2',$$

where $M_1' = [\hat{x}] \cap M_1$, $M_2' = [\hat{x}] \cap M_2$. Hence

$$\hat{x} = x_1 + x_2, \ x_1 \wedge x_2 = 0, \tag{5.1}$$

where $x_1 \in M_1'$, $x_2 \in M_2'$, $M_1' = x^{\perp}_{2[\hat{x}]}$ and $M_2' = x^{\perp}_{1[\hat{x}]}$. On the other hand, if we have (5.1) and put $S_1 = \{\theta \in X_G \mid X_1(\theta) \neq 0\}$, $S_2 = \{\theta \in X_G \mid X_2(\theta) \neq 0\}$, then the support of $\hat{x}$ satisfies

$$S(\hat{x}) = S = S_1 \cup S_2 \text{ and } S_1 \cap S_2 = \emptyset$$

since

$$\hat{x}^{\perp\perp} = \{g \in \hat{G} \mid S(g) \subseteq S(\hat{x})\}$$

(see [2], p. 609). Put

$$\begin{aligned}M_1 &= \{g \in \hat{x}^{\perp\perp} \mid \theta \in S_2 \Rightarrow g(\theta) = 0\},\\ M_2 &= \{g \in \hat{x}^{\perp\perp} \mid \theta \in S_1 \Rightarrow g(\theta) = 0\}.\end{aligned}$$

Then

$$\hat{x}^{\perp\perp} = M_1 \boxplus M_2, \ [\hat{x}] = M_1' \boxplus M_2',$$

where $M = x^{\perp}_{2\hat{x}^{\perp\perp}}$, $M_2 = x^{\perp\perp}_{2\hat{x}^{\perp\perp}} = x^{\perp}_{1\hat{x}^{\perp\perp}}$ and $M_1' = [\hat{x}] \cap M_1$, $M_2' = [\hat{x}] \cap M_2$. Hence the map $\varphi\colon M_1 \to M_1'$ is 1–1 from $P(\hat{x}^{\perp\perp})$ onto $P([\hat{x}])$ and the map $\varphi'\colon M_1' \to x_1$ is 1–1 from $P([\hat{x}])$ onto $P(x)$. By Lemma 5.3, $\varphi'(\varphi'\varphi)$ is an anti-isomorphism from $P([\hat{x}])$ $(P(\hat{x}^{\perp\perp}))$ onto $P(x)$, and φ is an isomorphism from $P(\hat{x}^{\perp\perp})$ onto $P([\hat{x}])$. □

Let P be a Boolean algebra and $0 < x \in P$. Put

$$P_1(x) = \{a \in P \mid 0 \leqslant a \leqslant x\}.$$

Then $P_1(x)$ is a subalgebra of P. We call $P_1(x)$ a section in P.

Proposition 5.5. *Let P and P' be two complete Boolean algebras and $\{x_\alpha \mid \alpha \in A\}$, $\{x'_\alpha \mid \alpha \in A\}$ maximal disjoint subsets in P and P', respectively. If $P_1(x_\alpha) \simeq P_1(x'_\alpha)$ as Boolean algebras for $\alpha \in A$, then P is isomorphic to P'.*

Proof. Since $\{x_\alpha \mid \alpha \in A\}$ is a maximal disjoint subset, we have $\bigvee_{\alpha\in A} x_\alpha = 1$. Indeed, if $\bigvee_{\alpha\in A} x_\alpha < 1$, then $\{x_\alpha, 1 - \bigvee_{\alpha\in A} x_\alpha \mid \alpha \in A\}$ is also disjoint. For any $y \in P$ let

$$y_\alpha = y \wedge x_\alpha \in P_1(x_\alpha)$$

for $\alpha \in A$. Then

$$\bigvee_{\alpha\in A} y_\alpha = \bigvee_{\alpha\in A} (y \wedge x_\alpha) = y \wedge \Big(\bigvee_{\alpha\in A} x_\alpha\Big) = y.$$

We denote y by $y = (y_\alpha)$ and call y_α the coordinate of y in the section $\{P_1(x_\alpha) \mid \alpha \in A\}$. Let φ_α be isomorphism between $P_1(x_\alpha)$ and $P_1(x'_\alpha)$. Let $y'_\alpha = \varphi_\alpha(y_\alpha) \in P_1(x'_\alpha)$. Then

$$\bigvee_{\alpha\in A} = y' \in P'.$$

So we get a map $\varphi\colon y \to y'$ from P to P'. We proceed in the following three steps.

(1) φ is 1–1. If $y, z \in P$ and $y \neq z$, then $y = \bigvee_{\alpha\in A} y_\alpha$, $z = \bigvee_{\alpha\in A} z_\alpha$ and there exists the least $\alpha_0 \in A$ such that $y_{\alpha_0} \neq z_{\alpha_0}$. Consequently, $y'_{\alpha_0} \neq z'_{\alpha_0}$ in $P_1(x'_{\alpha_0})$. Hence

$$y' = \bigvee_{\alpha\in A} y'_\alpha \neq \bigvee_{\alpha\in A} z'_\alpha = z'.$$

Otherwise, $y' = z'$ implies $y'_\alpha = y' \wedge x'_\alpha = z' \wedge x'_\alpha = z'_\alpha$ for all $\alpha \in A$.

(2) φ is from P onto P'. For any $y' \in P'$, we have $y' = \bigvee_{\alpha \in A} y'_\alpha$ with $y'_\alpha = y' \wedge x'_\alpha \in P_1(x'_\alpha)$. Now each y'_α corresponds to $y_\alpha = \varphi_\alpha^{-1}(y'_\alpha) \in P_1(x_\alpha)$. So y' is the image of $y = \bigvee_{\alpha \in A} y_\alpha$ under φ.

(3) φ preserves $\vee$ and $\wedge$. Let $y' = \varphi(y)$, $z' = \varphi(z)$. Then

$$\begin{aligned}\varphi_\alpha[(y \vee z)_\alpha] &= \varphi_\alpha[(y \vee z) \wedge x_\alpha] = \varphi_\alpha[(y \wedge x_\alpha) \vee (z \wedge x_\alpha)] \\ &= \varphi_\alpha(y_\alpha \vee z_\alpha) = \varphi_\alpha(y_\alpha) \vee \varphi_\alpha(z_\alpha) = y'_\alpha \vee z'_\alpha \\ &= (y' \wedge x'_\alpha) \vee (z' \wedge x'_\alpha) = (y' \vee z') \wedge x'_\alpha \\ &= (y' \vee z')_\alpha.\end{aligned}$$

So

$$\varphi(y \vee z) = \varphi(y) \vee \varphi(z).$$

Similarly, we have $\varphi(y \wedge z) = \varphi(y) \wedge \varphi(z)$. □

Theorem 5.6. *Let G and G' be two complete l-groups. If there exist maximal disjoint subsets $\{x_\alpha \mid \alpha \in A\}$ and $\{x'_\alpha \mid \alpha \in A\}$ in G and G', respectively, such that $[x_\alpha] \simeq [x'_\alpha]$ for $\alpha \in A$, then the essential closures G^e and G'^e are l-isomorphic.*

Proof. We need only to show that $P(G) \simeq P(G')$ as Boolean algebras. By the Bernau representation it is clear that $x_1 \wedge x_2 = 0$ if and only if $x_1^{\perp\perp} \wedge x_2^{\perp\perp} = 0$ in $P(G)$. Hence $\{x_\alpha^{\perp\perp} \mid \alpha \in A\}$ and $\{x_\alpha'^{\perp\perp} \mid \alpha \in A\}$ are maximal disjoint subsets in $P(G)$ and $P(G')$, respectively. Now $[x_\alpha] \simeq [x'_\alpha]$ as l-groups implies $P([x_\alpha]) \simeq P([x'_\alpha])$ as Boolean algebras for $\alpha \in A$. By Lemma 5.4 we have

$$P_1(x_\alpha^{\perp\perp}) = P(x_\alpha^{\perp\perp}) \simeq P(x_\alpha'^{\perp\perp}) = P_1(x_\alpha'^{\perp\perp})$$

for $\alpha \in A$. Similarly to the proof of Theorem 1 in [2] we can show that $P(G)$ and $P(G')$ are complete Boolean algebras. The theorem follows immediately from Proposition 5.5. □

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Author's address: Dept. of Math. & Phys., Hohai University, 210024 Nanjing, People's Republic of China.